ЖЭТФ/JETP, 2026, том 170, вып. 2 (8), стр. 174–203

OPTIMIZATION OF RANDOM PROCESSES COMPLETION BY RESTART

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Received June 25, 2026,
revised version June 30, 2026
Accepted for publication July 1, 2026

Restart – an interruption of a stochastic process before its completion, followed by the start of a new random realization of the same process – is a potential tool for control over the completion time statistical properties of stochastic dynamics (e. g., restart of randomized computational procedures). One of the key reasons underlying research interest in theoretical modelling of restart-induced effects on stochastic processes roots in observation that sometimes restart can accelerate process progress as evidenced by reduction of the mean completion time. After this observation was made, many subsequently resolved research questions arose, such as: the optimal schedule of restarting, the limits of the effectiveness of the restart method, the conditions under which this method works and the prospects for using this method to optimize performance measures other than the expected completion time. The most straightforward and general way of extracting theoretical prediction regarding completion time statistics of random processes in the presence of restart is provided by the renewal approach which is applicable to majority of particular stochastic models and restart protocols considered in the relevant literature. Supplemented by analysis of measures from descriptive statistics, linear response analysis, optimization analysis, and analysis based on probabilistic inequalities, the renewal approach allowed to pose and resolve many issues regarding completion statistics of generic random processes subject to various types of restart protocols as discussed in this review article.

Keywords: stochastic processes, random walk processes, kinetic models, restart, renewal equation method

DOI: 10.31857/S0044451026080088

1. INTRODUCTION

The average completion time $\langle T\rangle$ of a random process can be used as a measure characterizing the speed of various stochastic phenomena. Remarkably, and at first glance unexpectedly, this measure can in some cases be reduced by using restart — an episodic interruption of the process followed by the start of its new statistically independent realization (Fig. 1). The possibility of this effect was noted and explored in two papers, [1] and [2], devoted to the problem of increasing the speed of randomized computer algorithms (Fig. 2). After the work [3] which demonstrated that stochastic resetting of a particle to its starting position hastens diffusive search, this effect became a subject of study in statistical physics.

The origin of the restart-induced speed up can be qualitatively explained as follows. Consider stochastic dynamics of a random process in configuration space of its degrees of freedom/dynamical parameters. Depending on the context the corresponding state variable may represent, e. g., the coordinate of random walk process or a node in the tree of candidate solutions explored by computer algorithm. The process completion corresponds to its entry into a certain threshold region representing, e. g., an absorbing boundary or a true solution to the optimization problem. The time to reach threshold in the course of random motion of the process in its configuration space may exhibit significant statistical dispersion due to the external stochastic factors or intrinsic reasons. Restart could then potentially re-

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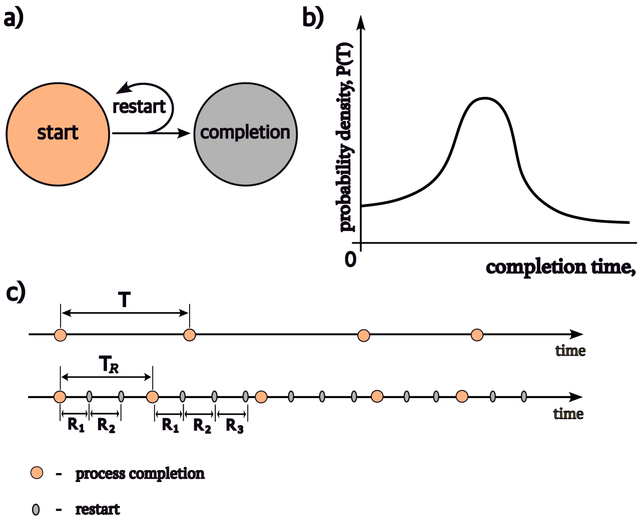

Fig. 1. *a* — General scheme of random process under restart. *b* — Probability density function of the process completion time under restart. *c* — Pattern of completion time moments in the absence of restart (top) and accelerating effect of restart on the random process (completion)

duce the average completion time by interrupting long-running process executions associated with its wandering in far-from-threshold or hard-to-get out parts of the configuration space.

It was also shown that, in addition to the average completion time $\langle T \rangle$, restart can be used to improve other statistical characteristics of random processes: the high-order statistical moments of completion time [4], the probability of meeting a deadline [5], the median completion time [6], the mode of the probability density function of the completion time [6], the entropy [7] and the diversity [8] of completion time, the probability of obtaining a desired outcome for a random process with alternative completion scenarios [9], the tail probability of completion time [2], the rate of obtaining the desired outcome, the mean completion time conditional to specific process outcome [10].

Two major routes can be distinguished in the topic of mean completion time optimization via restart: specific case studies and general analysis. The first type of works represents detailed investigations of how particular restart protocols (usually Poisson restart) influence completion time statistics of various stochastic phenomena: free diffusion [11], diffusion with mean drift component [12–16], diffusion in confined geometry [17–22], diffusion with partially absorbing boundary [23, 24], diffusion with spatially varying diffusion coefficient [25–27], many-particle diffusion [18], various models of continuous random walks [28–38] random walks

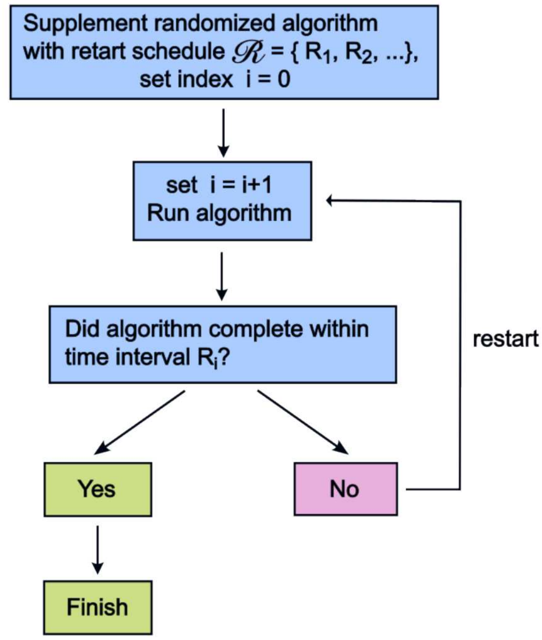

Fig. 2. The main area of research addressing the impact of restarts on the completion statistics of stochastic processes: speed-up of randomized algorithms via restart [1]. A randomized algorithm is an algorithm whose computational process possesses a degree of randomness resulting in runtime fluctuations

in regular lattices [39, 40], random walks in complex networks [41, 42], particular examples of runtime models for randomized computer algorithms [43]. From the methodological side, these studies typically (but not always) use kinetic equation on time evolution of the probability density function of the process state variable in the configuration space with the auxiliary terms incorporating effects of episodic restarts on the flow of probability. The restart-modified mean completion time is then extracted from process survival probability after solving kinetic equation with Fourier or Laplace transform method, or via Kolmogorov backward approach when applicable. The final answer is expressed via parameters of some kinetic model (i. e., diffusion coefficient, force fields parameters, geometric parameters of obstacles and targets shapes and arrangement, mean frequency of restarts, etc.). The calculation details, examples of results and more references to studies of this type can be found in the review articles [44–46].

The second type of theoretical works focuses on obtaining general conclusions regarding the effect of restart on stochastic processes of arbitrary nature and of arbitrary kinetic details. The key analytical tool in these works is the renewal approach, either explicitly used in the form of stochastic renewal equation(s) for random completion time in the presence of restart (see [47]) or implied in the probabilistic reasoning on the way to the closed-form expression for restart-modified expected completion time (see, e. g., [1]). The resulting statements regarding behaviour of the mean completion time are formulated in terms of the variables parametrizing restart schedule and via descriptive statistics measures of the completion time in the absence of restart rather than through parameters of some kinetic model. Due to this, the corresponding results turn out to be robust to the change of random process model details.

Methodologically, the renewal approach to modelling of restart-induced effects is typically accompanied by the linear response analysis, by the mean residual time analysis, by optimization analysis and by analysis based on implementation of statistical inequalities. The applicability of the conclusions obtained within this theoretical framework is usually limited only by the assumption of

(i) mutual statistical independence of repeated process runs,

and by the condition of

(ii) uncoupling of restart protocol from the process own dynamics meaning that inter-restart time intervals are prescribed deterministic or random and uncorrelated to process progress.

Also in some cases analysis assumes

(iii) smallness of the time required to perform restart in comparison with all relevant time scales of the process completion,

or

(iv) mutual independence and identical distribution of durations of restarts considered as non-instantaneous and their independence on process dynamics and inter-restart intervals.

Two other presuppositions, not crucial for applicability of renewal equations but relevant at extraction of some consequences, are the following:

(v) finiteness of the statistical moments of the random completion time in the absence of a restart,

(vi) specific form of the restart protocol applied to the random process.

It should be noted that properties (i)–(ii) are shared by the overwhelming majority of specific models considered in the existing literature on random processes with restart (examples of models where these assumptions are violated are discussed in Section 14).

The two lines of research, process details specific and general, complement each other together providing a wide range of modelling tools and firmly established theoretical conclusions that allow to predict how restart modifies completion time statistics of various stochastic phenomena. Being considered together, they offer a choice between two possible ways of analysis of stochastic process completion upon restart: starting from modelling of stochastic transitions in process configuration space or from descriptive statistics perspective focusing directly on the resulting process duration statistics. Renewal approach helps to overcome difficulties that may arise on the way of solving kinetic equation. When kinetic equation on the probability density function in the underlying space of the process state variables is unknown or analytically unavailable (like in the case of many randomized computational algorithms) the analysis starts with the restart-free runtime time distribution of some problem-specific form (obtained empirically or deduced theoretically). The same applies when kinetic equation is known, but restart events have non-Poissonian statistics so that they cannot be accounted by simple modification of probability flow in kinetic equation, The mean runtime under restart is then expressed via parameters of this distribution using probabilistic arguments based on the renewal nature of restart, see, e. g., [18, 48].

This paper is devoted to a review of results obtained within renewal approach concerning the problem of optimizing the mean completion time of random processes using the restart method. The work also includes results on restart-mediated optimization of other performance measures beyond the average completion time. The main focus is on the continuous time stochastic processes.

The reviewed material is structured as follows. Section 2 describes the problem statement. Section 3 outlines the essence of the renewal equation method, presents closed-form expression for the mean completion time of generic random process in the presence of an arbitrary deterministic restart protocol and discusses several particular types of restart strategies. Section 4 describes the sufficient conditions of restart efficiency. Section 5 considers the issue of globally optimal restart strategy. Section 6 presents upper bounds on the mean completion time under optimal restart protocol. Next, Section 7 describes lower limit below which the mean completion time of a random process cannot be reduced by restart. Section 8 presents constructivist criteria of restart efficiency. Universal properties of processes under optimally tuned periodic and Poisson restart protocols are discussed in Section 9. Some non-uniform restart strategies are discussed in 10. Section 11 presents review of the results regarding the impact of restarts on characteristics other than average completion time. Section 12 lists existing generalizations of the presented results to the case of non-zero time cost of restart. Section 13 additionally highlights some of the reviewed results. The final Section 14 contains very brief summary and discussion of scenarios under which basic assumptions adopted in this article can be violated.

2. PROBLEM FORMULATION: OPTIMIZATION OF THE MEAN COMPLETION TIME VIA RESTART

Consider a stochastic process that repeats over and over again. All runs are assumed to be statistically independent and identically distributed. The random completion time T of an individual run is characterized by the probability density function $P(T)$. This function is normalized to unity, $\int_0^\infty dTP(T) = 1$, which means that the process trials terminate in finite time with unit probability (scenarios of non-zero probability of never stopping is briefly discussed in Section 11.12). As might be expected from the definition of the sample mean and the law of large numbers, if $\langle T\rangle = \int_0^\infty dTP(T)T$ and $\langle T^2\rangle = \int_0^\infty dTP(T)T^2$ are finite, then the total number of completed process attempts N_W over a long observation period $W \to \infty$ only weakly fluctuates around the value $W/\langle T\rangle$. Therefore, to increase process speed in this situation, it is necessary to reduce the average completion time of trials. As discussed in the introduction, for some processes, such performance improvement can be achieved by restart method.

The restart protocol $\mathcal{R}$ is characterized by a sequence of inter-restarts time intervals $R_1, R_2, R_3, \dots$. In what follows we will always assume, if opposite is not specified, that this sequence consists of (potentially) infinite number of terms. It can be argued that protocols in which the flow of restart events does not terminate until the process reaches completion, no matter how long this takes, are more efficient for control applications aimed at optimizing the mean completion time than the protocols with a limit on the maximum number of restarts [48].

Unbounded restart protocol $\mathcal{R}$ works in the following way. If a random process is completed prior to the first scheduled restart, the story ends there. Otherwise, the current attempt is aborted, and the process starts from scratch. Similarly, next attempt may either complete prior to the second restart or not, with the same rules. This procedure repeats until the process finally reaches completion. The random completion time of the process in the presence of restart schedule $\mathcal{R}$ will be denoted as $T_\mathcal{R}$. The key quantity of interest for us is the mean value $\langle T_\mathcal{R}\rangle$.

Depending on the context, mechanisms that implement restart of stochastic dynamics in particular systems can be divided into deterministic and stochastic. In stochastic case, sequence of inter-restart intervals $R_1, R_2, R_3, \dots$ is random so that derivation of theoretical predictions for the mean completion time $\langle T_\mathcal{R}\rangle$ must also include averaging over statistics of $\mathcal{R}$.

Note that in practice, process initiation and restarts are accompanied by some time overheads. Furthermore, the time required to perform a restart can depend on the coordinate of the process in its configuration space prior to the beginning of restart (see, e. g., [49] where this issue is discussed in the context of experimental realizations of diffusion mediated search with resetting). The main part of this paper is devoted to results obtained under the assumption of (iii) negligible restart associated delays. Possible generalizations to the case of non-zero time penalties for restarts uncorrelated with process dynamics and inter-restart intervals are discussed in the Section 12.

