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HEAT KERNEL METHODS IN QUANTUM ANISOTROPIC YANG – MILLS THEORY

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We compute the renormalization of marginal operators in five-dimensional Yang – Mills theory exhibiting anisotropic scaling with dynamical critical exponent $z = 2$. Such a theory with the detailed balance condition is related via Parisi – Wu stochastic quantization to relativistic Yang – Mills theory in four spacetime dimensions. We use the background field method with zero temporal and static spatial components of the Yang – Mills background along with the heat kernel technique to derive the beta functions of all coupling constants of the theory.

Keywords: Lifshitz scaling, anisotropic Yang – Mills theory, heat kernel, background field method, beta functions, stochastic quantization

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1. INTRODUCTION

Quantum field theories of gauge fields in more than four spacetime dimensions are perturbatively nonrenormalizable and require an ultraviolet (UV) completion. One of the motivations for studying such theories comes from string theory, where gauge theories in higher dimensions naturally arise as effective descriptions of the low-energy dynamics of strings and branes. It is therefore interesting to explore field-theoretic mechanisms that can provide a UV completion of higher-dimensional gauge theories without necessarily relying on the string-theoretic framework.

A possible approach to making such a theory renormalizable is to consider anisotropic (Lifshitz) scaling [1] between time and space coordinates,

$$\begin{aligned} &t \to b^{-z} t, \\ &x_i \to b^{-1} x_i, \\ &i = 1, \ldots, d, \end{aligned} \tag{1}$$

with the dynamical critical exponent $z > 1$. Then, for the critical value of z, the quadratic action of the theory invariant under such scaling is allowed to contain higher spatial derivatives, which improve the convergence of loop integrals and make the theory power-counting renormalizable. At the same time, the absence of higher time derivatives renders the theory free from Ostrogradsky ghosts, so that unitarity is not violated [2]. Thus, in particular, it was shown that the projectable version of Hořava gravity is perturbatively power-counting renormalizable and unitary in the strict sense [3] and asymptotically free [4–6].

One example of such a higher-dimensional UV-complete theory is five-dimensional quantum electrodynamics with $z = 2$ Lifshitz scaling [7, 8]. It was shown that the theory possesses good UV properties: the gauge coupling constant is finite at all energy scales and at all orders of perturbation theory. This behavior follows from the fact that the photon anomalous dimension vanishes at all orders of perturbation theory, which implies that the electric charge is trivially finite.

We study a five-dimensional nonrelativistic Yang – Mills theory with $z = 2$ Lifshitz scaling between time and space, which is a non-Abelian generalization of the theory of [7, 8]. We consider the most general parity-even theory containing marginal operators only. Such a theory is also interesting in the framework of Parisi – Wu stochastic quantization [9–11], where the

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five-dimensional nonrelativistic Yang – Mills theory is related to relativistic Yang – Mills theory in four spacetime dimensions upon imposing the detailed balance condition [12]. As a consequence of this relation, the beta function for the kinetic part coupling constant, as well as part of the beta function for the potential part coupling constant of the five-dimensional theory, are known from the stochastic quantization results for the four-dimensional theory [13–15].

In this work, we employ the background-field method together with the heat-kernel technique [16] to compute the renormalization of the potential part of the action. At the one-loop level, this requires the calculation of the divergent part of the Lorentz-violating wave operator which, in a general gauge, is a nonminimal fourth-order differential operator whose time and space derivatives do not form a covariant five-dimensional Laplacian due to the anisotropic scaling (1). Consequently, the standard Schwinger – DeWitt [16–21] or Gilkey – Seeley [22–24] heat kernel techniques are not directly applicable. To evaluate the divergent part of the one-loop effective action, we therefore circumvent the noncovariance of the wave operator by performing a temporal Fourier transform, which reduces the problem to a purely spatial one for the higher-order differential operator. We treat the resulting operator using the technique developed in [25].

The paper is organized as follows. In Sec. 2 we formulate the theory, quantize it with the help of the background field formalism and perform the reduction to four dimensions to obtain the general expression for the one-loop effective action. In Sec. 3 we apply the heat kernel technique to a higher order operator which is a power of the minimal second order operator. In Sec. 4 we apply the commutator technique to a generic minimal higher order operator that includes, together with the power of the minimal one, also the lower derivative part. In Sec. 5 we derive the beta functions for the couplings of the potential part of the five-dimensional Yang – Mills action. We conclude in Sec. 6.

2. QUANTUM ANISOTROPIC YANG – MILLS THEORY

We consider a gauge theory with a gauge group chosen to be compact, to ensure a positive-definite kinetic term, and simple, to reduce the number of coupling constants. The action of the gauge theory, which is invariant under the scaling (1) and contains only parity-even marginal operators, reads

$$S = \frac{1}{g^2}\int dt\, d^4x\Big(\frac{1}{2}F^a_{i0}F^a_{i0} - \frac{\nu}{4}D_iF^a_{jk}\,D_iF^a_{jk} - \\ - \frac{\lambda}{2}D_iF^a_{ik}\,D_jF^a_{jk} - \frac{\kappa}{3}\,f^{abc}F^a_{ij}F^b_{jk}F^c_{ki}\Big), \quad (2)$$

where g, ν, λ, and κ are coupling constants and f^{abc} are the structure constants of the gauge group. This is the anisotropic counterpart of the Lorentz-invariant higher-derivative Yang – Mills theory [26] (see also [27–33]). The gauge field strength tensors and their spatial covariant derivatives have the form

$$F^a_{i0} = \partial_iA^a_0 - \partial_tA^a_i + f^{abc}A^b_iA^c_0\,, \\ F^a_{ij} = \partial_iA^a_j - \partial_jA^a_i + f^{abc}A^b_iA^c_j\,, \quad (3) \\ D_iF^a_{jk} = \partial_iF^a_{jk} + f^{abc}A^b_iF^c_{jk}\,.$$

