APPLICATION OF QUANTUM FIELD THEORY METHODS TO THE MANY BODY PROBLEM

V. M. GALITSKII and A. B. MIGDAL

Moscow Engineering-Physics Institute

Submitted to JETP editor July 12, 1957; resubmitted October 24, 1957.

J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 139-150 (January, 1958).

It is shown that the energy and damping of quasiparticles are determined by the poles of a single particle propagation function. The relation between the two-particle Green's function and the kinetic equation is established.

INTRODUCTION

IN many cases, weakly-excited states of a system of interacting particles can be described approximately as an aggregate of elementary excitations — quasiparticles. In such a treatment the excited state of the system is described by fewer parameters than are needed for an exact description. Thus the elementary excitation is not a stationary state, but is rather a packet of stationary states with a narrow energy spread. The washing out of the packet leads to damping of the excitation. A description of the states of a system in terms of elementary excitations is possible if the energy spread of the packet, which determines its damping, is small compared to the excitation energy.

We shall consider the case of a homogeneous unbounded system. In such a system, the momentum operator commutes with the Hamiltonian, so that the excited states are characterised by the value of the momentum of the system, in addition to the other parameters.

Apparently, in all Fermi systems there are excitations analogous to the excitations in an ideal Fermi gas. The energy of such an excitation is $E(p_1, p_2) = \epsilon(p_1) - \epsilon(p_2)$, where p_1, $\epsilon(p_1)$, and p_2, $\epsilon(p_2)$ are the momenta and energies of the particle and hole which constitute the excitation. Here $p_1 > p_0 > p_2$, and p_0 is the limiting Fermi momentum for the quasiparticles.

A quasiparticle with momentum p, near to p_0, can reduce its energy, transferring another quasiparticle from the Fermi sphere to a state with $p' > p_0$. From the limitations imposed by the Pauli principle and the laws of conservation of energy and momentum, it follows that the probability for such a process, which determines the damping of the quasiparticles, is proportional to $(p - p_0)^2$. Thus the description of excited states of a Fermi system by means of quasiparticles is the more exact the closer the momentum of the quasiparticles to p_0.

The properties of the excitations are conveniently studied by the methods of quantum field theory, by introducing the Green's function of the system. Then the single particle Green's function determines the energy and the damping of the quasiparticles. However there may be excitations in the system whose energy is not describable as a sum of energies of quasiparticles. The energy spectrum of such excitations can be found from the two-particle Green's function. The two-particle Green's function, as we shall show later, also enables us to determine the behavior of the system in a weak external field.

In addition to the Green's function of the particles, we can also introduce the propagation function for the interaction between the particles. For example, for the problem of electrons in a metal interacting with the lattice, this propagation function is the Green's function of the phonon. The phonon Green's function determines the energy and damping of excitations of the lattice.

SINGLE PARTICLE GREEN'S FUNCTION AND ENERGY SPECTRUM

1. The single particle Green's function is defined, as usual, by

$$G(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) = i\langle T\{e^{iHt_1}\psi(\mathbf{r}_1)e^{-iH(t_1-t_2)}\psi^+(\mathbf{r}_2)e^{-iHt_2}\}\rangle, \quad (1)$$

where $\psi(\mathbf{r}) = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{pr}}$; the average is taken over the ground state function of the Hamiltonian system H.

The Green's function of the field φ which provides the interaction between the particles is defined similarly:

$$D(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) = i\langle P\{e^{iHt_1}\varphi(\mathbf{r}_1)e^{iH(t_1-t_2)}\varphi(\mathbf{r}_2)e^{-iHt_2}\}\rangle. \quad (2)$$

In the case of the phonon field,

$$\varphi(\mathbf{r}) = \sum_{\mathbf{q}} \alpha_{\mathbf{q}} (b_{\mathbf{q}} + b^{+}_{-\mathbf{q}}) e^{i\mathbf{q}\mathbf{r}},$$

where $b_{\mathbf{q}}$ and $b^{+}_{\mathbf{q}}$ are the phonon annihilation and creation operators.