3. GENERAL METHOD OF ANALYSIS: RENEWAL APPROACH

The most generally applicable way of theoretical description of completion time statistics of random processes under restart is provided by renewal approach. This framework does not require incorporation of restart mechanism into the kinetic equation governing stochastic dynamics of the process of interest in its configuration space. Instead the renewal approach deals directly with stochastic realizations of the random completion time T, characterized by the probability density function $P(T)$, without looking into the kinetic details giving rise to this statistics, and shows how statistical properties of the completion time are affected by restart.

Namely, the random completion time $T_{\mathcal{R}}$ of the process in the presence of protocol $\mathcal{R}$ with unbounded number of restarts obeys the infinite chain of renewal relations

$$\begin{cases} T_{\mathcal{R}} = T_1 I(T_1 \leq R_1) + (R_1 + T_1^{\text{res}}) I(T_1 > R_1), \\ T_1^{\text{res}} = T_2 I(T_2 \leq R_2) + (R_2 + T_2^{\text{res}}) I(T_2 > R_2), \\ ... \\ T_n^{\text{res}} = T_{n+1} I(T_{n+1} \leq R_{n+1}) + \\ +(R_{n+1} + T_{n+1}^{\text{res}}) I(T_{n+1} > R_{n+1}), \\ ... \end{cases} \quad (1)$$

where $T_1, T_2, T_3, ...$ are mutually independent random variables sampled from $P(T)$, I is the indicator function which is equal to unity if the inequality in its argument is justified and is zero otherwise, and T_i^{res} represents the i-th residual time, i. e., the time remaining to the process completion immediately after the i-th restart provided that the process has not yet terminated by that moment. Intuition behind this set of equations is very simple: if process failed to finish before the next restart, it starts anew. Equations (1) are justified whenever assumptions (i), (ii) and (iii) listed in Section 1 are fulfilled.

By successively substituting T_i^{res} expressed via T_{i+1}^{res} with i running from 1 to $+\infty$ into the first equation of the set (1) and performing averaging over the statistics of mutually independent identically distributed random variables $T_1, T_2, T_3...$ with $R_1, R_2, R_3, \dots$ treated as fixed parameters, one obtains

$$\langle T_{\mathcal{R}} \rangle = \sum_{j=1}^{\infty} \left(\frac{\int_0^{R_j} dT P(T) T}{\int_{R_j}^{\infty} dT P(T)} + R_j \right) \prod_{i=1}^{j} \int_{R_i}^{\infty} dT P(T). \quad (2)$$

Equation (2) relates the expectation of the completion time $\langle T_{\mathcal{R}} \rangle$ in the presence of restart schedule $\mathcal{R}$ to the completion time probability density function $P(T)$ of the "bare" (i. e., restart-free) process. This result can be further generalized by adding time costs for process initialization and restart into (1). Here, however, we will keep the assumption of (iii) negligible time cost of restart, as was already explained above (see Section 12 for references to works where this assumption is relaxed and replaced by presupposition (iv)).

Dealing with a stochastic restart protocol one should additionally average renewal equations (1) over distribution of $R_1, R_2, R_3, ...$ to arrive at mean performance of schedules from a given statistical ensemble. In particular, in the case of stochastic protocols with time-homogeneous statistics, for which we will use the notation $\mathcal{R}_{t.h.}$, the completion time statistics can be extracted using a single renewal equation. Namely, if intervals $R_1, R_2, R_3, ...$ are independently sampled from the probability density $\rho(R)$, then the random completion time $T_{\mathcal{R}_{t.h.}}$ can be written as

$$T_{\mathcal{R}_{t.h.}} = T I(T \leq R) + (R + T'_{\mathcal{R}_{t.h.}}) I(T > R). \quad (3)$$

Here $T'_{\mathcal{R}_{t.h.}}$ is independent copy of the random variable $T_{\mathcal{R}_{t.h.}}$. Indeed, in the case of independent and identically distributed inter-restarts intervals, if the process failed to reach completion prior the first restart, then the residual time T_1^{res} just after the first restart is distributed in the same way as $T_{\mathcal{R}_{t.h.}}$. Note also that T, R and $T'_{\mathcal{R}_{t.h.}}$ in the right-hand side of Eq. (3) are uncorrelated.

After averaging of Eq. (3) over statistics of original random process and of stochastic inter-restarts intervals we get the closed-form expression for the mean process completion time

$$\langle T_{\mathcal{R}_{t.h.}} \rangle = \frac{\langle min(T,R) \rangle}{Pr[T \leq R]} = \\ = \frac{\int_0^{\infty} dR \rho(R) \int_0^R dT P(T) T + \int_0^{\infty} dR \rho(R) R \int_R^{\infty} dT P(T)}{\int_0^{\infty} dR \rho(R) \int_0^R dT P(T)}. \quad (4)$$

Equation (4) was obtained in Ref. [47] and can be also extracted from results of Ref. [50].

In the simplest case of periodic restart protocol (also known as sharp, strictly regular or fixed-cutoff restart) we have $R_1 = R_2 = R_3 = ... = \tau$, where τ is restart period, so that the expected completion time is given by (Refs. [5, 18], and [51])

$$\langle T_\tau \rangle = \frac{\int_0^\tau P(T) T dT + \tau \int_\tau^\infty P(T) dT}{\int_0^\tau P(T) dT}, \quad (5)$$

as can be seen from (4) by putting $\rho(R) = \delta(R - \tau)$. The periodic protocol is of particular practical importance as a control tool for computer science applications due to its property of global optimality discussed in Section 5. In this regard, it also has methodological significance in restart research, as understanding its impact on the average completion time of random processes allowed to determine the limits of the effectiveness of any restart procedure, see Ref. [52] and Section 7. Equation (5) can be also rewritten as

$$\langle T_\tau \rangle = \frac{\langle T \rangle + \tau - \langle |T-\tau| \rangle}{2Pr[T \le \tau]}, \tag{6}$$

see [53], or as

$$\langle T_\tau \rangle = \frac{\int\limits_0^\tau Pr[T > \tau'] d\tau'}{Pr[T \le \tau]}, \tag{7}$$

see [18] (or [1] for the discrete-time version).

Another important and rather simple case is the Poisson restart for which inter-restarts intervals represent mutually independent and identically distributed random variables with exponential probability density $\rho(R) = re^{-rR}$. Here r denotes the restart rate, i. e., the average frequency of random restarts. The mean completion time of a process subject to Poisson restart is given by (see [54] and [18])

$$\langle T_r \rangle = \frac{1-\tilde{P}(r)}{r\tilde{P}(r)}, \tag{8}$$

where

$$\tilde{P}(r) = \int\limits_0^\infty dT P(T) e^{-rT}$$

is the Laplace transform of $P(T)$ evaluated at r. Equation (8) can be derived from the Eq. (4) by substituting the exponential probability density for $\rho(R)$.

Besides, we will also discuss here the so-called Gamma restart which is characterized by random inter-restart time intervals independently sampled from the Gamma distribution

$$\rho(R) = \frac{\beta^k}{\Gamma(k)} R^{k-1} e^{-\beta R},$$

where β and k are rate and shape parameter, respectively. The expected completion time of a process under Gamma restart is described by the following expression (see [52])

$$\langle T_\beta \rangle = \frac{(-1)^{k-1}\beta^k}{\Gamma(k)} \frac{d^{k-1}}{d\beta^{k-1}} \frac{\tilde{P}(\beta)}{\beta}. \tag{9}$$

As in the case of Eq. (8), expression (9) can be derived from Eq. (4) by substituting particular form of probability density $\rho(R)$. Note also that at $k = 1$ Eq. (9) reproduces Eq. (8) since gamma distribution becomes exponential in this case. The Gamma-protocol is of general interest because examining its impact on random processes helped to establish some universal features of completion time statistics under optimal periodic restart, see Ref. [52] and Section 6.1. Moreover, it turns out that gamma restart with $k = 2$ and sufficiently small rate parameter β always either further reduces or leaves unchanged the average completion time of any processes subjected to optimal Poisson restart (see discussion in the Section 14).

Most of those theoretical results in this area that are insensitive to the fine details of the probability density function $P(T)$ have been obtained for the strictly periodic, Poisson, and Gamma restart protocols, discussed at length below. Practically and/or methodologically important restart scenarios are also represented by Luby restart strategy [1], by Poisson restart with hyperbolically decaying in time rate [55], and by geometric restart protocol, which are discussed in Section 10. Some of the conclusion overviewed here are valid for any restart strategy, see Sections 7 and 14.

We conclude this section with discussion of how restart affects random processes which do not have finite second or/and first statistical moments of the random completion time so that the number of process completions N_W over a very long observation period W either almost certainly zero or fluctuates very strongly around some non-zero mean. It turns out that with the aim of periodic restart the number of process completions observed within long time window can be made well reproducible non-zero quantity. Indeed, it is easy to see from expression (5) for $\langle T_\tau \rangle$ and relation

$$\langle T_\tau^2 \rangle = \frac{\int\limits_0^\tau dT T^2 P(T) + (\tau^2 + 2\tau\langle T_\tau \rangle) \int\limits_\tau^\infty dT P(T)}{\int\limits_0^\tau dT P(T)}, \tag{10}$$

(see, e. g., [5] or [47]), which can be derived, for example, by averaging renewal equation

$$\langle T_\tau^2 \rangle = T^2 I(T \le \tau) + (\tau + T_\tau')^2 I(T > \tau),$$

that irrespectively of the particular form of $P(T)$ there exist values of the restart periods $0 < \tau < \infty$ for which both $\langle T_\tau \rangle$ and $\langle T_\tau^2 \rangle$ are finite, and thus N_W is the relatively weakly fluctuating quantity inversely proportional to $\langle T_\tau \rangle$.

4. WHEN DOES RESTART WORK?

Once completion time probability density $P(T)$ is known, Eq. (2) allows us to determine whether a particular restart schedule improves the mean process performance or not. In practice however the full process statistics may be poorly specified. It would therefore be useful to identify some general conditions under which restart method is effective. These conditions should be formulated in terms of sufficiently small set of statistical characteristics of random completion time T whose reliable estimate needs much less statistical data than recovering probability density $P(T)$. The results in this direction obtained so far can be subdivided into two classes: existence criteria and constructive criteria of restart efficiency. This section is devoted to discussion of existence results representing sufficient conditions of when there exists a restart protocol of particular type decreasing the mean process completion time. Constructive criteria for restart efficiency, which not just detect that performance of the process under consideration can be improved by restart but also explicitly define parameters of effective restart strategies, will be presented in Section 8. It should be noted that among the many known existence conditions that can be found in the literature, here we list those expressed via relatively simple statistical characteristics. More results can be found in Refs. [56] and [53] (see also [43]).

As can be seen from discussion in the end of previous section, periodic restart with appropriately chosen period is always helpful for processes whose empirically estimated first and second statistical moments of random completion time do not converge with the growth of statistical sample size. However, restart method can be useful not only in such dramatic scenarios with extremely huge fluctuations. In this section we assume that restart-free mean completion time $\langle T\rangle$ is finite.

A variant of residual time analysis, presented in [54], provides a concise and practically important criteria to detect accelerating effect of rare Poisson restart on the expected completion time. More specifically, the inequality

$$\frac{\sigma(T)}{\langle T\rangle} > 1, \tag{11}$$

where $\sigma(T) = \sqrt{\langle T^2\rangle - \langle T\rangle^2}$ is the standard deviation of the completion time T from the mean value $\langle T\rangle$, represents the sufficient condition for when introduction of Poisson restart with small rate r leads to reduction of the mean completion time [54,57]. Equivalently, we can say that if the so-called CV-index defined as

$$\mathcal{J}_{CV} = 1 - \frac{1}{1 + (\frac{\sigma(T)}{\langle T\rangle})^2}$$

is large enough,

$$\mathcal{J}_{CV} \geq \frac{1}{2}, \tag{12}$$

then sufficiently small values of rate r improve mean performance [53]. Alternatively to the mean residual time estimate, the condition (11) (and, thus, (12)) can be deduced from analysis of the linear order term in series expansion of Eq. (8) in powers of the restart rate r (see supplementary material to [47]).

A simple sufficient condition of periodic restart efficiency reads

$$\frac{\langle T\rangle}{m} > 2. \tag{13}$$

Here m denotes the median completion time of restart-free process. As shown in [52], if this inequality is satisfied then there exists a period τ of strictly regular restart reducing the expected completion time.

A criterion similar to (13), but with different bound can be formulated for Poisson restart. Namely, if

$$\frac{\langle T\rangle}{m} > 2e, \tag{14}$$

then there exists a value of Poisson restart rate r which improves mean performance. In contrast to (11), derivation of conditions (13) and (14) is based on implementation of probabilistic inequalities rather than on linear response calculations or mean residual time analysis [52].

Next, the inequality

$$\frac{\langle T^{k+1}\rangle}{\langle T\rangle\langle T^k\rangle} > k + 1, \tag{15}$$

where k is positive integer, represents the sufficient condition for existence of efficient Gamma protocol with shape parameter k. Similarly to condition (11) this result can be obtained from analysis of linear order expansion of exact answer (9) upon the rate parameter β. If this inequality is satisfied, then application of Gamma restart protocol with sufficiently small rate parameter β and shape parameter k leads to reduction of expected completion time [52]. At $k = 1$ Eq. (15) becomes equivalent to Eq. (11).

In what follows, we will mainly focus on presenting the results themselves, whereas the details required for their derivation can be found in the cited literature.

The authors of work [53] presented criteria of periodic restart efficiency based on Gini index defined as

$$\mathcal{J}_{Gini} = 1 - \frac{1}{\langle T\rangle}\langle \min(T_1, T_2)\rangle,$$

where T_1 and T_2 are two identical copies of the random variable T. Similarly to CV-index this measure characterizes the statistical heterogeneity of the process completion time sample. As shown in [53], beneficial periodic restart strategy exists if the Gini index obeys the condition

$$\mathcal{J}_{Gini} \geq \frac{1}{2}. \tag{16}$$

Next, as can be deduced from analysis of Eq. (5) in low-τ limit, if the completion time statistics of random process in the absence of restart obeys the inequality

$$\langle T\rangle > \frac{1}{P(0)}, \tag{17}$$

then regular restart with sufficiently small period τ is beneficial [53, 53].