The theory is invariant under the gauge transformations

$$\delta_\epsilon A^a_i = \partial_i\epsilon^a + f^{abc}A^b_i\epsilon^c = D_i\epsilon^a, \\ \delta_\epsilon A^a_0 = \dot\epsilon^a + f^{abc}A^b_0\epsilon^c = D_t\epsilon^a\,. \quad (4)$$

In what follows we denote the temporal component of the gauge field by the calligraphic letter,

$$\mathcal{A}^a \equiv A^a_0. \quad (5)$$

Under the anisotropic scaling (1), the fields are assigned the following dimensions

$$\dim\mathcal{A}^a = 2, \\ \dim A^a_i = 1. \quad (6)$$

The Lagrangian in (2) consists of all possible nontrivial marginal terms which form a closed set under renormalization and determine the UV behavior of the theory. One can consider other parity-even terms[1)] of dimension 6 that are consistent with the scaling (1), for example, $F^a_{0i}D_iF^a_{ij}$. With the help of the time – space – space component of the Bianchi identity, one can show that this term is a total derivative [34]

$$F^a_{0j}D_iF^a_{ij} = \partial_j(F^a_{0i}F^a_{ji}) - \frac{1}{4}\partial_0(F^a_{ij}F^a_{ij}). \quad (7)$$

Also, it can be shown that the potential terms are related and therefore only two of them are independent

$$f^{abc}F^a_{ij}F^b_{jk}F^c_{ki} = D_iF^a_{ij}D_kF^a_{kj} - \frac{1}{2}D_iF^a_{jk}D_iF^a_{jk} - \\ - \partial_j(F^a_{ik}D_iF^a_{kj} - F^a_{ij}D_kF^a_{ki}). \quad (8)$$

[1)] In this work we do not consider parity-odd operators such as $\epsilon_{ijkl}D_mF^a_{ij}D_mF^a_{kl}$.

In (2) we set $\nu = 0$ to have a unique gradient term in view of the cyclic Jacobi identity for F^a_{ij}.

We perform a Wick rotation:

$$\tau = it, \quad \mathcal{A}^a_E = -i\mathcal{A}^a,$$

after which the action reads

$$S = \frac{1}{g^2}\int d\tau\, d^4x\Big(\frac{1}{2}F^a_{i0}F^a_{i0} + \frac{\lambda}{2}D_iF^a_{ik}\,D_jF^a_{jk} + \\ + \frac{\kappa}{3}\,f^{abc}F^a_{ij}F^b_{jk}F^c_{ki}\Big). \quad (9)$$

In what follows, we drop the Euclidean label E.

When $\kappa = 0$, the potential in (9) is the square of the equations of motion of the four-dimensional relativistic Yang – Mills theory. One says that d-dimensional and $(d+1)$-dimensional theories are related by the detailed balance condition when the potential part (without time derivatives) of the $(d+1)$-dimensional theory is equal to the square of the variation of the action of the d-dimensional theory. Within the framework of stochastic quantization, it has been shown that the renormalization group (RG) flows of theories related by the detailed balance condition coincide [11]. Therefore, the running of the kinetic coupling g and the κ-independent running of λ are known from the stochastic quantization results [13–15]

$$\beta_g = -\frac{3}{4}\frac{C_2}{(4\pi)^2}\frac{g^3}{\sqrt{\lambda}}, \\ \beta_\lambda\Big|_{\kappa=0} = \frac{13}{3}\frac{C_2}{(4\pi)^2}g^2\sqrt{\lambda}, \quad (10)$$

where C_2 is the quadratic Casimir in the adjoint representation.

2.1. Background field method

To find the renormalization of the coupling constants from the potential part of the action, we use the background field method and choose a zero temporal and general static spatial background

$$\mathcal{A}^a = 0 + b^a(t,x), \\ A^a_i = \bar{A}^a_i(x) + a^a_i(t,x). \quad (11)$$

Then we expand the action (9) up to the second order in quantum fluctuations and evaluate the resulting Gaussian path integral over b^a, a^a_i to obtain the renormalization of the coupling constants.

On this background the kinetic part of the action is zero, while the potential parts are non-zero and read

$$S[\bar{A}^a_i] = \frac{1}{g^2}\int d\tau\, d^4x\Big(\frac{\lambda}{2}\bar{D}_i\bar{F}^a_{ik}\,\bar{D}_j\bar{F}^a_{jk} + \\ + \frac{\kappa}{3}\,f^{abc}\bar{F}^a_{ij}\bar{F}^b_{jk}\bar{F}^c_{ki}\Big). \quad (12)$$

We calculate one-loop contributions to these operators to obtain the beta functions of λ and κ.

2.2. Gauge fixing and ghosts

Gauge fixing is implemented by adding to the action the term [3]

$$S_{gf} = \frac{1}{2g^2}\int d\tau\, d^4x\, F^a O^{-1}_{ab} F^b\,, \quad (13)$$

where the gauge-fixing function F^a and the operator O^{-1}_{ab} read

$$F^a = \partial_\tau b^a + \xi(-\bar{D}^2)\bar{D}_i a^a_i\,, \quad O^{-1}_{ab} = \frac{\sigma\delta_{ab}}{(-\bar{D}^2)}\,, \quad (14)$$

with σ and ξ being arbitrary gauge-fixing parameters and $\bar{D}^2 = \bar{D}_i\bar{D}_i$. Explicitly, the gauge-fixing action reads

$$S_{gf} = \frac{1}{2g^2}\int d\tau\, d^4x\Big(-b^a\frac{\sigma}{(-\bar{D}^2)}\partial^2_\tau b^a - \\ - \sigma\xi^2 a^a_i\,\bar{D}_i(-\bar{D}^2)\bar{D}_j a^a_j - 2\sigma\xi\, b^a\bar{D}_i\partial_\tau a^a_i\Big). \quad (15)$$

The non-locality in the b-sector can be eliminated by integrating in an auxiliary field [3].