In the absence of external fields, the functions G and D depend only on $r = |\mathbf{r}_1 - \mathbf{r}_2|$ and $\tau = t_1 - t_2$. Expanding the functions $G(r, \tau)$ and $D(r, \tau)$ in Fourier integrals, we get

$$\begin{aligned} G(r,\tau) &= \int \frac{dp\,d\varepsilon}{(2\pi)^4} G(p, \varepsilon) e^{i(\mathbf{p}\mathbf{r} - \varepsilon\tau)}, \\ D(r,\tau) &= \int \frac{dq\,d\omega}{(2\pi)^4} D(q, \omega) e^{i(\mathbf{q}\mathbf{r} - \omega\tau)}, \end{aligned} \tag{3}$$

where $G(\mathbf{p}, \epsilon)$ and $D(\mathbf{q}, \omega)$ are the Green's functions in momentum representation.

In the absence of interactions between the particles, we easily find from (1), for Fermi systems,*

$$G_0(r,\tau) = i \int \frac{dp}{(2\pi)^3} e^{i(\mathbf{p}\mathbf{r} - \varepsilon^0_p \tau)} \begin{cases} 1 - n_p, & \tau > 0 \\ -n_p, & \tau < 0, \end{cases}$$

where $n_{\mathbf{p}} = a^{+}_{\mathbf{p}} a_{\mathbf{p}}$. Going over to momentum representation by means of formula (3), we get

$$\begin{aligned} G_0(p,\varepsilon) &= 1/(\varepsilon^0_p - \varepsilon - i\Delta), \\ \Delta &\to \begin{cases} +0 & p > p_0 \\ -0 & p < p_0. \end{cases} \end{aligned} \tag{4}$$

Similarly, we find from (3),

$$D_0(q,\omega) = \alpha_q^2 \left\{ \frac{1}{\omega^0_q - \omega - i\delta} + \frac{1}{\omega^0_q + \omega - i\delta} \right\}, \quad \delta \to +0. \tag{5}$$

2. We now go on to the Fourier transforms with respect to $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. From (1), we find for

$$G(p,\tau) = \frac{1}{2\pi} \int d\varepsilon G(d, \varepsilon) e^{-i\varepsilon\tau}$$

the expression

$$G(p,\tau) = i \begin{cases} \langle a_p e^{-iH\tau} a^{+}_p \rangle e^{iE_0\tau} & \tau > 0 \\ -\langle a^{+}_p e^{iH\tau} a_p \rangle e^{-iE_0\tau} & \tau < 0. \end{cases} \tag{6}$$

If we express the operators which appear in $G(\mathbf{p}, \tau)$ in the energy representation, we have

$$G(p,\tau) = \begin{cases} i \sum\limits_s |(a^{+}_p)_{s0}|^2 \exp\{-i(E_s - E_0)\tau\} & \tau > 0, \\ -i \sum\limits_s |(a_p)_{s0}|^2 \exp\{i(E_s - E_0)\tau\} & \tau < 0. \end{cases} \tag{7}$$

Since the operator $a^{+}_{\mathbf{p}}$ increases the momentum of the system by the amount $\mathbf{p}$, and the number of particles in the system by unity, the summation for $\tau > 0$ extends over all states with momentum $\mathbf{p}$ and particle number $N + 1$, if the number of particles in the ground state was N and the momentum was equal to zero. Similarly, the summation for $\tau < 0$ is taken over states with particle number $N - 1$ and momentum $-\mathbf{p}$. We use the notation

$$\begin{aligned} &E_s(N+1) - E_0(N) \\ &= \varepsilon_s(N+1) + E_0(N+1) - E_0(N) = \varepsilon_s + \mu, \end{aligned}$$

where $\mu = E_S(N+1) - E_0(N)$ is the chemical potential. The excitation energy $\epsilon_S = E_S(N+1) - E_0(N+1)$ is, by definition, positive. Similarly

$$\begin{aligned} &E_s(N-1) - E_0(N) \\ &= \varepsilon_s(N-1) - E_0(N) + E_0(N-1) = \varepsilon'_s - \mu'. \end{aligned}$$