Analysis of Eq. (5) in opposite large-τ limit shows that if the restart-free random process satisfies the condition

$$\frac{1}{\langle T\rangle} < \frac{1}{H(\infty)}, \tag{18}$$

where

$$H(t) = \frac{P(t)}{\int\limits_t^\infty P(T)dT}$$

is the Hazard rate function, and

$$H(\infty) = \lim_{t\to\infty} H(t),$$

then mean performance can be improved by applying regular restart with sufficiently large period τ [53, 53]. Clearly, the scenario of equation (18) applies to random processes with power-law behaviour of tails of the probability density function $P(T)$.

It should be emphasized that any restart efficiency criterion, regardless of the type of strategy for which it was originally proposed, is also applicable to the case of periodic restart. This directly follows from the fact that if process performance can be improved by some type of restart strategy, then there also exists beneficial periodic restart protocol (see section 4). Therefore, the conditions (11) and (15) first derived for Poisson and Gamma protocols, respectively, are also sufficient for existence of beneficial periodic strategy (see also [56]).

The existence conditions described in the literature are very practical since they do not require knowledge of full completion time statistics or details of kinetic model giving rise to this statistics. The general lesson following from these results is that restart is helpful in those cases when process completion time exhibits significant fluctuations manifesting in sufficiently large values of coefficient of variation, of high order statistical moments, of diversity indexes, of the ratio of mean over median, of probability of extreme events, etc.

At the same time it should be remembered that being based on partial statistical information these efficiency conditions are only sufficient but not necessary. Let us provide here a simple example of the model process for which well-known relative fluctuation based condition (11) of Poisson restart efficiency is not met, i. e., $\sigma(T) < \langle T\rangle$, but effective restart rates r, reducing the expectation of the random completion time still exist. A possible example is given by the following completion time probability density:

$$P(T) = p\delta(T - t_1) + (1 - p)\delta(T - t_2)$$

with $p = 0.55$ and $t_2 = 10t_1$ [52].

5. WHICH RESTART PROTOCOL IS THE MOST EFFICIENT?

Of course, restart-induced speed up is not a prerogative of a particular class of restart protocols: many different restart strategies can produce accelerating effect on the same stochastic process (see, e. g., [47, 55, 58–60] for some examples of protocols whose effect on the mean completion time of diffusion mediated search has been investigated). Which restart strategy is most effective in reducing the expected completion time? Addressing this question, Luby et al. [1] showed universal optimality of periodic restart for random processes with discrete completion time distributions. For continuous time models, Pal and Reuveni [47] proved dominance of periodic restart over any stochastic restart protocol with independent and identically distributed inter-restarts intervals (see also [61]). Finally, the dominance of periodic restart in the space of all possible restart protocols for continuous models has been shown in the Lorenz's paper [51]. Formally, the periodic restart is optimal in the following sense: if we find for a given process the restart period $\tau_* \geq 0$ (possibly $\tau_* = +\infty$) such that for any $\tau \geq 0$ one has $\langle T_{\tau_*}\rangle \leq \langle T_\tau\rangle$, then $\langle T_{\tau_*}\rangle \leq \langle T_{\mathcal{R}}\rangle$ for all restart protocols $\mathcal{R}$. This also means that if some restart protocol attains a certain reduction of the mean completion time, then there exists a periodic restart protocol that can, at least, match this reduction. Note however that a periodic protocol with a non-optimal period $\tau \neq \tau_*$ may be inferior to other restart strategies [54].

Taking into account Eq. (5) for the mean completion time $\langle T_\tau\rangle$ in dependence on restart period τ and optimal property of periodic restart we may conclude that, when assumptions (i), (ii) and (iii) are justified, the inequality

$$\min_{\tau>0} \frac{\int\limits_0^\tau P(T)TdT + \tau\int\limits_\tau^\infty P(T)dT}{\int\limits_0^\tau P(T)dT} < \langle T\rangle, \tag{19}$$

represents a sufficient and necessary condition for existence of restart protocol decreasing the average completion time of the random process with specified completion time probability density $P(T)$. Practical application of this condition, however, requires knowledge of the full completion time statistics, while the sufficient conditions from the previous section are less demanding.

6. HOW WELL DOES RESTART WORK?

Given the opportunities providing by the method of restart in the completion time optimization, a natural question of efficiency estimate arises. How good is the process performance under the optimized restart strategy of specified type? The answer is known for the case of a globally optimal strategy (i. e., periodic restart, see previous section) and for the Poisson restart. The relevant results are discussed in this section.

6.1. Periodic restart protocol

Let τ_* be the optimal restart period, i. e., $\langle T_{\tau_*}\rangle \leq \langle T_\tau\rangle$ for all τ. As shown in [52], in the presence of optimally tuned periodic restart, the average completion time $\langle T_{\tau_*}\rangle$ does not exceed twice the median m of the unperturbed process, i. e.,

$$\langle T_{\tau_*}\rangle \leq 2m. \tag{20}$$

m is defined by the condition

$$\int\limits_0^m P(T)dT = \frac{1}{2}$$

and is finite for any $P(T)$. The bound dictated by Eq. (20) is sharp: for

$$P(T) = \frac{1}{2}\delta(T-t) + \frac{1}{2}\delta(T-3t)$$

one obtains $m = t$ and $\langle T_{\tau_*}\rangle = 2t$ with $\tau_* = t$. Equation (20) can be generalized to the arbitrary order statistical moments of the process duration under optimal periodic restart as follows (see [52])

$$\sqrt[n]{\langle T^n_{\tau_*}\rangle} \leq 2\sqrt[n]{n!}m, \tag{21}$$

where n is positive integer. Clearly, at $n = 1$ Eq. (21) reduces to (20).

Thus, even when completion time of original process exhibits infinite statistical moments (say, due to power-law tails of probability density $P(T)$), the moments $\langle T^n_{\tau_*}\rangle$ of completion time under optimally tuned periodic restart always exist and do not exceed the well-defined universal bounds determined by restart-free median completion time m.

Let us note by passing that sufficient condition of periodic restart efficiency, provided by Eq. (13), represents a consequence of the upper bound (20). Indeed, if $\langle T\rangle > 2m$, then the choice $\tau = +\infty$, corresponding to absence of restart, cannot represent the best possible period τ_*, and, therefore, there must exist values of τ decreasing the expected completion time.

6.2. Poisson restart protocol

Here we will use notation $r_* \geq 0$ for the optimal rate of Poisson restart which is defined by the extremum conditions

$$\frac{d\langle T_r\rangle}{dr}|_{r=r_*} = 0 \quad \text{and} \quad \frac{d^2\langle T_r\rangle}{dr^2}|_{r=r_*} > 0.$$

The mean performance of the random process under optimally tuned Poisson protocol satisfies the following upper bound

$$\langle T_{r_*}\rangle \leq 2em, \tag{22}$$

which is $e = 2,7182\ldots$ times higher than the bound (20) associated with best periodic restart strategy. Note that due to relation (43) (see below) the standard deviation $\sigma(T_{r_*})$ of the completion time T_{r_*} at optimal restart rate r_* also obeys the upper bound dictated by Eq. (22).

Besides, assuming that first three statistical moments of T are finite (assumption (v), see Section 1) and condition (11) holds, we can write

$$\langle T_{r_*}\rangle \leq \frac{\langle T^2\rangle}{2\langle T\rangle} - \frac{\langle T^3\rangle}{3\langle T\rangle^2} + \\ + \frac{1}{3}\sqrt{\frac{\langle T^3\rangle^2}{\langle T\rangle^4} - \frac{3\langle T^2\rangle\langle T^3\rangle}{\langle T\rangle^3} + \frac{6\langle T^3\rangle}{\langle T\rangle}}. \tag{23}$$

Due to estimate $\langle T^2\rangle \leq \sqrt{\langle T\rangle\langle T^3\rangle}$, which follows from the inequality $\langle |X_1X_2|\rangle \leq \sqrt{\langle X_1^2\rangle\langle X_2^2\rangle}$ (for any pair of random variables X_1 and X_2), the expression under the square root in the right-hand side of Eq. (23) is positive for any completion time distribution $P(T)$. Also, it can be easily checked that the right-hand side does not exceed the restart-free mean completion time $\langle T\rangle$, so this bound is informative.

Note that due to relation (43) (see below) the standard deviation $\sigma(T_{r_*})$ of the completion time T_{r_*} at optimal restart rate r_* also obeys the upper bounds dictated by Eqs. (22) and (23).

7. WHAT ARE THE LIMITS OF RESTART EFFECTIVENESS?

Thus, if restart is helpful at all for the process under consideration, application of periodic strategy necessarily reduces the mean completion time below the upper bound limit dictated by Eq. (20) (or (22) in the case of Poisson protocol). Now let us pass to the issue of lower bounds. What, if any, are the principal limitations of the optimization via restart? This question has been addressed in Ref. [52]. It turns out that restart performance is limited to a quarter of the harmonic mean completion time $h = \langle T^{-1}\rangle^{-1}$ of original process, i. e.

$$\langle T_{\mathcal{R}}\rangle \geq \frac{1}{4}h, \tag{24}$$

for any $\mathcal{R}$ (including periodic restart with best possible period τ_*). Arguments underlying derivation of Eq. (24) do not impose any restrictions on the form of $P(T)$ or of the restart protocol $\mathcal{R}$ and, thus, this result is universally valid for any setting if the assumptions (i), (ii) and (iii), described in section 1, are satisfied. Obviously, for any given process, periodic strategy provides the closest approach to this universal limit among all possible restart protocols due to its global optimality. Note that if the mean harmonic completion time h of a process is zero (this is the case when $P(0) \neq 0$), this, of course, does not mean that its average completion time can be made arbitrarily small by tuning the restart period. The latter is only possible in the case of delta-functional behavior of $P(T)$ at $T = 0$.

Another useful result can be obtained if we consider smooth completion time distribution $P(T)$ with single maximum at some non-zero value of T. The efficiency of any restart protocol in this case is subject to the following bound [52]

$$\langle T_{\mathcal{R}}\rangle \geq \frac{1}{4}M, \tag{25}$$

where $M = \mathrm{argmax}_T P(T) > 0$ is the mode of the probability distribution $P(T)$, i. e., the value of the random completion time T that occurs most frequently. More generally, if the probability distribution $P(T)$ has several local maxima, then $\langle T_{\mathcal{R}}\rangle \geq \frac{1}{4}M^{min}$, where M^{min} denotes the leftmost mode.

It remains unknown whether the bounds determined by Eqs. (24) and (25) are sharp. Namely, one cannot exclude that the constant 1/4 in the right-hand sides of these equations can potentially be replaced by one larger. Numerical data on model processes presented in [52] indicate that the unknown best possible constants C_1 and C_2 in the inequality constraints $\langle T_{\mathcal{R}}\rangle \geq C_1 h$ and $\langle T_{\mathcal{R}}\rangle \geq C_2 M$ defining the exact lower bounds on $\langle T_{\mathcal{R}}\rangle$ lie in the ranges $1/4 \leq C_1 < 5/4$ and $1/4 \leq C_2 < 5/3$.

A simple high-order generalization of Eqs. (24) and (25) can be arrived at by implementation of the estimate $\sqrt[k]{\langle T^k_{\mathcal{R}}\rangle} \geq \langle T_{\mathcal{R}}\rangle$ which follows from inequality $\phi(\langle X\rangle) \leq \langle\phi(X)\rangle$ (for any random variable X and any convex function $\phi(x)$). Applying this we immediately obtain

$$\sqrt[k]{\langle T^k_{\mathcal{R}}\rangle} \geq \frac{1}{4}h, \tag{26}$$

in general case, and

$$\sqrt[k]{\langle T^k_{\mathcal{R}}\rangle} \geq \frac{1}{4}M, \tag{27}$$

in the case of unimodal completion time distribution.

Note that the lower (Eqs. (24) and (25)) and upper (Eq. (20)) bounds do not contradict each other since the relations $h \leq 2m$ and $M \leq 2m$ take place [52]. Reference [52] also contains analysis excluding several hypothetical forms of inequality constraints on the average performance. More specifically, it is shown that in contrast to harmonic mean h and mode M, fixed values of the expectation $\langle T\rangle$ and the standard deviation $\sigma(T)$ of the restart-free completion time do not produce any non-trivial lower bound for $\langle T_{\mathcal{R}}\rangle$. Next, hypothetical upper bounds of the form $\langle T_{\tau_*}\rangle < C_1 h$ and $\langle T_{\tau_*}\rangle < C_2 M$, where C_1 and C_2 are some positive constants, cannot be universally valid. Last but not least, the inequality $\langle T_{\mathcal{R}}\rangle \geq Cm$, with universal non-zero numerical constant C also cannot exist. These conclusions were drawn by appropriately choosing the model probability densities $P(T)$, which play the role of counterexamples for the corresponding hypothetically universal inequalities [52].

8. HOW TO FIND EFFICIENT RESTART STRATEGY?

The recent papers [53, 62] presented first examples of the constructive criteria for the effectiveness of restart, expressed through the simple statistical characteristics of restart-free completion time which in practice can be estimated, for example, from sufficiently large sample of the random variable T trials (the size of which is nevertheless small in comparison with that required to determine the probability density function $P(T)$). Similar to existence results described in section 4, the constructive criteria do not demand knowledge of the full probability distribution $P(T)$. Unlike existence results, the constructive criteria serve not just as indicators of potential restart effectiveness, but also provide specific examples of protocol parameters that are guaranteed to reduce the average completion time at partially specified statistical characteristics of the process completion. Moreover, many of the known constructive criteria allows to estimate the resulting efficiency of the protocols they propose. Efficiency of the restart strategy $\mathcal{R}$ is introduced as the ratio

$$\eta_{\mathcal{R}} = \frac{\langle T\rangle - \langle T_{\mathcal{R}}\rangle}{\langle T\rangle}. \tag{28}$$

Clearly, $0 \leq \eta_{\mathcal{R}} < 1$ for those restart protocols that reduce the mean completion time.