The kinetic term of the Lagrangian (9) quadratic in quantum fields reads on the background (11)

$$\frac{1}{2g^2}\int d\tau\, d^4x\, F^a_{i0}F^a_{i0} \simeq \frac{1}{2g^2}\int d\tau\, d^4x\Big(b^a(-\bar{D}^2)b^a - \\ - a^a_i\partial^2_\tau a^a_i + 2b^a\bar{D}_i\partial_\tau a^a_i\Big). \quad (16)$$

To cancel the cross terms between b^a and a^a_i, we choose the gauge-fixing parameter to be $\sigma = 1/\xi$.

The Faddeev – Popov ghost action can be obtained as a BRST transformation of the gauge-fixing function

$$S_{gh} = -\frac{1}{g^2}\int d\tau\, d^4x\, \bar{c}^a(\mathbf{s}F^a). \quad (17)$$

The BRST transformation is a gauge transformation (4) with the gauge parameter ϵ^a replaced by the ghost field c^a. On the background (11) the BRST transformation has the form

$$\mathbf{s}b^a = \partial_\tau c^a, \\ \mathbf{s}a^a_i = \bar{D}_i c^a. \quad (18)$$

After the substitution of (18) into (17) one obtains the ghost action

$$S_{gh} = \frac{1}{g^2}\int d\tau\, d^4x\,\left(-\bar{c}^a\partial^2_\tau c^a + \xi\,\bar{c}^a(-\bar{D}^2)^2 c^a\right). \quad (19)$$

2.3. Effective action

The one-loop effective action is given by the Gaussian path integral over the fluctuations,

$$\exp\left(-\Gamma^{1-\text{loop}}\right) = \sqrt{\text{Det}\, O^{-1}_{ab}} \times \\ \times \int Da_i DbDcD\bar{c}\, \exp(-S^{(2)}[a_i, b, c, \bar{c}]). \quad (20)$$

The normalization factor $\sqrt{\text{Det}\, O^{-1}_{ab}}$ comes from the smearing of the gauge-fixing conditions with a Gaussian weight which leads to the gauge-fixing term (15) [3]. The quadratic action reads[2)]

$$S^{(2)}[a_i, b, c, \bar{c}] = \\ = \int d\tau\, d^4x \Big(\frac{1}{2} a^a_i \left(-\delta^{ab}\delta_{ij}\partial^2_\tau + \mathbb{E}^{ab}_{ij}\right) a^b_j + \\ + \frac{1}{2} b^a \frac{1}{\xi(-\bar{D}^2)} \left(-\delta^{ab}\partial^2_\tau + \xi\delta^{ab}(-\bar{D}^2)^2\right) b^b + \\ + \bar{c}^a \left(-\delta^{ab}\partial^2_\tau + \xi\delta^{ab}(-\bar{D}^2)^2\right) c^b \Big), \quad (21)$$

and the operator $\mathbb{E}^{ab}_{ij}$ has the form

$$\mathbb{E}^{ab}_{ij} = \lambda\delta_{ij}\delta^{ab}(-D^2)^2 + (\lambda-\xi)\delta^{ab}D_iD_j(-D^2) + 2(2\lambda+\kappa)f^{abc}F^c_{ij}(-D^2) + 3\lambda f^{abc}F^c_{ik}D_jD_k + \\ + (\lambda - 4\kappa + 2\xi)f^{abc}F^c_{jk}D_kD_i + 2\kappa f^{acb}\delta_{ij}F^c_{kl}D_kD_l + 2(\lambda+2\kappa-\xi)f^{abc}(D_kF^c_{kj})D_i - \\ - 2(\lambda+\kappa)f^{abc}\delta_{ij}(D_kF^c_{kl})D_l - 2\kappa f^{abc}(D_kF^c_{ij})D_k + (\lambda+2\xi)f^{abc}(D_iF^c_{jk})D_k + \\ + (2\kappa-\lambda-2\xi)f^{abd}f^{dce}F^c_{ik}F^e_{jk} + 2(\lambda+\xi)f^{acd}f^{bed}F^c_{ik}F^e_{jk} + (2\lambda-\xi)f^{abc}(D_iD_kF^c_{kj}). \quad (22)$$

Here and below, the covariant derivatives and the field strength tensors are defined with respect to the background field.

The result of the integration

$$e^{-\Gamma^{1-\text{loop}}} = \frac{\sqrt{\text{Det}\frac{1}{\xi(-D^2)}}\,\text{Det}\left(-\delta^{ab}\partial^2_\tau + \xi\delta^{ab}(-D^2)^2\right)}{\sqrt{\text{Det}\left[\frac{1}{\xi(-D^2)}\left(-\delta^{ab}\partial^2_\tau + \xi\delta^{ab}(-D^2)^2\right)\right]}\sqrt{\text{Det}\left(-\delta^{ab}\delta_{ij}\partial^2_\tau + \mathbb{E}^{ab}_{ij}\right)}} \quad (23)$$

can be transformed into the sum of two terms

$$\Gamma^{1-\text{loop}} = \frac{1}{2}\text{Tr}\ln\left(-\delta^{ab}\delta_{ij}\partial^2_\tau + \mathbb{E}^{ab}_{ij}\right) - \\ - \frac{1}{2}\text{Tr}\ln\left(-\delta^{ab}\partial^2_\tau + \xi\delta^{ab}(-D^2)^2\right). \quad (24)$$

The non-locality in the b-sector factors out and cancels with the normalization factor which arises from the gauge-fixing procedure. After such a cancellation, the resulting b-determinant coincides with the ghost determinant.