The quantities ϵ'_S and μ' are identical with ϵ_S and μ to terms of order $1/N$. We introduce the functions

$$\begin{aligned} A(p,E)\,dE &= \sum_s |(a^{+}_p)_{s0}|^2, \quad E < \varepsilon_s < E + dE, \\ B(p,E)\,dE &= \sum_s |(a_p)_{s0}|^2, \quad E < \varepsilon_s < E + dE \end{aligned} \tag{8}$$

and carry out a Fourier transformation with respect to τ in (7). We get

$$G(p,\varepsilon) = \int_0^\infty dE \left\{ \frac{A(p,E)}{E - \varepsilon + \mu - i\delta} - \frac{B(p,E)}{E + \varepsilon - \mu - i\delta} \right\}. \tag{9}$$

Formula (9) is Lehmann's[1] expansion for the single particle Green's function of a system consisting of a finite number of fermions. Using it, we can obtain some relations between the real and imaginary parts of the function $G(\mathbf{p}, \epsilon)$. In fact, from the equality

$$\frac{1}{E - \varepsilon + \mu - i\delta} = \mathrm{P} \frac{1}{E - \varepsilon + \mu} + i\pi\delta(E - \varepsilon + \mu)$$

if follows that

$$\mathrm{Im}\, G(p,\varepsilon) = \pi \begin{cases} A(p, \varepsilon - \mu) & \varepsilon > \mu \\ -B(p, \mu - \varepsilon) & \varepsilon < \mu, \end{cases} \tag{10}$$

i.e., the imaginary part of the Green's function changes sign at the point $\epsilon = \mu$. Using (9) and (10), it is easy to obtain the formula* which gives the relation between the real and imaginary parts of G:

$$\mathrm{Re}\, G(p,\varepsilon) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{P} \frac{\mathrm{Im}\, G(p,\varepsilon')}{\varepsilon' - \varepsilon} d\varepsilon'. \tag{11}$$

*We shall give the formulas for Fermi systems. As is easily seen, most of the results also are valid for the case of Bose particles.

*This formula was obtained by L. D. Landau.

We remark that in the case of a system of bosons the expression for $G(p, \epsilon)$ differs from (9) only in a change in sign of $B(p, \epsilon)$. Thus for bosons, unlike (10), the imaginary part of $G(p, \epsilon)$ is positive for all p and ϵ. Formula (11) remains valid for bosons.

Similar formulas can be gotten for $D(q, \omega)$:

$$D(q,\tau)=\begin{cases} i\sum\limits_s |(\varphi_{-q})_{s0}|^2 \exp\{-i(E_s-E_0)\tau\} & \tau>0 \\ i\sum\limits_s |(\varphi_q)_{s0}|^2 \exp\{i(E_s-E_0)\tau\} & \tau<0. \end{cases} \quad (12)$$

Here we have made use of the reality of the field φ : $\varphi_q^+ = \varphi_q$. Unlike (7), the number of particles, N, in the states of the sum (12) is equal to the number of particles in the ground state. We use the notation

$$\Phi(q,\omega')\,d\omega' = \sum_s |(\varphi_{-q})_{s0}|^2 = \sum_s |(\varphi_q)_{s0}|^2, \quad \omega' < E_s - E_0 < \omega' + d\omega', \quad (13)$$

Then

$$D(q,\tau) = \int_0^\infty d\omega' \Phi(q,\omega')\, e^{-i\omega'|\tau|}. \quad (14)$$

Taking the Fourier transform of (14) with respect to τ, we get

$$D(q,\omega) = \int_0^\infty d\omega' \Phi(q,\omega') \left\{\frac{1}{\omega'-\omega-i\delta} + \frac{1}{\omega'+\omega-i\delta}\right\}. \quad (15)$$

From (15) we have

$$\operatorname{Im} D(q,\omega) = \pi\Phi(q,|\omega|) > 0; \quad (16)$$

$$\operatorname{Re} D(q,\omega) = \frac{1}{\pi}\int_0^\infty d\omega' \operatorname{Im} D(q,\omega') \operatorname{P}\left\{\frac{1}{\omega'-\omega} + \frac{1}{\omega'+\omega}\right\}. \quad (17)$$