This section discusses the constructive criteria for periodic, Poisson and Gamma restart. Let us outline the basic methodological idea, presented in Ref. [62], which guided derivation of many of constructive criteria listed below. For simplicity we explain this idea using example of a periodic restart protocol. First of all, applying to Eq. (5) (or Eq. (6)) some (specially chosen) probabilistic inequality, one can estimate from above the mean completion time of the process under periodic restart as $\langle T_\tau\rangle < \mathcal{T}$, where the time scale $\mathcal{T}$ is expressed via restart period τ and some statistical measures of the restart-free completion time (e. g., mean $\langle T\rangle$ and median values m). Then the solution of the inequality $\mathcal{T} < \langle T\rangle$ with respect to $\tau > 0$ defines a range of guaranteed efficient values of τ with range boundaries expressed via the descriptive statistics characteristics of process completion in the absence of restart. With an appropriate choice of auxiliary statistical inequalities, this line of reasoning easily extends to analysis of Poisson and Gamma restart strategies, see [62] for details.

8.1. Periodic restart protocol

Eliazar and Reuveni proved in Ref. [53] that if the so called Vdiam-index (vertical diameter) defined as

$$\mathcal{J}_{Vdiam} = \frac{1}{\langle T\rangle}\langle|T-m|\rangle$$

satisfies the inequality

$$\mathcal{J}_{Vdiam} > \frac{m}{\langle T\rangle}, \tag{29}$$

then periodic restart at $\tau = m$ is helpful.

Next, if the Pietra index

$$\mathcal{J}_{Pietra} = \frac{1}{2\langle T\rangle}\langle|T-\langle T\rangle|\rangle$$

of the random completion time obeys the condition

$$\mathcal{J}_{Pietra} > 1 - K(\langle T\rangle), \tag{30}$$

where $K(T)$ denotes the cumulative distribution function of T, then regular restart with period $\tau = \langle T\rangle$ improves the mean performance [53].

Another constructive criterion of periodic restart efficiency is expressed via the Hdiam-index (horizontal diameter) defined as

$$\mathcal{J}_{Hdiam} = \frac{1}{m_{dol}}\langle|T-m_{dol}|\rangle.$$

Here m_{dol} is the median value of the random variable T_{dol} having probability density function $\frac{T_{dol}}{\langle T\rangle}P(T_{dol})$. Fulfilment of the inequality

$$\mathcal{J}_{Hdiam} > \frac{m_{dol}}{m_{dol}+\langle T\rangle}, \tag{31}$$

guarantees that restart with period $\tau = m_{dol}$ leads to reduction of the mean completion time, also see [53].

Extending the results on Vdiam-index analysis reported in [53] , the work [62] demonstrated that if the condition (29) is met, then not only a particular value $\tau = m$ is beneficial, but any period belonging to the interval

$$m \leq \tau < \langle|T-m|\rangle, \tag{32}$$

turns out to be effective. Herewith, the value $\tau = m$ proposed in [53] provides the highest proved guaranteed efficiency, $\eta = \frac{\langle|T-m|\rangle - m}{\langle T\rangle}$, among other periods from the interval (32).

Finally, fulfilment of the inequality

$$m < \frac{1}{2}\langle T\rangle, \tag{33}$$

guarantees that mean performance is improved by application of regular restart with a period belonging to the interval

$$m \leq \tau < \frac{1}{2}\langle T\rangle, \tag{34}$$

see [62]. The later result represents constructive version of the existence condition mentioned in Section 4, see Eq. (13). Among values enclosed inside the interval (34), Ref. [62] recommends to choose $\tau = m$ as for this case the lower estimate for guaranteed efficiency is maximal:

$$\eta \geq 1 - \frac{2m}{\langle T\rangle}.$$

A remark similar to that made in the end of Section 4 in discussion of existence criteria should be added also here. Namely, the applicability conditions outlined in the constructivist criteria for restart efficiency (see

Eqs. (29), (30), (31) and (33)) are sufficient for constructing beneficial protocols, but not necessary: violation of these conditions does not mean that a beneficial restart period for the process in question does not exist (see Ref. [52] for discussion and example). Furthermore, it should be remembered that, in general, the values of periods proposed by constructive criteria cover only a subset of the entire set of efficient protocols. This is because the full range of efficient periods may be wider than the intervals outlined in the constructive criteria. The same comments apply to the criteria for stochastic protocols described below.

8.2. Poisson restart protocol

Let us pass to discussion of the constructive criteria of efficiency for Poisson restart. If inequality (14) is satisfied, then Poisson restart with rate belonging to the interval

$$2e\langle T\rangle^{-1} \le r \le m^{-1}, \tag{35}$$

leads to reduction of the mean completion time. The resulting efficiency at choice $r = m^{-1}$ is estimated as $\eta \ge 1 - \frac{2em}{\langle T\rangle}$. It can be also shown that choosing instead $r = 2e\langle T\rangle^{-1}$ we get completion time reduction with efficiency $\eta > 1 - \frac{2(e-1)m}{\langle T\rangle}$.

Next, if the condition (11) (which can also be written as $\langle T^2\rangle > 2\langle T\rangle^2$) is met and the third order statistical moment $\langle T^3\rangle$ is finite, then Poisson restart with rate

$$0 < r < \frac{3(\langle T^2\rangle - 2\langle T\rangle^2)}{\langle T^3\rangle}, \tag{36}$$

is proved to be beneficial, see [62]. Within the derivation procedure performed in [62], among all rates enclosed inside the interval (36), the value

$$r_0 = \frac{\langle T^3\rangle - \sqrt{\langle T^3\rangle^2 - 3\langle T^2\rangle\langle T^3\rangle\langle T\rangle + 6\langle T^3\rangle\langle T\rangle^3}}{\langle T\rangle\langle T^3\rangle}, \tag{37}$$

provides the greatest guaranteed efficiency which is estimated as

$$\eta \ge 1 - \frac{3\langle T^2\rangle - 2\langle T^3\rangle r_0}{6\langle T\rangle^2}. \tag{38}$$

Exploiting probabilistic inequality $\langle T^2\rangle \le \sqrt{\langle T\rangle\langle T^3\rangle}$, it can be checked that the expression under the square root in the right-hand side of Eq. (37) is positive. Besides, using assumed condition $\langle T^2\rangle > 2\langle T\rangle^2$ one can show that r_0 indeed belongs to the interval (36).

Let us stress that the well-known condition (11) can just detects existence of effective restart strategy without presenting it. Moreover, if we look closely at the details of the derivation of this condition, it becomes clear that knowing the first two statistical moments of the random completion time T without restart is not enough to find the guaranteed efficient restart rate. The point is that the pair $\langle T\rangle$ and $\langle T^2\rangle$ determines only the slope of $\langle T_r\rangle$ in its dependence on r at the point $r = 0$, saying nothing about its behavior for non-zero values of r. The inequality (11) indicates that the slope is negative so that introduction of small restart rate r decreases the mean completion time. Too large values of restart rate r, however, can degrade performance in the case of processes with non-monotonic dependencies $\langle T_r\rangle$. Addition of information about the third-order moment $\langle T^3\rangle$ allows us to strictly define how small the restart rate should be to ensure that $\langle T_r\rangle < \langle T\rangle$.

Reference [62] also contains two other variants of constructive criteria for Poisson restart expressed via $\langle T\rangle$, $\langle T^2\rangle$ and $\langle T^3\rangle$ (and not exploiting any other statistical measures) which differ from each other by the ranges of effective rates they establish and by estimates on efficiency at recommended values of rate. This alternativeness arises from the difference in the auxiliary probability inequalities on Laplace transform $\tilde{P}(r)$ used to estimate the expectation $\langle T_r\rangle$ which is determined by Eq. (8).

Interestingly, the values of statistical moments $\langle T\rangle$, $\langle T^2\rangle$ and $\langle T^3\rangle$ allows also to estimate from below the optimal restart rate r_* which minimizes the expected completion time $\langle T_r\rangle$. Namely, if condition (11) is met, then

$$r_* \ge \sqrt{\frac{\langle T^3\rangle^2}{\langle T^2\rangle^4} - \frac{4\langle T\rangle\langle T^3\rangle}{\langle T^2\rangle^3} + \frac{2}{\langle T^2\rangle}} - \frac{\langle T^3\rangle}{\langle T^2\rangle^2} + \frac{2\langle T\rangle}{\langle T^2\rangle}. \tag{39}$$

The assumed inequality $\langle T\rangle < \sqrt{\langle T^2\rangle/2}$ guarantees that the right-hand side of Eq. (39) represents real positive quantity.

To avoid possible confusions, let us note that inequality (39) cannot be used for the purpose of choosing restart rate when one need to build efficient restart protocol based on partial statistical information. This inequality provides lower bound for the best possible restart rate r_* at given $\langle T\rangle$, $\langle T^2\rangle$ and $\langle T^3\rangle$, but does not tell us which values of r are guaranteed to reduce the expected completion time. Also, the recommended rate r_0, see Eq. (37), should not be confused with the optimal value r_*.

8.3. Gamma restart protocol

Finally, the constructive criterion for the restart efficiency in the case of mutually independent Gamma-distributed inter-restart intervals with shape parameter $k = 2$ is as follows. Let the statistical moments $\langle T\rangle$, $\langle T^2\rangle$, $\langle T^3\rangle$ and $\langle T^4\rangle$ exist. If the condition (15) is met for $k = 2$, then corresponding Gamma restart with rate β belonging to the interval

$$0 < \beta < \frac{\langle T^3\rangle - 3\langle T\rangle\langle T^2\rangle}{\langle T\rangle\langle T^3\rangle + \langle T^4\rangle}, \tag{40}$$

improves the mean performance, see Ref. [62]. Line of arguments presented in Ref. [62] allows also to determine (numerically) the rate parameter providing maximum guaranteed efficiency among the values specified by Eq. (40) and the estimate for the resulting efficiency.

Note also, that derivation procedure exploited in Ref. [62] can be adapted to build the constructive criteria of efficiency of gamma protocol with arbitrary integer shape parameters; however, the larger k, the higher the order of statistical moments that will appear in the resulting criteria.

9. OPTIMALLY RESTARTED PROCESSES EXHIBIT UNIVERSAL FEATURES

The expected completion time $\langle T_r\rangle$ as a function of Poisson restart rate r may exhibit a global minimum at some optimal value r_*. The analysis based on the renewal approach presented in the Ref. [54] showed that although the optimal rate r_* is model-dependent the resulting completion time statistics exhibit some universal features common to all stochastic process subject to optimally tuned Poisson restart regardless of their nature and fine statistical details. Also, universal properties can be found in completion time statistics of random processes under optimal periodic restart. Below we overview the known results relevant to this subtopic.

9.1. Optimal periodic restart

As was shown in [54], the relative fluctuation of the completion time of the process restarted with optimal period τ_*, defined by the condition $\langle T_{\tau_*}\rangle \le \langle T_\tau\rangle$ for any τ, universally obeys the inequality [54]

$$\frac{\sigma(T_{\tau_*})}{\langle T_{\tau_*}\rangle} \le 1. \tag{41}$$

Equation (41) is direct consequence of optimal property of periodic restart and relation (11).

More generally, for the high-order statistical moments of the random completion time at optimal periodic restart one universally obtains (see [52] for the proof)

$$\frac{\sqrt[n]{\langle T^n_{\tau_*}\rangle - \langle T_{\tau_*}\rangle^n}}{\langle T_{\tau_*}\rangle} \le \sqrt[n]{n! - 1}. \tag{42}$$

At $n = 1$ Eq. (42) becomes equivalent to Eq. (41).

For generic non-negative random variable T high-order moment $\langle T^n\rangle$ can be arbitrary larger than $\langle T\rangle^n$. Thus, among all possible processes with fluctuating completion times, those processes that are restarted with an optimal period are characterized by relatively moderate completion time fluctuations bounded by inequality (42).

9.2. Optimal Poisson restart

Next we pass to description of universal features shared by random processes subject to optimally tuned rate of Poisson restart r_*. Consider a three-dimensional configuration space, where each statistical distribution law of random process duration T_r is characterized by three parameters: coefficient of variation, skewness and kurtosis coefficients, which are defined as dimensionless centralized moments

$$CV_r = \frac{\sigma(T_r)}{\langle T_r\rangle}, \quad A_r = \frac{\langle (T_r - \langle T_r\rangle)^3\rangle}{\sigma^3(T_r)},$$

$$K_r = \frac{\langle (T_r - \langle T_r\rangle)^4\rangle}{\sigma^4(T_r)},$$

respectively. The first of these characteristics represents a dimensionless measure of statistical data dispersion. The second quantity plays a role of the asymmetry degree of the unimodal probability density relatively to its average value. The last metric measures the extremality of the deviations of a random variable from its average value. In other words, coefficient of variation, skewness and kurtosis coefficients are integral statistical characteristics of T_{r_*} describing typical amplitude of fluctuations, shape of probability density function and its "taildness" degree.