The operator $\mathbb{E}$ is a fourth order differential operator and is non-minimal in the general gauge. To simplify the calculation, we work in the gauge

$$\xi = \lambda, \quad (25)$$

where the operator $\mathbb{E}$ is minimal, i.e. its principal part is a power of a Laplacian. Calculations in the general gauge are possible [35, 36] and rely on the recently developed technique [37].

2.4. Perturbation theory

We denote the purely derivative part of the operator $\mathbb{E}$ as $\mathbb{E}^{(0)}$ and the lower derivative part ("perturbation") as $\bar{\mathbb{E}}$

$$\mathbb{E} = \mathbb{E}^{(0)} + \bar{\mathbb{E}}, \\ \mathbb{E}^{(0)} = \lambda\delta^{ab}\delta_{ij}(-D^2)^2. \quad (26)$$

The vector part of the one-loop effective action (24) can be transformed in the following way

[2)] We redefined the fields as $(a^a\, b^a\, c^a\, \bar{c}^a) \to g(a^a\, b^a\, c^a\, \bar{c}^a)$ to get rid of the overall factor $1/g^2$.

$$\Gamma^{1-\text{loop}} = \frac{1}{2}\mathrm{Tr}\ln\left(-\hat{1}\partial_\tau^2 + \mathbb{E}\right) = -\frac{1}{2}\int d\tau\, d^4x \int\limits_0^\infty \frac{ds}{s}\mathrm{tr}\, e^{-s\left(-\hat{1}\partial_\tau^2+\mathbb{E}\right)}\delta(\tau,\tau')\delta(x,x')\Big|_{\substack{\tau'=\tau\\x'=x}} =$$

$$= -\frac{1}{2}\int d\tau\, d^4x \int\limits_0^\infty \frac{ds}{s}\,\mathrm{tr}\, e^{s\hat{1}\partial_\tau^2}\, e^{-s\mathbb{E}} \int\limits_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')}\delta(x,x')\Big|_{\substack{\tau'=\tau\\x'=x}} =$$

$$= -\frac{1}{2}\int d\tau\, d^4x \int\limits_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-s\omega^2} \int\limits_0^\infty \frac{ds}{s}\mathrm{tr}\, e^{-s\left(\mathbb{E}^{(0)}+\bar{\mathbb{E}}\right)}\delta(x,x')\Big|_{x'=x}. \quad (27)$$

Note that in the transition to the third line, the static nature of the background allowed us to disentangle the time derivatives in the operator exponent without any Baker – Campbell – Hausdorff commutators.

The renormalizability of the theory requires that the counterterms have the same structure as the terms in the bare action, whose background dimensionality is 6. The lowest background dimensionality of the terms in the operator $\bar{\mathbb{E}}$ (22) is 2. Therefore, we expand the spatial operator exponential up to the third order in the perturbation $\bar{\mathbb{E}}$

$$e^{-s\left(\mathbb{E}^{(0)}+\bar{\mathbb{E}}\right)} \simeq e^{-s\mathbb{E}^{(0)}} - \int\limits_0^s ds_1\, e^{-(s-s_1)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-s_1\mathbb{E}^{(0)}} +$$

$$+ \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2\, e^{-(s-s_1)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-(s_1-s_2)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-s_2\mathbb{E}^{(0)}} -$$

$$- \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2 \int\limits_0^{s_2} ds_3\, e^{-(s-s_1)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-(s_1-s_2)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-(s_2-s_3)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-s_3\mathbb{E}^{(0)}}, \quad (28)$$

where $\simeq$ means equality up to terms of higher background dimensionality. Substitution of this expansion into (27) yields

$$\Gamma^{1-\text{loop}} \simeq -\frac{1}{2}\int d\tau\, d^4x \int\limits_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-s\omega^2} \int\limits_0^\infty \frac{ds}{s}\mathrm{tr}\Big[e^{-s\mathbb{E}^{(0)}} - \int\limits_0^s ds_1\, \bar{\mathbb{E}}e^{-s\mathbb{E}^{(0)}} +$$

$$+ \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2\, \bar{\mathbb{E}}e^{-(s_1-s_2)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{(s_1-s_2)\mathbb{E}^{(0)}}e^{-s\mathbb{E}^{(0)}} -$$

$$- \int\limits_0^s ds_1 \int\limits_0^{s_1} ds_2 \int\limits_0^{s_2} ds_3\, \bar{\mathbb{E}}e^{-(s_1-s_2)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-(s_2-s_3)\mathbb{E}^{(0)}}\bar{\mathbb{E}}e^{-(s_3-s_1)\mathbb{E}^{(0)}}e^{-s\mathbb{E}^{(0)}}\Big]\delta(x,x')\Big|_{x'=x}, \quad (29)$$

where in the last transition we used the cyclicity of the trace. Now we make use of the formula

$$e^{-tA}Be^{tA} = \sum_{n=0}^\infty \frac{(-t)^n}{n!}[\underbrace{A,[A,\ldots,[A}_n,B],\ldots]], \quad (30)$$

for $t = s_1 - s_2$, integrate over s_1, s_2, s_3 and take into account that in the third line of (29) all operators effectively commute to obtain

$$\Gamma^{1-\text{loop}} \simeq -\frac{1}{4\sqrt{\pi}}\int d\tau \int\limits_0^\infty \frac{ds}{s^{3/2}}\mathrm{Tr}_4\Big(\hat{1} - s\bar{\mathbb{E}} + \frac{s^2}{2}\bar{\mathbb{E}}^2 - \frac{s^3}{3!}\bar{\mathbb{E}}[\mathbb{E}^{(0)},\bar{\mathbb{E}}] + \frac{s^4}{4!}\bar{\mathbb{E}}[\mathbb{E}^{(0)},[\mathbb{E}^{(0)},\bar{\mathbb{E}}]] - \frac{s^3}{3!}\bar{\mathbb{E}}^3\Big)e^{-s\mathbb{E}^{(0)}}. \quad (31)$$

Thus, the evaluation of the one-loop effective action reduces to the calculation of powers of the perturbation $\bar{\mathbb{E}}$, the nested commutators of $\bar{\mathbb{E}}$ with $\mathbb{E}^{(0)}$, and the subsequent action of these operators on the heat kernel of the purely differential minimal fourth-order operator $\mathbb{E}^{(0)}$.