3. Let us examine the properties of the Green's function in the complex ϵ plane. Replacing E by $-E$ in the second term of (9), we get

$$G(p,\varepsilon) = \int_C \frac{F(p,E)}{E-\varepsilon+\mu}\, dE. \quad (18)$$

The integration contour C is shown in Fig. 1. The expression on the right side of (18) is an integral of the Cauchy type. Functions defined by such integrals are known[2] to be analytic throughout the plane except for the points on the contour of integration. In our case the integration contour C divides the plane of the complex variable $\epsilon - \mu$ into two regions, and the integral (18) defines two different functions: $f_I(\epsilon-\mu)$ which is analytic in region I, and $f_{II}(\epsilon-\mu)$ which is analytic in region II. The Green's function $G(p, \epsilon)$, which is defined by the values of the integral (12) on the real axis, coincides with f_{II} for $\epsilon < \mu$, and with f_I for $\epsilon > \mu$. Thus $G(p, \epsilon)$ is not an analytic function of ϵ, but has a singularity at $\epsilon = \mu$.

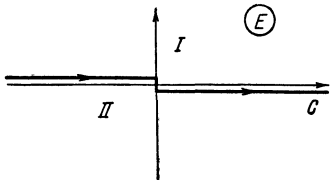

FIG. 1

Using the reality of the function $F(p, E)$, it is not difficult to show that the values of the integral (18) for points lying infinitely close to one another on opposite sides of the contour C are complex conjugates. Thus the function f_I, for negative values of $\epsilon - \mu$ lying above the contour C, is the complex conjugate of $G(p, \epsilon)$:

$$f_I(\varepsilon - \mu + 2i\delta) = G^*(p,\varepsilon), \quad \varepsilon < \mu. \quad (19)$$

Thus $G(p, \epsilon)$ for $\epsilon > \mu$, when continued analytically into the upper half-plane, coincides for $\epsilon < \mu$ with $G^*(p, \epsilon)$, or in other words, $G(p, \epsilon)$ for $\epsilon > \mu$ and $G^*(p, \epsilon)$ for $\epsilon < \mu$ comprise the analytic function f_I. Similarly, $G(p, \epsilon)$ for $\epsilon < \mu$, when continued analytically into the lower half-plane, coincides for $\epsilon > \mu$ with $G^*(p, \epsilon)$.

4. We shall now establish the connection of the single-particle Green's function with the spectrum of excitations. The Green's function $G(p, \tau)$ has a simple physical meaning. Suppose that initially the system is in the state $\Psi(0) = a_p^+\Phi_0$, where Φ_0 is the ground state of the system of N particles (the physical "vacuum"). At the time $\tau > 0$, the wave function of the system is

$$\Psi(\tau) = e^{iH\tau} a_p^+ \Phi_0.$$

The function $G(p, \tau)$ is the probability for finding the system in state $\Psi(0)$ at time τ.

In fact,

$$(\Psi(0), \Psi(\tau)) = (\Phi_0 a_p e^{-iH\tau} a_p^+ \Phi_0) = -iG(p,\tau). \quad (20)$$

A similar relation holds for $\tau < 0$. According to (7) and (8), for $\tau > 0$

$$(\Psi(0), \Psi(\tau)) = e^{-i\mu\tau}\int_0^\infty A(p,E)\, e^{-iE\tau} dE. \quad (21)$$

In the presence of interaction, for values of p greater than p_0,

$$A(p,E) = \delta(E+\mu-\varepsilon_p^0) \text{ and } (\Psi(0), \Psi(\tau)) = e^{-i\varepsilon_p^0\tau}.$$

When we switch on the interaction between particles, the δ function in $A(p, E)$ is replaced by a function having a sharp maximum near $E = \epsilon_p - \mu$, where ϵ_p is the energy of the quasiparticles.