Results of works [54] indicate that stochastic processes under optimal Poisson restart occupy two-dimensional manifold of the three-dimensional space of parameters (CV, A, K). Namely, the relative fluctuation of the random completion time T_{r_*} is always equal to unity, i. e.

$$CV_{r_*} = 1, \tag{43}$$

while skewness and kurtosis are limited from below as

$$A_{r_*} \ge 2. \tag{44}$$

and

$$K_{r_*} \geq 5. \tag{45}$$

Note that in general case for non-negative random variable we have [63, 64]

$$CV \geq 0, \quad A \geq \frac{CV^2 - 1}{CV}, \quad K \geq A^2 + 1.$$

The relation (43) was obtained in work [54] by revealing a connection between $\langle T_r \rangle$ and $\langle T_r^2 \rangle$ at the point r_* which satisfy the condition

$$\frac{d\langle T_r\rangle|_{r=r_*}}{dr} = 0$$

by the virtue of definition. Lower bounds determined by Eqs. (44) and (45) follow from Eq. (43) and the inequality

$$\frac{\langle T_{r_*}^3\rangle}{\langle T_{r_*}\rangle^3} \geq 6, \tag{46}$$

(see [52]) which can be obtained from (8), by imposing the condition

$$\frac{d^2\langle T_r\rangle|_{r=r_*}}{dr^2} \geq 0$$

required as we assume that extrema at point $r = r_*$ is minima. Relation (45) follows from (44) and estimate $K \geq A^2+1$. Derivation of Eq. (43), (44) and (45) does not assume any restrictions concerning the particular form of the completion time density $P(T)$ and, thus, these results hold for any random process for which optimal rate r_* exists.

In descriptive statistics, skewness and kurtosis are often used to characterize the possible deviations of statistical data from normality. In the context considered here it is more interesting to adopt the exponential distribution, for which

$$CV_{exp} = 1, \quad A_{exp} = 2, \quad K_{exp} = 9,$$

as a referent probability density. We see that completion time of optimally restarted processes exhibit unit coefficient of variation as exponential random variables do. What is more, the exponential distribution belongs to the region defined by the conditions (44) and (45). Next, we may recall that Poisson restart turns the power-law tails of a completion time probability density into exponential tails [65]. These observations may suggest that for any random process under the optimally tuned restart rate the statistics of the completion time is exactly exponential. This conclusion, however, is refuted by counterexamples, see [54].

Note that the third order statistical moment of completion time of a process optimized via periodic protocol obeys inequality which is opposite to (46): $\langle T_{\tau_*}^3\rangle/\langle T_{\tau_*}\rangle^3 \leq 6$, see Eq. (42) for $k = 3$. Also, note the difference between strict identity (43) and inequality relation $CV_{\tau_*} \leq 1$ (see Eq. (41)). These observations and inequality $\langle T_{\tau_*}\rangle \leq \langle T_{r_*}\rangle$, which is a consequence of the global optimality of periodic restart, highlight that Poisson restart is inferior in its optimization potential to the periodic protocol, both in terms of average completion time and its resulting fluctuations, when the latter are characterized by the second and third statistical moments.

10. SOME NON-UNIFORM RESTART PROTOCOLS

Determination of the optimal restart period τ_* providing maximum possible mean completion time reduction for a given process requires knowledge of the completion time probability density $P(T)$. Not necessarily optimal, but still guaranteed beneficial values of restart period can be predicted based on partial statistical information (e. g., restart-free mean and median values of the random completion time T estimated from statistical sample $T_1, T_2, T_3, ..., T_l$ of sufficiently large size l), as discussed in Section 8. In the absence of any information about the statistical characteristics of process completion, the only way to guarantee that mean performance will not be reduced by a poorly chosen restart strategy is to not restart the process at all. In this section, we describe several non-uniform patterns of restart events proposed to address the problem of unknown process statistics by using a broad spectrum of inter-restart intervals, differing by orders of magnitude, within a single strategy.

10.1. Luby restart protocol

Long before the first rigorously deduced rules of constructing effective restart strategies based on partial information were developed [53, 62], in work [1] Luby et al investigated how well one can do when absolutely no information is available on the original completion time distribution. Considering discrete time random processes, they proposed the following deterministic sequence of inter-restart periods

$$R_i = \begin{cases} 2^{k-1}\tau_0, & \text{if } \; i = 2^k - 1, \\ \\ R_{i-2^{k-1}+1}, & \text{if } \; 2^{k-1} \leq i < 2^k - 1, \end{cases} \tag{47}$$

with natural k, whose beginning looks like τ_0, τ_0, $2\tau_0$, τ_0, τ_0, $2\tau_0$, $4\tau_0$, τ_0, τ_0, $2\tau_0$, τ_0, τ_0, $2\tau_0$, $4\tau_0$, $8\tau_0$, τ_0, ...

Here τ_0 denotes the discrete time step of random process model. All inter-restart intervals are powers of two times τ_0, and each time a pair of intervals of a given duration has appeared, an interval of twice that length immediately follows. The strategy can be called scale-free because instead of a fixed value of the restart period, it uses a set of periods with ever-increasing values such that at any (sufficiently large) point in time after the start of the process, the total time spent on the intervals of each duration among those that have already appeared is roughly equal. This protocol, nowadays known as Luby strategy, possess following remarkable property: regardless of the statistical details of the original process, the average completion time of the process subject to this strategy cannot exceed the expectation of completion time providing by the optimal periodic protocol by no more then a logarithmic factor. More specifically, for any discrete completion time distribution one universally has (see [1] for the proof)

$$\langle T_{Luby}\rangle \leq \langle T_{\tau_*}\rangle \cdot 192 \log_2 \frac{32\langle T_{\tau_*}\rangle}{\tau_0}. \tag{48}$$

What is more, the logarithmic functional form of the worst-case pre-factor, i. e., $C_1 \log_2 \frac{C_2\langle T_{Luby}\rangle}{\tau_0}$, is the best in the entire space of restart protocols. This is because for any strategy $\mathcal{R}$ there is an example of completion time model for which the resulting expectation $\langle T_{\mathcal{R}}\rangle$ is larger by a logarithmic factor than the minimal possible value $\langle T_{\tau_*}\rangle$:

$$\langle T_{\mathcal{R}}\rangle \geq \frac{1}{8}\langle T_{\tau_*}\rangle \log_2 \frac{\langle T_{\tau_*}\rangle}{\tau_0}$$

(Theorem 7 in [1]). The analysis presented in work [1] also provides the lower estimate for the tail probability of strategy (47). Reference [51] argues that numerical constants in Eq. (48) can be replaced smaller ones:

$$\langle T_{Luby}\rangle \leq 128 \log_2 \frac{16\langle T_{\tau_*}\rangle}{\tau_0}.$$

The sharp values of numerical constants which cannot be further reduced remain unknown.

Unfortunately, the Luby strategy property that it provides the mean performance which is not too far from the best possible does not survive in the case of continuous-time processes because of the absence of any universal short-time cut-off τ_0 in the continuous models [51]. There are probability densities $P(T)$ for which the expected completion time under Luby's strategy with $\tau_0 \to 0$ is arbitrarily worse than that under the optimal periodic protocol.

Another problematic issue associated with practical implementation of Luby strategy is that even for discrete-time processes, by applying this protocol, we can only be sure that the resulting average completion time will not be too bad compared to the best possible value $\langle T_{\tau_*}\rangle$ achieved by optimally tuned periodic restart. There is no guarantee that the Luby strategy will not slow down process completion as compared to the restart-free case [51]. In other words, inequality $\langle T_{Luby}\rangle \leq \langle T\rangle$ is not universally valid for any completion time model.

10.2. A non-uniform Poisson protocol with a rate that decreases inversely with time

A promising variant of a truly scale-free restart strategy in continuous-time settings is time-inhomogeneous Poisson restart with hyperbolically decaying rate,

$$r(t) = \frac{\alpha}{t}, \tag{49}$$

where t is time elapsed since the start of the process, and α is a dimensionless constant. This protocol has been proposed in Ref. [55] and represents an example of stochastic restart strategy with mutually dependent random inter-restart intervals. Note that in practice non-integrable singular behaviour of the decay rate at $t \to +0$, implying non-physically infinite number of restarts just after the start of the process, should be regularized by an introducing a short-scale cut-off τ_0, e. g., in the following way $r(t) = \frac{\alpha}{t+\tau_0}$. To ensure that properties of the regularized strategy will reproduce those of the idealized one, there must be empirically or theoretically justified confidence that the separation of time scales $\tau_0 \ll a$ is fulfilled for all reasonable values of the unknown parameter a for the system under consideration, given the chosen cut-off value τ_0.

Authors of Ref. [55] used renewal ideas and properties of Poisson flow of random events to obtain integral equation on the survival probability of the process under restart protocol (49) which is defined as the probability that the process has not finished up to a given time moment. The equation can be used to obtain closed-form expression for Laplace transform of survival probability for model with arbitrary $P(T)$, and the mean completion time $\langle T_\alpha\rangle$ is then obtained as the corresponding Laplace transform estimated at zero argument. The final answer reported in Ref. [55] looks like

$$\langle T_\alpha\rangle = \alpha \int_0^\infty du \frac{1-\tilde{P}(u)}{u^2} e^{-\alpha \int_u^\infty dv \frac{\tilde{P}(v)}{v}}, \tag{50}$$

where

$$\tilde{P}(u) = \int_0^\infty dT e^{-uT} P(T).$$

The work [55] also predicts that the Laplace transformed probability density of random completion time under restart protocol (49) is given by expression

$$\tilde{P}_\alpha(s) = 1 - \\ - \alpha s \int_s^\infty du \left(\frac{u-s}{u}\right)^{\alpha-1} \frac{1-\tilde{P}(u)}{u^2} \times \\ \times \exp\left\{-\alpha \int_u^\infty dv \frac{\tilde{P}(v)}{v}\right\}. \quad (51)$$

Analysis of Eq. (50) shows that for random processes whose completion time statistical law $P(T)$ is parametrized by a single time scale $a > 0$, i. e.,

$$P(T) = a^{-1} f(T/a),$$

where f is some non-negative function obeying the condition

$$\int_0^\infty dz f(z) = 1,$$

the ratio $\langle T_{\alpha_*}(a)\rangle / a$ is constant for any a, and, therefore, the protocol efficiency η defined by Eq. (28) is independent of a, since

$$\langle T\rangle = \int_0^\infty dT f(T/a) T/a = a \int_0^\infty dz f(z) z$$

if finite for a given shape function f, scales linearly with a. This also means that the optimal parameter α_*, which brings the expected decay time $\langle T_\alpha(a)\rangle$ to a minimum (if any), is determined by the shape of the function f and is insensitive to a. In other words, if $\langle T_{\alpha_*}(a)\rangle \le \langle T_\alpha(a)\rangle$ for any $\alpha > 0$, then one also has $\langle T_{\alpha_*}(ga)\rangle \le \langle T_\alpha(ga)\rangle$, where $g > 0$ is arbitrary dimensionless factor.

Thus, once the functional form of the probability density $P(T)$ is known one can (at least numerically) compute optimal value α_* (possibly $\alpha_* = 0$) and use it to achieve the best possible within this strategy value of the mean completion time $\langle T_\alpha(a)\rangle$ even in the absence of reliable estimate of the time scale a.

In [55] the identity $\langle T_\alpha^{res}(ga)\rangle = g\langle T_\alpha^{res}(a)\rangle$, which is basic for establishing the properties of the protocol (49) described above, was obtained from Eq. (50). Let us provide an alternative derivation in the spirit of stochastic renewal equations presented in Section 3.

Given restart at the moment t, the time till next restart $R_{next|t}$ is a random variable having probability density [55]

$$\rho(R_{next}|t) = \frac{\alpha t^\alpha}{(R_{next}+t)^{a+1}}.$$

It can be checked that $R_{next|t} \sim g R_{next|\frac{t}{g}}$, where $g > 0$ is arbitrary number, and symbol $\sim$ means that the random variables have the same distribution.

Consider again the case when the probability density function $P(T)$ of restart-free random process is parametrized by a single time scale $a > 0$. Given the process has not finished upon the moment t, the time remaining to the completion $T_\alpha^{res}(t,a)$ obeys the equation

$$T_\alpha^{res}(t,a) = T_a I(T_a \le R_{next|t}) + \\ + \left(R_{next|t} + T_\alpha^{res}(t+R_{next|t}, a)\right) \times \\ \times I(T_a > R_{next|t}). \quad (52)$$

Here we specify the characteristic time scale of the model $P(T)$ by adding lower subscript a to notation T for the restart-free random completion time. Now consider the same stochastic phenomena but with different characteristic time scale of evolution. Replacing a with ga in the above equation we get

$$T_\alpha^{res}(t,ga) = T_{ga} I(T_{ga} \le R_{next|t}) + \\ + \left(R_{next|t} + T_\alpha^{res}(t+R_{next|t}, ga)\right) \times \\ \times I(T_{ga} > R_{next|t}). \quad (53)$$

Next, let us also consider $g$times larger time moment replacing t with gt to obtain

$$T_\alpha^{res}(gt,ga) = T_{ga} I(T_{ga} \le R_{next|gt}) + \\ + \left(R_{next|gt} + T_\alpha^{res}(gt+R_{next|gt}, ga)\right) \times \\ \times I(T_{ga} > R_{next|tg}). \quad (54)$$

Since $T_{ga} \sim gT_a$ by assumed single-parametric form of $P(T)$ and $R_{next|gt} \sim gR_{next|t}$ by the construction of the restart protocol under consideration, one has

$$T_{ga} I(T_{ga} \le R_{next|gt}) \sim g T_a I(T_a \le R_{next|t}),$$

$$R_{next|gt} I(T_{ga} > R_{next|tg}) \sim g R_{next|t} I(T_a > R_{next|t}),$$

$$T_\alpha^{res}(gt + R_{next|gt}, ga) I(T_{ga} > R_{next|tg}) \sim \\ \sim T_\alpha^{res}(g(t+R_{next|t}), ga) I(gT_a > gR_{next|t}) \sim \\ \sim g T_\alpha^{res}(t+R_{next|t}, a) I(T_a > R_{next|t})$$

(since the role of factor g in these expressions is equivalent to change of units of measurements of time).