Applying the same procedure to the scalar trace, we obtain

$$\Gamma^{1-\text{loop}}_{\text{scalar}} = -\frac{1}{4\sqrt{\pi}}\int d\tau \int\limits_0^\infty \frac{ds}{s^{3/2}} \operatorname{Tr}_4 e^{-s\,\xi(-D^2)^2} = \\ = \frac{1}{2}\int d\tau \operatorname{Tr}_4\left(\sqrt{\xi}(-D^2)\right). \quad (32)$$

In the last step, we performed the proper time integration and extracted the square root of the operator. The divergent part of the functional trace of a local operator gives rise to a power divergence, which vanishes in dimensional regularization. Therefore,

$$\operatorname{Tr}_4\left(\sqrt{\xi}(-D^2)\right)\Big|^{\text{div}} = 0\,. \quad (33)$$

Consequently, the scalar trace does not contribute to the divergent part of the one-loop effective action.

3. HEAT KERNEL FOR A POWER OF THE MINIMAL OPERATOR

As discussed above, in the gauge (25) the operator $\mathbb{E}^{(0)}$ is the integer power of a minimal operator $\mathbb{F}$

$$\mathbb{E}^{(0)} = \lambda\mathbb{F}^2, \\ \mathbb{F} = \delta_{ij}\delta^{ab}(-D^2). \quad (34)$$

The heat kernel of the operator $\mathbb{E}^{(0)}$ is therefore given by [38]

$$\hat{K}_{\mathbb{E}^{(0)}}(s\,|\,x,x') = e^{-s\lambda\mathbb{F}^2}\delta(x,x') = \\ = \frac{\Delta^{1/2}(x,x')}{(4\pi)^2\lambda s}\sum_{l=0}^{\infty}\mathcal{E}_{2,2-l}\left(-\frac{\sigma(x,x')}{2(\lambda s)^{1/2}}\right)\times \\ \times(\lambda s)^{l/2}\hat{a}_l(\mathbb{F}|x,x')\,, \quad (35)$$

where $\sigma(x,x')$ is the Synge world function (one-half of the square of the geodesic distance between the points x and x'), $\Delta^{1/2}(x,x')$ is the dedensitized Pauli – Van Vleck – Morette determinant built of $\sigma(x,x')$[3] and $\hat{a}_l(\mathbb{F}|x,x')$ are the Schwinger – DeWitt or Gilkey – Seeley coefficients of the operator $\mathbb{F}$ (the hat indicates their matrix nature, $\hat{a}_l = a_{l\,ij}$). The generalized exponential functions $\mathcal{E}_{\nu,\alpha}(z)$ are defined by the series

$$\mathcal{E}_{\nu,\alpha}(z) = \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\alpha+k}{\nu}\right)}{\Gamma(\alpha+k)}\frac{z^k}{k!}. \quad (36)$$

Their multiple derivatives and particular value are

$$\frac{d^n}{dz^n}\mathcal{E}_{\nu,\alpha}(z) = \mathcal{E}_{\nu,\alpha+n}(z), \\ \mathcal{E}_{\nu,\alpha}(0) = \frac{1}{\nu}\frac{\Gamma\left(\frac{\alpha}{\nu}\right)}{\Gamma(\alpha)}. \quad (37)$$

After the substitution of (35) into (31), the evaluation of the functional trace boils down to finding the coincidence limits of the expressions of the form

$$\int d\tau\, d^4x\, \mathcal{F}\,\nabla\nabla\ldots\nabla\Big(\frac{\Delta^{1/2}(x,x')}{(4\pi)^2\lambda s}\times \\ \times\sum_{l=0}^{\infty}\mathcal{E}_{2,2-l}\left(-\frac{\sigma(x,x')}{2(\lambda s)^{1/2}}\right)(\lambda s)^{l/2}\hat{a}_l(\mathbb{F}|x,x')\Big)\Big|_{x'=x}, \quad (38)$$

where $\mathcal{F}$ are background field tensors built of the gauge field strength and its derivatives. Using the tabulated coincidence limits for the Synge world function, the Pauli – Van Vleck – Morette determinant, and the Schwinger – DeWitt coefficients [20] as well as the standard Casimir identities for the structure constants [39]

$$f^{abc}f^{abd} = C_2(G)\delta^{cd}, \\ f^{ade}f^{bef}f^{cfd} = \frac{C_2(G)}{2}f^{abc}, \quad (39)$$

we find the divergent part of the one-loop effective action on the background (11)

$$\Gamma^{1-\text{loop}}\big|^{\text{div}} = \frac{\ln L^4}{(4\pi)^2}\int d\tau\, d^4x\,\big(C_\lambda D_i F^a_{ik} D_j F^a_{jk} + \\ + C_\kappa f^{abc}F^a_{ij}F^b_{jk}F^c_{ki}\big)\,, \quad (40)$$

where $\ln L^4$ is the logarithmic divergence

$$\ln L^4 = \int\frac{ds}{s} \simeq \ln\left(\frac{\Lambda^4_{UV}}{k^4_*}\right), \quad (41)$$

and the counterterms have the form

$$C_\lambda = -\frac{C_2\lambda(35\lambda^2+6\lambda\kappa+\kappa^2)}{48\lambda^{3/2}}\,, \\ C_\kappa = -\frac{C_2\kappa(32\lambda^2+2\lambda\kappa-\kappa^2)}{64\lambda^{5/2}}\,. \quad (42)$$

[3] We work in flat space, where $\sigma(x,x') = (x-x')^2/2$ and $\Delta(x,x') = 1$. The general expressions are retained to keep the formulation applicable to curved spacetimes.