Let us look at the behavior of the Green's function for large positive times. Suppose that the singularity of the analytic continuation of $A(p, E)$ into the lower half-plane, which is closest to the real axis is a simple pole at $E = \epsilon_p - \mu - i\Gamma$. Then, by shifting the contour of integration in (21) into the lower half-plane, we get

$$G(p, \tau) = ie^{-i\mu\tau} \int_C Ae^{-iE\tau} dE. \qquad (21')$$

The integration contour C is shown in Fig. 2. The non-exponential term in the function $G(p, \tau)$,

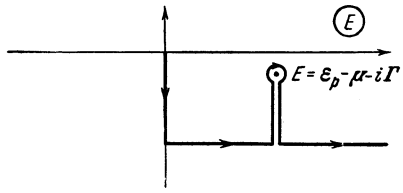

FIG. 2

which arises from the integration along the imaginary axis near $E = 0$, is of order $(\Gamma/\epsilon_p)^2$ for $\tau \gg 1/\Gamma$. Thus

$$G(p, \tau) = c_p e^{-i\varepsilon_p\tau - \Gamma\tau} + O[(\Gamma/\varepsilon_p)^2]. \qquad (22)$$

This result can be interpreted in the following way: the state $\Psi(0)$ contains, with amplitude c_p, a packet describing a quasiparticle with energy ϵ_p and damping Γ. The values of ϵ_p and Γ are determined by the position of the pole of $A(p, E)$, i.e., by the imaginary part of $G(p, \epsilon)$ in the lower half-plane. The poles of Im G coincide with the poles of G or G*, but the analytic continuation of the latter is $f_{II}(\epsilon - \mu)$, which is analytic in the lower half-plane. Thus the energy and damping of the excitations are determined by the real and imaginary parts of the poles of the analytic continuation of $G(p, \epsilon)$ for $\epsilon > \mu$ in the lower half-plane. Similarly, the energy and damping of holes in the Fermi distribution are given by the poles of the analytic continuation of $G(p, \epsilon)$ for $\epsilon < \mu$ into the upper half-plane.

Let us introduce the irreducible part of the proper energy of the particles, $\Sigma(p, \epsilon)$:

$$G^{-1}(p, \varepsilon) = \varepsilon_p^0 - \varepsilon - \Sigma(p, \varepsilon)$$
$$= \varepsilon_p^0 - \varepsilon - \Sigma_0(p, \varepsilon) - i\Sigma_1(p, \varepsilon), \qquad (23)$$

where Σ_0 and Σ_1 are the real and imaginary parts of Σ. The energy and damping of the quasiparticle are determined from the equation

$$\varepsilon_p^0 - (\varepsilon_p - i\Gamma) - \tilde{\Sigma}(p, \varepsilon_p - i\Gamma) = 0, \qquad (24)$$

where $\tilde{\Sigma}(p, \epsilon)$ is the analytic continuation of $\Sigma(p, \epsilon)$ for $\epsilon > \mu$. For $\Gamma/\epsilon_p \ll 1$, we have approximately:

$$\varepsilon_p^0 - \varepsilon_p - \Sigma_0(p, \varepsilon_p) = 0,$$
$$\Gamma_p = \Sigma_1(p, \varepsilon_p) \Big/ \left[1 + \left(\frac{\partial\Sigma_0}{\partial\varepsilon}\right)_{\varepsilon=\varepsilon_p}\right]. \qquad (24')$$

Analogous results hold for $D(q, \omega)$. Let $\Pi(q, \omega)$ be the irreducible part of the proper energy of the phonon

$$D(q, \omega) = \alpha_q^2 / [\omega_q^{02} - \omega^2 - 2\omega_q^0\alpha_q^2\Pi(q, \omega)]$$
$$= \alpha_q^2 / [\omega_q^{02} - \omega^2 - 2\omega_q^0\alpha_q^2\Pi_0(q, \omega) - 2i\omega_q^0\alpha_q^2\Pi_1(q, \omega)].$$