Taking into account the mentioned statistical equivalence and comparing Eqs. (52) and (54) we then conclude that $T_\alpha^{res}(gt, ga) \sim gT_\alpha^{res}(t, a)$. Taking the limit $t \to +0$ one can drop subscript "res" and obtain $T_\alpha(ga) \sim gT_\alpha(a)$ for the random completion time of the same stochastic phenomenon at different values of time scale parameter. Therefore the mean completion time of the process under geometric restart strategy with factor q possesses the property $\langle T_\alpha^{res}(ga)\rangle = g\langle T_\alpha^{res}(a)\rangle$.

10.3. Geometric restart strategy

Consider deterministic restart strategy where inter-restarts intervals are given by the following sequence

$$R_i = q^i\tau \tag{55}$$

with $\tau > 0$ (having the dimension of time), $q > 1$ (dimensionless factor) and $-\infty < i < +\infty$ (integer number). The ith restart within this strategy occurs at time moment

$$t_i = \sum_{j=-\infty}^{i-1} R_i = \frac{q^i}{q-1}\tau.$$

As in the case of stochastic restart protocol from previous example, this schedule implies infinite number of restarts in infinitely small time slice just after process initialization and, thus, in practice requires short time regularization by introducing $i_{min} = \log_q \frac{\tau_0}{\tau}$ [6]. Here τ_0 is some time scales which is assumed to be much smaller than relevant time scales of restart-free process completion. This protocol is a continuous-time variant of geometric strategy for stochastic process models with discrete time, see, e. g., [66].

The two-side infinite sequence (55) does not changes if we replace τ by $q^n\tau$ with natural n keeping q constant. Therefore, identity $\langle T_{q,\tau}\rangle = \langle T_{q,q^n\tau}\rangle$ trivially holds for any random process. In other words, the mean completion time $\langle T_{q,\tau}\rangle$ plotted as a function of $\ln\tau$ at fixed q is periodic with period $\ln q$.

Consider the case when the probability density function $P(T)$ of restart-free random process is parametrized by a single time scale $a > 0$, i. e.,

$$P(T) = a^{-1}f(T/a),$$

where $f(z)$ is non-negative function normalized by unity at $0 \le z < +\infty$. It is easy to see that

$$\frac{\langle T_{q,\tau|a}\rangle}{a} = \frac{\langle T_{q,\tau|q^n a}\rangle}{q^n a}.$$

Therefore, if there exists an optimal value $q_*(\tau)$ such that $\langle T_{q_*(\tau),\tau|a}\rangle \le \langle T_{q,\tau|a}\rangle$ for all $q > 1$ at fixed τ, then one also has $\langle T_{q_*(\tau),\tau|q_*^n(\tau)a}\rangle \le \langle T_{q,\tau|q_*^n(\tau)a}\rangle$. Similarly, if we find optimal $\tau^*(q)$ such that $\langle T_{q,\tau^*(q)|a}\rangle \le \langle T_{q,\tau|a}\rangle$ for all $\tau > 0$ at given q, then $\langle T_{q,\tau^*(q)|q^n a}\rangle \le \langle T_{q,\tau|q^n a}\rangle$.

These properties are reminiscent to the time scale independence of the optimal α_* from previous example, where we discussed impact of time non-uniform Poisson strategy with hyperbolically decaying restart rate on processes with completion time probability density characterized by a single time scale parameter. It should be noted, however, that in this case constancy of the optimal factor $q_*(\tau)$ or optimal time interval $\tau^*(q)$ is observed only on a very specific discrete set of time scale values:

$$\ldots, q_*(\tau)^{-2}a, q_*(\tau)^{-1}a, q_*(\tau)a, q_*(\tau)^2a, q_*(\tau)^3a, \ldots$$

and

$$\ldots, q^{-2}a, q^{-1}a, qa, q^2a, q^3a, \ldots,$$

respectively. These sets are related to all possible time unit scaling transformations that leave the full spectrum of numerical values of the intervals between restarts, defined by (55), unchanged.

From practical perspective, given the known shape function f one can try to find out if there are values of τ and q for which geometric restart (55) is guaranteed to reduce the expected completion time for any $a \gg \tau_0$. If such values do not exist, then implementation of geometric restart to the process with finite $\langle T\rangle$, determined by unknown time scale a, can degrade mean performance.

11. IMPACT OF RESTART ON OTHER MEASURES

Although the mean completion time plays a central role for some applications, in many other settings, one may be interested in improving of other statistical characteristics of random process completion [6, 66, 67]. In other words, the statistical characteristics that restart based optimization should target obviously depends on specific practical situation. In this section we review the general results in the body of literature concerning restart-induced effects on median completion time (Subsection A), on mode of completion time probability distribution (Subsection B), on high-order statistical moments of completion time (C), on tail behaviour of completion time probability density (D), on Shannon entropy (E) and diversity of completion time (F), on deadline meeting probability (G), on success probability in random experiment with alternative completion scenario (I), on the rate of obtaining the desired process outcome (K), and (I) on probability of never-stopping.

Before we proceed to discussion of this result let us note that not only particular statistical quantity, but full statistics of the completion time $T_{\mathcal{R}}$ under restart protocol $\mathcal{R}$, encoded via probability density function $P_{\mathcal{R}}(t) = \langle \delta(t - T_{\mathcal{R}}) \rangle$, characteristic function $\langle e^{-ikT_{\mathcal{R}}} \rangle$, Laplace transform $\langle e^{-sT_{\mathcal{R}}} \rangle$, or survival probability $Pr[T_{\mathcal{R}} > t]$, can potentially be described analytically. In particular, see Ref. [47] for details of derivation of $\langle e^{-sT_{\mathcal{R}}} \rangle$ for any $P(T)$ and arbitrary stochastic protocol with independent and identically distributed restart intervals. Note that the completion time statistics of processes with an exponential $P(T)$ remain absolutely unchanged under any $\mathcal{R}$ because of the memoryless nature of their completion kinetics.

In the case of Poisson restart protocol the Laplace transform of random completion time is given by

$$\langle e^{-sT_r} \rangle = \frac{\tilde{P}(s+r)}{\frac{s}{s+r} + \frac{r}{s+r}\tilde{P}(s+r)}, \tag{56}$$

as proven in [54]. It was derived in [6] from (56), that the probability density function $P_r(t)$ of random completion time T_r for a process experiencing Poisson restarts obeys the relation

$$\frac{P_r(t) - P(t)}{r} \approx -P(t)t - \int\limits_t^\infty dTP(T) + \\ + \int\limits_t^\infty dT \int\limits_0^T dT' P(T')P(T-T') \tag{57}$$

at $r \to 0$. This formula describes the linear response of the entire process completion time histogram to the introduction of a small restart rate.

Let us also note here that the survival probability of a process under regularly repeated restarts at period τ has the following form:

$$Pr[T_{\mathcal{R}} > t] = \left(\int\limits_\tau^\infty dTP(T) \right)^{[\frac{t}{\tau}]} \int\limits_{t-[\frac{t}{\tau}]}^\infty dTP(T), \tag{58}$$

see, e. g., [5, 68]. Here $[\ldots]$ denotes the integer part of the number.

11.1. Median completion time

In work [6] the effect of Poisson restart on the median completion time of a generic random process has been investigated. The median completion time is a right quantity for optimization in situations where one deals with large ensemble of parallel trials of the same random process. The analysis provided a sufficient condition for when the introduction of a small restart rate reduces the median completion time, or more generally, a given quantile of the completion time distribution. The corresponding criteria reads [6]

$$2\langle T \rangle_{T_q} + K_2(T_q)T_q < \langle T_1 + T_2 \rangle_{T_q} + qT_q. \tag{59}$$

Here

$$\langle T \rangle_{T_q} = \int\limits_0^{T_q} dTP(T)T, \ \langle T_1 + T_2 \rangle_{T_q} = \int\limits_0^{T_q} dTP_2(T)T$$

are the so-called truncated (or partial) moments [4]; $P_2(T)$ is the probability density, and

$$K_2(T) = \int\limits_0^{T_q} dTP_2(T)$$

is the cumulative distribution function of the random variable $T_1 + T_2$ representing the sum of two random copies T_1 and T_2 independently sampled from $P(T)$; T_q is the qth quantile, i. e., the time such that the process has the probability $0 < q < 1$ to finish before T_q. Result (59) has been obtained from calculation of leading order correction to restart-free quantile T_q induced by rare Poisson restarts based on general expression (57) [6].

In the case of median, $q = 1/2$, Eq. (59) takes the following form

$$2\langle T \rangle_m + mK_2(m) < \langle T_1 + T_2 \rangle_m + \frac{m}{2}. \tag{60}$$

Taking the limit $q \to 0$ and assuming that $P(T) \approx CT^m$ at $T \to 0$, where C is positive dimensional number and $m > -1$, it can be shown from condition (59) that introduction of Poisson restart with very small rate inhibits the short-time completion unless the probability density function $P(T)$ is singular at $T = 0$. In the opposite limit $q \to 1$, the response of the quantile to rare Poisson restarts is determined by the behaviour of the tails of the FPTD. It follows from (59) that restart at small rate reduces the large-q quantiles of the heavy-tailed completion time probability density, while increasing them in the case of the Gaussian tails.

It also follows from the results of the work [6] that if there exists the optimal rate $r_*(q)$ of the Poisson restart, which brings the q-quantile of the random process completion time to the minimum value $T_q(r_*)$, then the following identity must hold

$$\frac{2\langle T_{r_*} \rangle_q + T_q(r_*)K_{2r_*}(T_q(r_*))}{\langle T_{r_*1} + T_{r_*2} \rangle_{T_q(r_*)} + qT_q(r_*)} = 1. \tag{61}$$

A simple example of model where quantiles of random completion time can be reduced via restart is given by one-dimensional diffusive search [6]. Numerical analysis shows that the sufficient condition of restart efficiency (59) is satisfied for $q > q_0$, where $q_0 \approx 0.4123$. Stochastic simulations indicate that the median search time m_r as a function of restart rate attains minimum value at $r_* \approx 1.5D/L^2$, where D is the particle diffusivity, L is the starting distance to the target.

11.2. Mode of completion time probability density

As already was mentioned in Section 6, the mode of the completion time probability density $P(T)$ is the time M at which this function takes its global maximum. Effect of Poisson restart on the mode of a smooth probability density function $P(T)$ of random completion time T has been analysed in Ref. [6] based on Eq. (57). Results indicate that introduction of Poisson restart always decreases or leaves unchanged the mode in such models. Note that this effect is non-perturbative in the sense that it takes place for all values of restart rate r, not necessarily small enough: the mode M_r as a function of r is non-increasing. Although it is not stated explicitly in [6], the logic of the analysis presented there indicates that this property holds for every local maximum of any smooth completion time probability density, if there are several of them. In contrast, positions of the local minimums (if any) of any smooth $P(T)$ are non-increasing functions of restart rate r.

11.3. Non-linear statistical moments of the completion time

Several works addressed optimization of non-linear statistical moments of random completion time via restart [4, 48, 69, 70]. The general recurrent relation for the statistical moments of arbitrary natural order for a generic random process in the presence of periodic restart schedule were presented in [5, 48, 69] and looks like

$$\langle T_\tau^n\rangle = \frac{\int\limits_0^\tau T^n P(T)dT + \int\limits_\tau^\infty P(T)dT \cdot \sum_{l=0}^{n-1} C_n^l \tau^{n-l} \langle T_\tau^l\rangle}{\int\limits_0^\tau P(T)dT}, \quad (62)$$

where $n \geq 1$ and $C_n^l = \frac{n!}{l!(n-l)!}$ is binomial coefficient.

Noteworthy, in contrast to the problem of mean completion time optimization, periodic restart protocol is typically not optimal for the purpose of control over non-linear statistical moments. This conclusion was made in [4] (see also [48]) based on numerical analysis of particular model distributions $P(T)$.

11.4. Entropy of completion time distribution

Analysis [7] explored how restart affects differential entropy

$$S(T) = \int\limits_0^\infty dT P(T) \ln P(T)$$

of the completion time distribution — a measure that can be interpreted as a measure of stochasticity of random variable T. Simple criteria established in [7] describe when regular restart with sufficiently small or sufficiently large period could render duration of random process more predictable. Beside existence results, that work also describes constructive criteria specifying the intervals of restart periods effective from the point of view of the entropy reduction.

11.5. Diversity of completion time

Diversity of random completion time is defined as

$$\mathcal{D}_\varepsilon(T) = \left[\frac{1}{\int\limits_{-\infty}^{+\infty} dT P^\varepsilon(T)}\right]^{\frac{1}{\varepsilon-1}},$$

where $\varepsilon > 1$. The integral in definition of diversity gives the probability that the sample of independent copies $T_1, T_2, ..., T_\varepsilon$ of size ε will appear as if it is deterministic. The reciprocal of this probability can be used as a quantitative gauge of randomness of random completion time T. Thus, similarly to entropy, diversity measures inherent stochasticity but in different manner. Reference [8] reported criteria that determine existence of periods of regular restart decreasing diversity of random completion time.

11.6. Tail behaviour of completion time probability density

Work [53] presents tail-behavior analysis of completion times in the presence of periodic restart. The quantity targeted by optimization in this case is the long-term survival probability of the process. Restart procedure $\mathcal{R}$ can be called to be beneficial with respect to this characteristic if

$$\lim_{t\to\infty} \frac{Pr[T_\mathcal{R} > t]}{Pr[T > t]} < 1.$$

The analysis presented in [53] established universal criteria that determine when periodic restart makes the probability tails of the process completion duration lighter thus rendering the occurrence of extremely prolonged realizations less likely. The corresponding criteria are expressed via the hazard rate function of the restart-free process defined as

$$H(t) = \frac{P(t)}{Pr[T > t]},$$

or, equivalently,

$$H(t) = -\frac{d \ln Pr[T > t]}{dt}.$$

An example of such a criterion is given by the inequality

$$H(\infty) < H(0). \tag{63}$$

If process completion time statistics exhibits property (63), then introduction of periodic restart with sufficiently small period τ leads to reduction of the long-term survival probability.