These counterterms represent the main result of the calculation.

In this section, we took into account the contribution of the perturbation $\bar{\mathbb{E}}$ of the operator $\mathbb{E}$ by expanding the exponential of the full operator up to the required order in the background dimensionality (28). Although this approach is straightforward, it is not systematic – the explicit form of (31) depends on the operator, and the entire procedure must be repeated for every new one. In the next section, we use the commutator technique [25], which allows one to run the perturbation theory in a systematic way. Moreover, the computation done by another method provides a powerful check of the results.

4. COMMUTATOR TECHNIQUE

4.1. Review of the method

We start with a brief review of the commutator technique for the case of the operator $\mathbb{E} = \lambda\mathbb{F}^2 + \bar{\mathbb{E}}$. The diagonal heat kernel expansion has the form

$$\hat{K}_{\mathbb{E}}(s|x,x) = \sum_{m=0}^{\infty} s^{\frac{m}{2}-1}\hat{E}_{2m}(\mathbb{E}|x), \tag{43}$$

where the Gilkey – Seeley coefficients $\hat{E}_{2m}(\mathbb{E}|x)$ equal

$$\hat{E}_{2m}(\mathbb{E}|x) = \frac{1}{(4\pi)^2}\sum_{k=0}^{2m}\frac{\Gamma\left(k-\frac{m}{2}+1\right)}{2\Gamma\left(2k+2-m\right)}\left[\,\hat{a}_{m,k}(\mathbb{E})\,\right]. \tag{44}$$

The square brackets denote the coincidence limit. According to [25], the double-indexed coefficients of this expansion $\hat{a}_{m,k}(\mathbb{E}|x,x')$ are given by the expression

$$\hat{a}_{m,k}(\mathbb{E}|x,x') = \sum_{l\geq L_{m,k}}^{m+k}\left\{\hat{U}_k(D_x)\right\}_{2k+l-m}\times$$
$$\times\,\Delta^{1/2}(x,x')\,\hat{a}_l(\mathbb{F}|x,x'),$$
$$L_{m,k} = \max\left\{0, m-2k\right\}, \quad -\infty < m < \infty, \tag{45}$$

in terms of the Schwinger – DeWitt coefficients $\hat{a}_l(\mathbb{F}|x,x')$ of the operator $\mathbb{F}(D)$ weighted by the Pauli – Van Vleck factor and acted upon (on the argument x) by special differential operators $\{\hat{U}_k(D_x)\}_{2k+l-m}$.

The construction of these operators consists of three steps. First, one introduces the multiple nested commutator of k-th order

$$\hat{V}_k(D) = [\![\bar{\mathbb{E}},\mathbb{F}^2]\!]_k \equiv \big[\ldots\big[[\bar{\mathbb{E}},\underbrace{\mathbb{F}^2],\mathbb{F}^2\big],\ldots,\mathbb{F}^2}_{k}\big]. \tag{46}$$

The second step is the construction of the sequence of local differential operators $\hat{U}_k(D)$ defined by the relations

$$\hat{U}_0(D) = \hat{1}\,,$$
$$\hat{U}_k(D) =$$
$$\sum_{\substack{n,k_1,\cdots k_n\\ n+|k|=k}}(-1)^n\frac{\hat{V}_{k_1}(D)\cdots\hat{V}_{k_n}(D)}{k_1!\cdots k_n!c(k)},\ k>0, \tag{47}$$
$$c(k) = (k_1+1)(k_1+k_2+2)\cdots$$
$$\cdots(k_1+\cdots+k_n+n),$$

where the summation runs over the indicated domain of integer indices $n, k_1, \ldots, k_n$, $n+|k|=k$, and we use the standard multi-index notation:

$$k = (k_1,\ldots,k_n), \quad |k| = k_1+\cdots+k_n.$$

The order of the differential operators $\hat{U}_k(D)$ is $3k$, their physical dimension is $4k$, and their background dimensionality is

$$\dim\hat{U}_k(D) = \sum_{i=1}^{n}\dim\hat{V}_{k_i} = n+|k| = k.$$

The last step is the transition

$$\hat{U}_k(D)\to\{\hat{U}_k(D)\}_{2k+l-m}.$$

It is the so-called syngification operation,

$$\hat{F}(D)\to\{\hat{F}(D)\}_n,$$

denoted by curly brackets with the subscript $n = 2k+l-m$. This operation can be defined for any N-th order local differential operator

$$\hat{F}(D) = \sum_{k=0}^{N}\hat{F}_k^{i_1\ldots i_k}(x)D_{i_1}\cdots D_{i_k}$$

as

$$\{\hat{F}\,\}_n = \sum_{j=n}^{N}\hat{F}^{i_1\ldots i_j}\left\{D_{i_1}\ldots D_{i_j}\right\}_n, \tag{48}$$

where $\{D_{i_1}\ldots D_{i_j}\}_n$ represents the sum of all possible monomials $D_{i_1}\ldots D_{i_j}$ in which n covariant derivatives D_i are replaced by the relevant derivatives of the Synge world function

$$-D_i\sigma(x,x')/2 = -\sigma_i(x,x')/2.$$

Such a replacement should be performed without changing the ordering of the remaining $(j-n)$ derivatives and the n newly introduced factors.