11.7. Deadline meeting probability

Another possible metric of interest is the probability that a pre-determined deadline T_d is met. In the absence of restart, probability that a stochastic process with the completion time distribution $P(T)$ will finish before the time moment T_d is expressed as

$$p_d = \int_0^{T_d} P(T)dT.$$

Clearly,

$$p_d = 1 - Pr[T > T_d],$$

and thus the tail survival-probability optimization, discussed in the previous subsection, can be considered as a special case of deadline meeting optimization problem.

The prospects of restart method in the deadline meeting optimization were a subject of query in Refs. [6, 48, 69–71]. Considering all possible protocols with fixed total number of restarts within the time interval of duration T_d, author of work [70] showed that the extrema of deadline meeting probability are at equihazard inter-restart intervals, with as special case the equidistant intervals corresponding to strictly periodic restart. However, periodic strategy does not always give global maxima [66]. As for the stochastic protocols, analysis presented in [6] demonstrated that Poisson restart increases the chance to meet deadline whenever the inequality

$$2\langle T\rangle_{T_d} + K_2(T_d)T_d < \langle T_1 + T_2\rangle_{T_d} + p_d T_d \tag{64}$$

is satisfied. At optimal restart rate r_* which maximizes probability to meet deadline the random completion time obeys the equality

$$2\langle T\rangle_{T_d} + K_2(T_d)T_d = \langle T_1 + T_2\rangle_{T_d} + p_d T_d. \tag{65}$$

Note similarity between Eqs. (64), (65) and corresponding results in quantile optimization problem, Eqs. (59) and (61).

11.8. Splitting probabilities

Most works on restart-induced optimization focuses on random processes with a single end result of stochastic dynamics (e. g., the problem is solved, the target is found, etc.), that is not further subdivided into any types. In some settings, however, multiple alternative completion scenarios, corresponding to different outcomes of stochastic dynamics, are possible. The multiplicity of possible outcomes may be due, for example, to competition between different completion mechanisms coexisting within a single process, as in the case of random search with decay, or due to multiple thresholds for the same dynamic variable – with the random search in the presence of two absorbing targets as a canonical example. The frequencies of occurrence of certain outcomes in a long series of multiple independent process trials are called splitting probabilities. In those settings where outcomes are not equally valuable one may be interested in optimizing the chances for observing particular completion scenarios.

The work [9] demonstrated that probability of getting desired process outcome can potentially be increased by restart. Consider Bernoulli-like random process, i. e., of a process which can end with one of two outcomes – "success" and "failure". The probability density $P(T)$ of a random completion time T of such process can be represented as the sum

$$P(T) = P_s(T) + P_f(T),$$

where $P_s(T)$ and $P_f(T)$ denote the contributions of successful and unsuccessful process trials, respectively. The normalization of the function $P_s(T)$ gives the "undisturbed" (i. e., restart-free) probability of success:

$$p = \int_0^{\infty} P_s(T)dT.$$

As in the case of the mean completion time optimization, the restart protocol $\mathcal{R}$ is determined by a sequence of inter-restarts time intervals $R_1, R_2, R_3, \ldots$. Assume that these intervals are independently sampled from the probability density $\rho(R)$. Then for the success probability in the presence of restart one gets (see [9])

$$p_{\mathcal{R}_{t.h.}} = \frac{\int\limits_0^\infty d\tau \rho(\tau) \int\limits_0^\tau dT P^s(T)}{\int\limits_0^\infty d\tau \rho(\tau) \int\limits_0^\tau dT P(T)}. \tag{66}$$

The regular restart protocol with period τ corresponds to $\rho(R) = \delta(R-\tau)$. In this case the probability of observing a successful outcome is given by [9]

$$p_\tau = \frac{\int\limits_0^\tau P^s(T) dT}{\int\limits_0^\tau P(T) dT}. \tag{67}$$

Both regular and stochastic restart strategies could potentially improve the chances for observing desirable process outcome, see examples in [9]. As was discussed in Section 5, the optimally tuned periodic restart always outperforms other restart strategies in terms of attaining the lowest expected completion time. Similar result has been proved in the context of success probability optimization: if there exists such τ_* that the protocol with restart time distribution $\rho(R) = \delta(R-\tau_*)$ brings the probability of success to a maximum p_{τ_*}, i. e., $p_{\tau_*} \geq p_\tau$ for any τ, then no other choice of $\rho(R)$ can provide a probability of success $p_{\mathcal{R}}$ greater than p_{τ_*} [9]. Obviously, maximum of success probability corresponds to minimum in probability of getting failure, since the sum of splitting probabilities is equal to unity due to normalization condition.

Beyond the question of optimal strategy, renewal approach based research on success probability optimization helped to identify universality in the statistics of optimally restarted processes, allowed us to establish sufficient conditions for the existence of effective restart strategies, and formulate constructive criteria for restart efficiency. The corresponding results cover the cases of periodic, Poisson and gamma restart strategies as overviewed below.

If τ_* is the optimal period of regular restart bringing success probability to maximum value p_{τ_*} then the resulting mean completion time conditional to success $\langle T^s_{\tau_*}\rangle$ and the mean completion time conditional to failure $\langle T^f_{\tau_*}\rangle$ are universally ordered as (see [9])

$$\langle T^s_{\tau_*}\rangle \geq \langle T^f_{\tau_*}\rangle. \tag{68}$$

Thus, unsuccessful trials of a process under optimally tuned periodic restart are shorter (in average) than successful ones. This conclusion can be deduced from Eq. (72) (see below) recalling that periodic restart outperforms the Poisson restart in terms of success probability improvement.

Paper [62] presented a simple constructive criterion for the efficiency of a periodic restart protocol beneficial for the purpose of success probability optimization. Namely, if

$$m_s < m \tag{69}$$

(or, equivalently $m_s < m_f$), then regular restart with period, belonging to the interval

$$m_s < \tau < m, \tag{70}$$

improves the chances for success. Here m_s and m_f are the median completion times of successful and failed trials, correspondingly, while m is the unconditional median completion time.

Next, the probability of success p_r for the Bernoulli-like random process under Poisson protocol with mean frequency of restarts r is given by the formula

$$p_r = \frac{\tilde{P}^s(r)}{\tilde{P}(r)}, \tag{71}$$

where

$$\tilde{P}^s(r) = \int\limits_0^\infty e^{-rT} P^s(T) dT, \quad \tilde{P}(r) = \int\limits_0^\infty e^{-rT} P(T) dT$$

denote the Laplace transforms of, respectively, $P^s(T)$ and $P(T)$ evaluated at r [9]. It can be shown from Eq. (71) that the inequality

$$\langle T^s\rangle < \langle T\rangle, \tag{72}$$

or, equivalently, $\langle T^s\rangle < \langle T^f\rangle$, since

$$\langle T\rangle = p\langle T^s\rangle + (1-p)\langle T^f\rangle$$

gives a simple criteria for existence of effective rate of Poisson restart increasing the success probability [9]. Here we assume that the mean completion times conditional to success $\langle T^s\rangle$ and failures $\langle T^f\rangle$ and the high-order moments of completion times are finite. In work [62] the existence result (72) has been converted into constructive criteria of Poisson restart efficiency. More specifically, if the condition (72) is met, then restart with any rate enclosed inside the interval

$$0 < r < \frac{\langle T\rangle \left(\langle T\rangle - \langle T_s\rangle\right)}{\langle T^2\rangle \langle T_s\rangle}, \tag{73}$$

increases chances for success.

In situations when success probability p_r as a function of restart rate r attains a maximum for some r_* we universally get (see [9])

$$\langle T^s_{r_*}\rangle = \langle T^f_{r_*}\rangle, \qquad (74)$$

and

$$\langle {T^s_{r_*}}^2\rangle \leq \langle {T^f_{r_*}}^2\rangle. \qquad (75)$$

Equations (74) and (75) tell us that, as in the case of optimizing the mean completion time, optimal control over the success probability via a restart imposes certain universal imprints on the statistics of random process.

For the Gamma protocol with rate parameter β and shape parameter $k = 2$, success probability is given by the following expression

$$p_\beta = \frac{\beta\partial_\beta \tilde{P}^s(\beta) - \tilde{P}^s(\beta)}{\beta\partial_\beta \tilde{P}(\beta) - \tilde{P}(\beta)}, \qquad (76)$$

presented in [62]. A remarkably simple sufficient condition for existence of Gamma strategy improving the chances for getting desirable outcome reads

$$\langle T^2\rangle > \langle T^{s2}\rangle, \qquad (77)$$

as can be derived from Eq. (76) by linear order expansion of the right-hand side of this equation upon rate parameter β. Since

$$\langle T^2\rangle = p\langle T^{s2}\rangle + (1-p)\langle T^{f2}\rangle,$$

this condition can also be rewritten as $\langle T^{f2}\rangle > \langle T^{s2}\rangle$.

Beside specifying conditions sufficient for existence of beneficial gamma strategy, it is also possible determine the corresponding range of efficient rate values β. Namely, if Eq. (77) is satisfied then Gamma restart with $k = 2$ and rate belonging to the interval

$$0 < \beta < \frac{3(\langle T^2\rangle - \langle T^2_s\rangle)}{3\langle T^3\rangle + \langle T^3_s\rangle}, \qquad (78)$$

increases the probability of success. More examples of constructive criteria for Gamma restart efficiency in success probability optimization problem can be found in [62].

Comparing conditions (69), (72) and (77), we see that restart provides advantageous for those processes with alternative outcomes for which the following time scale ordering takes place: the characteristic time associated with completion of successful trials is short compared with the characteristic completion time of trials producing undesirable outcomes. Qualitatively this can be explained in the following way. If prolonged process trials typically end with failure then it could be better (in average) to stop the process has not finished during time of characteristic successful trials duration, and try again.

Obviously, results discussed here are also relevant to processes with arbitrary number of possible outcomes sub-divided into two groups — desirable (classified as success) and undesirable (collectively considered as failure).

11.9. Rate of obtaining desirable outcome

Suppose that a random experiment with two alternative outcomes is repeated over and over again, and we want to optimize the rate of successful outcomes ν_r, rather than the proportion p of the total number of trials that end in the desired way, discussed in the previous section. The former is defined as

$$\nu_r = \frac{N^s_r}{W}, \qquad (79)$$

where N^s_r is the number of successful outcomes observed during large time interval $W \gg \langle T\rangle$. When both $\langle T\rangle$ and $\langle T^2\rangle$ are finite, N^s_r represent reproducible (i. e., small fluctuating) non-zero quantity. Since $N^s_r = p_r N_r$ and $W = N_r\langle T\rangle$ at $W \to \infty$, where N_r is the total number of all process completions (both successes and failures) during the interval W and p_r is the probability of success, then

$$\nu_r = \frac{p_r}{\langle T\rangle}. \qquad (80)$$

Substituting relations (8) and (71) into (80) one gets

$$\nu_r = \frac{r\tilde{P}^s(r)}{1-\tilde{P}(r)}. \qquad (81)$$

Let r_* be the optimal value of restart rate bringing ν_r to its maximum. The identity

$$\langle T^s_{r_*}\rangle = \frac{1}{2}\frac{\langle T^2_{r_*}\rangle}{\langle T_{r_*}\rangle}, \qquad (82)$$

and inequality

$$\langle T^s_{r_*}\rangle^2 \leq \frac{1}{3}\frac{\langle T^3_{r_*}\rangle}{\langle T_{r_*}\rangle}, \qquad (83)$$

inevitably hold for random processes in the presence of a Poisson restart with a rate tuned to maximize the rate of obtaining the desired outcome.

11.10. Conditional mean completion time

Considering random process with multiple competitive completion pathways under Poisson restart, authors of work [10] investigated optimization of the mean duration of those trials that end with desirable outcome, $\langle T_r^s \rangle$. For this type of optimization problem the protocol $\mathcal{R}$ is called to be beneficial if $\langle T_{\mathcal{R}}^s \rangle < \langle T^s \rangle$. The reference reports a sufficient condition for existence of Poisson restart strategy providing acceleration of the successful trials. Namely, if the inequality

$$\frac{\sigma(T^s)}{\langle T^s \rangle} > \sqrt{\frac{\langle T \rangle^2}{2\langle T^s \rangle^2}\left(1+\frac{\sigma(T)^2}{\langle T \rangle^2}\right)} \tag{84}$$

is met then introduction of rare Poisson restart must reduce the mean completion time conditional to desirable process outcome. What is more, if there exists a value r_* for which $\langle T_r^s \rangle$ reaches minimum then resulting completion time statistics obeys universal equality

$$\frac{\sigma(T_{r_*}^s)}{\langle T_{r_*}^s \rangle} = \sqrt{\frac{\langle T_{r_*} \rangle^2}{2\langle T_{r_*}^s \rangle^2}\left(1+\frac{\sigma(T_{r_*})^2}{\langle T_{r_*} \rangle^2}\right)}. \tag{85}$$

The full statistics of the random completion time of successful trials can be extracted from the following expression

$$\langle e^{-sT_r^s} \rangle = \frac{\tilde{P}^s(r+s)}{\tilde{P}^s(r)} \frac{(s+r)\tilde{P}(r)}{s+r\tilde{P}(r+s)}. \tag{86}$$

11.11. Joint optimization of two statistical characteristics

Random process performance, quantified in a variety of ways beyond the mean completion time analysis, could potentially be improved by restart. It should be noted, however, that a positive effect of a given restart schedule on one statistical metric of process completion does not guarantee improvement in others. For example, as noted in papers [8] and [7] restarting can reduce the average completion time, while increasing its uncertainty measured by Shannon entropy or diversity. Also, restart may have an opposite effect on average behavior and on tail behavior of random completion time, see [53].

Also noteworthy, even when effects of restart on two different characteristics are in accord, the optimal rates/periods corresponding to the optimization of these quantities generally do not coincide. Say, analysis presented in the work [6] demonstrates that the optimal rates that minimize the mean and median search time of immobile target in one-dimensional diffusion are different. Beside, as noted in [66] (chapter 4.3.1), optimality in deadline meeting probability of the process under restart does not mean to optimality in expected runtime. Similarly, according to results of work [5] restart time that minimises the non-linear statistical moments of the completion time is typically not identical to optimal period bringing to minimum the mean completion time.