4.2. Computation

The one-loop effective action in (27) is expressed as an integral over time and Schwinger proper time, with the integrand given by the purely spatial functional trace of the heat kernel of the operator $\mathbb{E}$, weighted by $s^{-3/2}$

$$\Gamma^{1-\text{loop}} = -\frac{1}{4\sqrt{\pi}} \int d\tau \int\limits_0^\infty \frac{ds}{s^{3/2}} \text{Tr}_4\, e^{-s\mathbb{E}}. \tag{49}$$

Substituting the diagonal heat kernel expansion (43)–(44) and singling out the logarithmically divergent part, we obtain the following expression

$$\begin{aligned}\Gamma^{1-\text{loop}}\Big|^{\text{div}} = -\frac{\ln L^4}{2(4\pi)^{5/2}} \sum_{k=0}^{6} \frac{\Gamma\left(k-\frac{1}{2}\right)}{2\Gamma\left(2k-1\right)} \times \\ \times \int d\tau\, d^4x\, [\,\hat{a}_{3,k}(\mathbb{E})\,]\,.\end{aligned} \tag{50}$$

First, we note that the functional trace of a single $\hat{V}_k(D)$ vanishes, since every $\hat{V}_k(D)$ for $k \geq 1$ is a commutator (46),

$$\text{Tr}\,\hat{V}_k = \text{Tr}\,([\hat{V}_{k-1}, \mathbb{F}^2]) = 0, \quad \text{for} \quad k \geq 1. \tag{51}$$

Thus, the relevant $\hat{U}_k$ operators (47) take the form

$$\begin{aligned}&\hat{U}_0 = \hat{1}, \quad \hat{U}_1 = -\hat{V}_0, \quad \hat{U}_2 = \frac{1}{2}\hat{V}_0^2 + (\ldots),\\ &\hat{U}_3 = \frac{1}{3}\hat{V}_0\hat{V}_1 + \frac{1}{6}\hat{V}_1\hat{V}_0 - \frac{1}{6}\hat{V}_0^3 + (\ldots),\\ &\hat{U}_4 = \frac{1}{8}\hat{V}_0\hat{V}_2 + \frac{1}{24}\hat{V}_2\hat{V}_0 + \frac{1}{8}\hat{V}_1^2 + (\ldots)\,,\\ &\hat{U}_5 = 0 + (\ldots)\,, \quad \hat{U}_6 = 0 + (\ldots)\,,\end{aligned} \tag{52}$$

where the ellipses denote the terms of the above type that do not contribute to the functional trace.

There are five coefficients $\hat{a}_{3,k}(\mathbb{E}|x,x)$ which contribute to the divergent part of $\Gamma^{1-\text{loop}}$

$$\begin{aligned}&\hat{a}_{3,0}(\mathbb{E}|x,x) = \hat{a}_3(\mathbb{F}|x,x),\\ &\hat{a}_{3,1}(\mathbb{E}|x,x) =\\ &= \sum_{l=1}^{2} \left[\left\{\hat{U}_1\right\}_{l-1} \Delta^{1/2}(x,x')\, \hat{a}_l(\mathbb{F}|x,x')\right],\\ &\hat{a}_{3,2}(\mathbb{E}|x,x) =\\ &= \sum_{l=0}^{1} \left[\left\{\hat{U}_2\right\}_{l+1} \Delta^{1/2}(x,x')\, \hat{a}_l(\mathbb{F}|x,x')\right],\\ &\hat{a}_{3,3}(\mathbb{E}|x,x) = \left[\left\{\hat{U}_3\right\}_3 \Delta^{1/2}(x,x')\, \hat{a}_0(\mathbb{F}|x,x')\right],\\ &\hat{a}_{3,4}(\mathbb{E}|x,x) = \left[\left\{\hat{U}_4\right\}_5 \Delta^{1/2}(x,x')\, \hat{a}_0(\mathbb{F}|x,x')\right].\end{aligned} \tag{53}$$

Therefore, the evaluation of the divergent part of the one-loop effective action (50) boils down to the computation of the coincidence limits of the form

$$\left[\left\{D^x_{i_1}\ldots D^x_{i_n}\right\}_k \Delta^{1/2}(x,x')\, \hat{a}_l(\mathbb{F}|x,x')\right]. \tag{54}$$

The syngified derivative monomial $\{D^x_{i_1}\ldots D^x_{i_n}\}_k$ is a differential operator of order $n-2k$ which acts on $\Delta^{1/2}(x,x')$ and $\hat{a}_l(\mathbb{F}|x,x')$. Therefore, the objects (54) can be expressed through the coincidence limits for the Synge world function, the Pauli – Van Vleck – Morette determinant, and the Schwinger – DeWitt coefficients.

In the case of (53), one needs to compute coincidence limits of the following syngified derivative monomials

$$\begin{aligned}&\left[\{D_iD_jD_kD_l\}_1\right], \quad \left[\{D_iD_jD_k\}_1\right],\\ &\left[\{D_{i_1}\ldots D_{i_{2n}}\}_n\right], \quad n = 1,2,3,4,5.\end{aligned} \tag{55}$$

The first two objects are differential operators of the second and first order, respectively. Using the coincidence limit $[D_iD_j\sigma] = \delta_{ij}$, we obtain

$$\left[\{D_iD_jD_k\}_1\right] = -\frac{1}{2}\big(\delta_{ij}D_k + \delta_{ik}D_j + \delta_{jk}D_i\big), \tag{56}$$

$$\begin{aligned}\left[\{D_iD_jD_kD_l\}_1\right] = -\frac{1}{2}\big(&\delta_{kl}D_iD_j + \delta_{jl}D_iD_k +\\ &+ \delta_{jk}D_iD_l + \delta_{ik}D_jD_l + \delta_{il}D_jD_k + \delta_{ij}D_kD_l\big).\end{aligned} \tag{57}$$

For the last objects in (55) one can derive the general formula

$$[\{D_{i_1}\ldots D_{i_{2n}}\}_n] = \left(-\frac{1}{2}\right)^n \delta^{(n)}_{i_1i_2\ldots i_{2n}}, \tag{58}$$

where $\delta^{(n)}_{i_1i_2\ldots i_{2n}}$ is the symmetrized sum of Kronecker symbols

$$\begin{aligned}&\delta^{(1)}_{i_1i_2} = \delta_{i_1i_2},\\ &\delta^{(2)}_{i_1i_2i_3i_4} = \delta_{i_1i_2}\delta_{i_3i_4} + \delta_{i_1i_3}\delta_{i_2i_4} + \delta_{i_1i_4}\delta_{i_2i_3},\\ &\delta^{(n)}_{i_1i_2\ldots i_{2n}} = \sum_{k=2}^{2n} \delta_{i_1i_k}\delta^{(n-1)}_{i_2i_3\ldots\hat{i}_k\ldots i_{2n}}.\end{aligned} \tag{59}$$

Application of the above machinery shows that the commutator technique result (50) confirms the expression (42) for the potential part divergences (40) obtained in the previous section.

5. BETA FUNCTIONS

The beta functions are defined as the logarithmic derivatives of the renormalized couplings with respect to the RG reference scale k_*,

$$\beta_\chi \equiv \frac{d\chi_R}{d\ln k_*}, \quad \chi = g, \lambda, \kappa. \tag{60}$$

We extract the renormalized couplings by comparing the action (12) with the divergent part of the one-loop effective action (40)

$$\begin{aligned}\left(\frac{\lambda}{2g^2}\right)_{\rm R} &= \frac{\lambda}{2g^2} + C_\lambda \ln\left(\frac{\Lambda_{UV}}{k_*}\right)^4,\\ \left(\frac{\kappa}{3g^2}\right)_{\rm R} &= \frac{\kappa}{3g^2} + C_\kappa \ln\left(\frac{\Lambda_{UV}}{k_*}\right)^4.\end{aligned} \tag{61}$$

Therefore, the beta functions of the couplings λ and κ have the form

$$\beta_\lambda = \frac{13}{3}c_2 g^2\sqrt{\lambda}\left(1+\frac{3}{13}\frac{\kappa}{\lambda}+\frac{1}{26}\frac{\kappa^2}{\lambda^2}\right), \tag{62}$$

$$\beta_\kappa = \frac{9}{2}c_2 g^2\frac{\kappa}{\sqrt{\lambda}}\left(1+\frac{1}{12}\frac{\kappa}{\lambda}-\frac{1}{24}\frac{\kappa^2}{\lambda^2}\right). \tag{63}$$

It is convenient to introduce a new set of couplings,

$$\mathcal{G} = \frac{g^2}{2\sqrt{\lambda}}, \qquad \eta = \frac{1}{\sqrt{\lambda}}, \qquad \varrho = \frac{\kappa}{\lambda}. \tag{64}$$

Recalling the expression (10) for β_g, we obtain the β-functions in terms of the new variables

$$\beta_{\mathcal{G}} = -\frac{c_2}{6}\mathcal{G}^2\left(44+6\varrho+\varrho^2\right), \tag{65}$$

$$\beta_\eta = -\frac{c_2\mathcal{G}}{6}\eta\left(26+6\varrho+\varrho^2\right), \tag{66}$$

$$\beta_\varrho = \frac{c_2\mathcal{G}}{24}\varrho\left(8-30\varrho-17\varrho^2\right). \tag{67}$$

We observe that the couplings $\mathcal{G}$ and η enter only as overall coefficients. The beta function for ϱ, apart from the overall factor $\mathcal{G}$, depends only on the variable ϱ. The quadratic polynomials in ϱ appearing in $\beta_{\mathcal{G}}$ and β_η are strictly positive for all real values of ϱ. This feature is very important for the detailed analysis of the RG flow of the theory, which will be reported in [35].

6. CONCLUSIONS

In this work, we analyzed the renormalization of the marginal parity-even potential part operators in the five-dimensional Yang – Mills theory exhibiting anisotropic scaling with dynamical critical exponent $z = 2$. We employed the background field method and expanded around a vanishing temporal and generic static spatial background. This choice is suitable for studying the potential sector because the kinetic part operators vanish on this background while the potential part operators are non-zero and explicitly distinct from one another.

At the one-loop level, determining the renormalization of the potential part of the action requires calculating the divergent part of the Lorentz-violating wave operator. A Fourier transform in the temporal coordinate reduces the original five-dimensional noncovariant problem to a four-dimensional one involving a spatially covariant higher-order differential operator. The latter is nonminimal in the general gauge. To simplify the calculations, we employed the gauge in which this operator is minimal and applied two versions of the heat kernel technique for the higher-order operators. The first one is a straightforward perturbation theory supplied with the heat kernel technique for a higher order operator which is a power of the minimal second order operator. The second one is the commutator technique [25], which is applicable to a general higher order minimal operator. The agreement between the results obtained by two independent methods provides a powerful check for the obtained counterterms. Our work also lays the groundwork for developing the general heat-kernel technique for theories exhibiting anisotropic scaling between time and space coordinates [36].

Having obtained the divergent part of the one-loop effective action, we derived the beta functions for the potential part couplings λ and κ. We leave a detailed analysis of the RG flow of the theory for future work [35].

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