11.12. Probability of halting

So far we have considered a scenario in which restart-free process trials eventually terminate with unit probability, i. e.,

$$\int_0^\infty P(T)dT = 1.$$

Now let us overview what is known regarding effect of restart on completion statistics of processes with non-zero probability of never ending. This discussion is relevant to computational algorithms with non-halting problem instances [5] which go on forever getting stuck in infinite loops and to diffusion process with absorbing targets in spatial dimensions of 3 and higher where particle has chances to escape to infinity without touching the surface of the targets.

Assume that original stochastic process ends for finite time with probability $0 < q < 1$. This probability can be represented as

$$q = Pr[T < \infty] = \int_0^\infty dT P(T),$$

where $P(T)$ should be now understood as the envelope of the histogram constructed on the basis of a large sample of the completion times of those trials that terminate. Most importantly, periodic restarting completely eliminates the possibility of never stopping, making the average completion time finite. This can be checked using Eq. (5) which follows from the renewal equation (3) remaining valid for the processes with non-zero probability on non-halting. Indeed, Eq. (5) predict that $\langle T_\tau \rangle$ is finite for any $0 < \tau < +\infty$. Therefore

$$Pr[T_\tau > W] \le \langle T_\tau \rangle / W \to 0 \quad \text{at} \quad W \to +\infty.$$

Next, the inequality constraint (24) for the expected completion time $\langle T_{\mathcal{R}} \rangle$ remains valid when random process exhibits non-zero probability of never ending, if h is understood as the harmonic mean completion time of the halting trials. Moreover, this estimate can be improved as: $\langle T_{\mathcal{R}}\rangle \geq (1/4)(h/q)$ [52] . Similarly, the lower bound (25) is applicable to the processes with non-zero statistical weight of never-stopping evolution scenarios if M is thought of as the most frequent completion time of halting trials [52].

Work [52] also provides generalization of the upper estimate for mean completion time optimized via periodic restart, see Eq. (5), accounting non-zero probability of non-halting: $\langle T_{\tau_*}\rangle \leq 2m/q$ with m denoting the median completion time of terminating trials.

12. ACCOUNTING FOR NON-ZERO TIME COST OF RESTART

In any practical scenarios restarts are non-instantaneous. Say, in computer science applications there is a time cost relating to reinitializing variables and other possible memory operations accompanying restart of randomized algorithm [66].

Assume that start of the process and each restart are accompanied by mutually independent and identically distributed random time costs $\Delta t_0, \Delta t_1, \Delta t_2, ...$ which do not correlate with random process dynamics and inter-restarts intervals (if they are stochastic). How does this affect the results discussed above? Corresponding generalization of Eq. (4) can be found in [47]. Generalizations of the expressions (5) (and 6), (8) and (9) for the expected completion time under specific restart strategies in the presence of non-zero time overheads associated with restart can be found, respectively, in supplementary material of work [52] (and [62]) (for periodic restart), in work [54] (for Poisson restart), and in work [62] (for Gamma restart). The paper [57] formulates a sufficient condition for the efficiency of Poisson restart in the presence of time costs which generalizes criterion (11). The constructive criteria of restart efficiency at non-zero time costs of restart are presented in Ref. [62] for periodic, Poisson and Gamma protocols. The work [52] explains how account of non-zero duration of restarts changes the upper bound (20) on the average completion time optimized via periodic restarts.

Reference [5] includes results for the arbitrary order statistical moments of completion time of generic random process subject to periodic protocol with non-instantaneous restarts of fixed duration.

The work [70] contains expression for deadline meeting probability of generic random process in the presence of finite number of non-instantaneous restarts.

Interestingly, some of the conclusions made in the framework of zero-duration restarts are completely robust to the modification of the original problem incorporating time costs. More specifically, the analysis presented in [52] suggests that periodic protocol remains the optimal strategy for optimizing mean completion time in the presence of generally distributed restart-associated time costs. In particular case of exponentially distributed time delays, the relation (43) for the relative fluctuation at optimal rate of Poisson restart minimizing expected completion time remains true without any corrections as shown in [54].

Next, the expressions for the success probability in Bernoulli-like experiment given by Eqs. (66), (71) and (76) are valid for arbitrary distributed time costs for restarting as long as the costs are uncorrelated with process dynamics and restart times, see [9]. What is more, the sufficient condition of restart efficiency (72) and relation (74) for the conditional statistical moments of completion time under optimal restart rate in Bernoulli-like experiment also hold true at arbitrary time delays statistics (see supplementary material to [9]). Finally, conclusion of global optimality of periodic restart for the purpose of success probability optimization is robust to introduction of time penalties (also see [9]).

Thus, in some situations (rare given the entire spectrum of possible pairs stochastic process + restart protocol), non-zero time costs do not change the final conclusions. However, in general they lead to corrections to the answers obtained in the instantaneous restart limit, which should be taken into account by appropriately modifying the renewal equations. It should be stressed, that all generalizations discussed in this section correspond to the situation when time penalty for restarting is uncoupled from the process dynamics and durations of inter-restarts periods.

13. DISCUSSION

Table 1 lists key literature sources that provide an idea of the application of the renewal approach to the problem of random process optimization via restart. Broadly, one can see that control-oriented research in this area consider four different types of optimization goals: how to speed up process completion, how to reduce variability in process duration, how to increase the chances that the process will end is desirable way, and how to improve some time scale associated with getting desired process outcome. The main focus of

Table 1. Types of restart strategies: deterministic protocols with homogeneous in time structure, deterministic protocols with non-homogeneous in time structure, stochastic protocols with time-independent statistical properties, stochastic protocols with time-dependent statistical properties. Periodic restart with optimally tuned period (whose value is problem-dependent) outperforms all other restart strategies in the context of expected completion time optimization

	Deterministic	**Stochastic**
Uniform in time	Examples: Periodic protocol [1, 47, 51, 52]	Examples: Poisson protocol [3, 54] Gamma protocol [62, 72] Restart at power-law times [59, 60]
Non-uniform in time	Examples: Luby protocol [1, 51] Geometric protocol [6, 66, 73]	Examples: Poisson restart with time varying rate [6, 55]

research efforts through more than three decades since first published study on restart mediated control were concentrated on the mean completion time optimization. The second most thoroughly studied issue is the restart-induced optimization of splitting probabilities for random processes with competitive outcomes.

Some of the results considered here are particularly noteworthy due to their exceptionally broad applicability. For example, the inequality constraint (24) for average process performance under restart quantified by the mean completion time is relevant to any process under any restart protocol, provided the assumptions (i), (ii) and (iii) presented in Section 1 are justified. Also, it is worth mentioning such a phenomenon as the universal response of the process completion time mode to a Poisson restart: the introduction of a Poisson restart or an increase in its rate in situation where it is already present always leads to a leftward shift of the leftmost peak position of the completion time probability density, regardless of the nature and fine details of the process under consideration if the assumptions (i)–(iii) are justified, see Section 11.2. (in contrast, such time scales as mean completion time and its median vale may decrease or increase in response to the introduction of restart depending on whether completion time fluctuations are sufficiently pronounced or not.) Strictly periodic restart is universally optimal strategy for the mean completion time optimization under assumptions (i), (ii) and (iv). Another exceptionally general rigorous result: optimal Poisson restart makes relative fluctuation in random completion time equal to unity, Eq. (43), whenever presuppositions (i), (ii) are valid and time cost of restart are negligible (assumption (iii)) or represent random variables independently sampled from the same exponential distribution.

Relation (74) is universally valid for any Bernoulli-like random process under optimally tuned Poisson restart protocol with any time costs for restart, if the assumptions (i) and (ii) are reasonable and if there are no correlations between restart costs and durations of inter restart intervals and free process dynamics (assumption (iv)). The dominance of periodic protocol over all possible time-homogeneous stochastic restart strategies in splitting probabilities optimization problem is proved under the same set of assumptions: (i), (ii) and (iv).

Let us also mention a remarkable property of gamma restart with shape parameter $k = 2$: the mean completion time of any random process whose performance was maximized by Poisson restart with optimally tuned rate r_* (whose particular value is problem-specific, of course) can potentially be further reduced via application of $k = 2$ gamma restart with sufficiently low rate parameter β. This conclusion follows from Eqs. (15) and (46) but has not been emphasized previously. What is more, similar observation takes place for splitting probabilities optimization problem, see Eqs. (75) and (77): chances for getting desirable outcome in Bernoulli-like random experiment which is already subject to optimally tuned Poisson restart can potentially be even further increased by additional application of gamma restart strategy with shape parameter $k = 2$ and sufficiently small rate β. In those special cases when inequalities in Eqs. (46) and (75) turn into identities at optimal Poisson restart conditions, imposing rare gamma distributed restarts with $k = 2$ leaves the process performance unchanged. Clearly, for any process which is subject to optimally tuned Poisson restart

Table 2. Statistical characteristics of stochastic processes studied in the literature in the context of restart induced optimization

Metric of interest	**References**
Mean completion time, $\langle T \rangle$	[1], [54], [47], [56], [53], [51], [57], [52], [62]
Median completion time, m	[6]
Mode of completion time distribution, M	[6]
High-order statistical moments, $\langle T^n \rangle$	[69], [48]
Tail behaviour of completion time distribution	[2], [53]
Deadline meeting probability, p_D	[69], [70], [6]
Shannon entropy of random completion time, S	[7]
Diversity of random completion time, $\mathcal{D}$	[8]
Success probability, p_s	[9], [62]
Conditional mean completion time, $\langle T^s \rangle$	[10]

mechanism aimed at optimization of mean completion time of splitting probabilities there exists a periodic restart strategy further improving performance or leaving it unchanged. Gamma restarting, thus, shares this property with the periodic strategy. However, it should be noted that even when useful, the gamma protocol cannot compete with the periodic in terms of maximum achievable effectiveness.

Finally, let us highlight the simplest constructive criteria for the effectiveness of restart method known at the moment. The criterion defined by Eqs. (33) and (34) allows to find a restart period guaranteed to reduce mean completion time based on the knowledge of just two time scales of restart-free completion — median and mean values. Criterion determined by (72) for success probability optimization also requires only two time scales and, moreover, remains valid in the presence of generally distributed mutually independent time costs associated with restart which do not correlate with process dynamics and inter-restart intervals.

14. CONCLUSION

To sum up, the use of renewal approach to restart problem (rooted in probabilistic arguments of early computer science work [1] and explicitly formulated in [47]) provided a simple way to analytically calculate the average completion time of an arbitrary stochastic process under an arbitrary restart protocol (Section 3); helped to establish remarkably simple sufficient conditions for when restart is beneficial (Section 4) and to formulate simple recipes how to design restart schedule in the absence of detailed knowledge of the process statistics (Section 8); allowed to pose and resolve broad optimization issues (Section 5); allowed to find upper bound for the resulting expected completion time (Section 6); provided insights into fundamental efficiency limits of any restart protocol (Section 7); helped to reveal somewhat unexpected universality in statistics of optimally restarted processes (Section 9); helped to reveal properties of some scale free restart protocols (Section 10); helped to analyse effect of restart on other time scales of random process completion beyond the mean value $\langle T \rangle$ (Section 11 and Table 2; allowed to investigate effect of non-zero time cost associated with restart (Section 12).

Such a wide variety of theoretical results was achieved through the use of a unified analytical technique to describing the restart-induced effects and the use of descriptive statistics language in formulating theses. In conclusion let us discuss examples of stochastic processes models with restart in which the key assumptions (i) and (ii) (see Section 1) underlying derivation of the conclusions reviewed here are violated.

The assumption (i) can fail when only part of relevant degrees of freedom, but not a system as a whole, resets to its initial state (precise or belonging to chosen statistical ensemble) upon restart event. This is the case in the models of random walks under Poisson position resetting with gated searcher-target interaction whose stochastic dynamics having long time correlations is unaffected by resetting [74]. Another example is given by random walk with time varying diffusion coefficient [75]. Besides, the process runs between successive restarts can be not independent and not identically distributed due to slowly decaying or persistent imprints of the process path on the structure or dynamics of environment [76].

This review paper considers exclusively static restart protocols: restarts occur according to prescribed deterministic schedule or at random time moments uncorrelated to process dynamics. There is also a type of protocols that can be called dynamic restarts [66]. In the simplest scenario of this sort successive process trials are still mutually statistically independent and identically distributed (i. e., assumption (i) is satisfied), but restart is coupled to the process dynamics (assumption (ii) is not met): restart mechanism exploits information on the current process state (or trajectory) in configuration space to make probabilistic or deterministic decision when to restart based on real-time observations. As an example, in-fly monitoring of randomized algorithm features has been proposed to build runtime predictors guiding process behaviour dependent restart decisions in computer science (see Ref. [66] and references therein). In the context of random walk models, examples of dynamic restart strategies is Poisson resetting with position dependent rate [65, 77–79] and threshold passing driven resetting [80].

Finally let us take a closer look at scenarios where assumption (iii), allowing treat restart as a process with zero time cost, fails. As was discussed in Section 12, existing literature provides examples of studies generalizing some of the presented here results to the case of non-zero time costs for restarting by replacing (iii) with (iv). However, there are situations in which admission (iv) does not reflect the actual properties of time costs. Namely, the time penalty for restart may depend on the state of the process immediately before the beginning of the next restart event, see, e. g., description of resetting phase of motion in reported experimental realizations of random walk processes with force induced resetting [49]. The analysis of the effect of restart on completion time statistics in such situations can be conducted with use of renewal approach like ideas, see Refs. [49, 81], but extracting of quantitative predictions inevitably demands also modelling of process kinetics in its configuration space.

Acknowledgments. The author thanks D. Starkov and I. Nikitin for discussions.

Funding. The work was supported by the Ministry of Science and Higher Education of the Russian Federation (Project No. FFWR-2024-0017).

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