As above, the energy and damping of the phonon excitation are given by the poles of the analytic continuation of $D(q, \omega)$ for $\omega > 0$:

$$\omega_q^{02} - (\omega_q - i\gamma)^2 - 2\omega_q^0\alpha_q^2\tilde{\Pi}(q, \omega_q - i\gamma) = 0, \qquad (25)$$

where $\tilde{\Pi}(q, \omega)$ is the analytic continuation of $\Pi(q, \omega)$ for $\omega > 0$. Approximately, for $\omega_q \gg \gamma$:

$$\omega_q^2 = \omega_q^{02} - 2\omega_q^0\alpha_q^2\Pi_0(q, \omega_q),$$
$$\gamma_q = \alpha_q^2 \frac{\omega_q^0}{\omega_q} \Pi_1(q, \omega_q) \Big/ \left[1 + 2\omega_q^0\alpha_q^2 \left(\frac{\partial\Pi_0}{\partial\omega}\right)_{\omega=\omega_q}\right]. \qquad (25')$$

We determine the momentum p_0 from the condition $\mu = \epsilon_p$, where μ is the chemical potential of the system. Then, assuming that Im G is continuous at $\epsilon = \mu$, we find from (10) and (24') that $\Gamma(p_0) = 0$. Thus the damping of the excitations goes to zero at the point p_0, which is determined by the equation $\epsilon_{p_0} = \mu$.

5. The single-particle Green's function also enables us to find other characteristics of the system. Thus the momentum distribution of the particles is related to the Green's function by[3]

$$n_p = i \int_C G(p, \varepsilon) \frac{d\varepsilon}{(2\pi)}, \qquad (26)$$

where the contour C consists of the real axis and a semicircle of infinite radius in the upper half-plane.

As was shown in Ref. 3, when we pass through the point $p = p_0$, the pole of $G(p, \epsilon)$ which lies nearest to the real axis moves into the lower half of the ϵ plane, and is thus outside of the contour C of formula (26). Thus the jump in n_p at $p = p_0$ remains even in the presence of an arbitrary interaction.

Let us express the energy of the ground state of the system in terms of the function G. Differentiating $G(p, \tau)$ with respect to the time t, it is not difficult to obtain the formula

$$\left(i\frac{\partial}{\partial t} - \varepsilon_p^0\right) G(p, \tau) = -\delta(\tau) - i\langle T\{[H'(t), a_p(t)]\, a_p^+(t')\}\rangle, \qquad (27)$$

where $H'(t)$ is the interaction Hamiltonian between particles, and $[H', a_p]$ is the commutator of the operators H' and a_p. Comparing (27) in the (p, ϵ) representation with the expression relating G and G_0, we find the general form of the product $\Sigma(p, \epsilon) G(p, \epsilon)$:

$$\Sigma(p, \varepsilon) G(p, \varepsilon) = i \int d\tau e^{i\varepsilon\tau} \langle T \{[H'(t), a_p(t)] a_p^+(t')\}\rangle. \tag{28}$$

By integrating (28) with the factor $e^{i\epsilon\Delta}$ and going to the limit $\Delta \to +0$, we get the formula

$$\lim_{\Delta \to +0} \int \frac{d\varepsilon}{2\pi} e^{i\varepsilon\Delta} \Sigma(p, \varepsilon) G(p, \varepsilon)$$

$$\equiv \int_C \frac{d\varepsilon}{2\pi} \Sigma(p, \varepsilon) G(p, \varepsilon) = -i \langle a_p^+ [H', a_p] \rangle,$$

or,

$$i \int_C \frac{d^4p}{(2\pi)^4} \Sigma(p, \varepsilon) G(p, \varepsilon) = \int \langle a_p^+ [H', a_p] \rangle \frac{dp}{(2\pi)^3}, \tag{29}$$

where the contour C coincides with the integration contour in (26).

In the case of pair interaction between particles, the right side of (29) reduces to the average value of the interaction Hamiltonian

$$\langle H' \rangle = -\frac{i}{2} \int_C \frac{d^4p}{(2\pi)^4} \Sigma(p, \varepsilon) G(p, \varepsilon). \tag{30}$$

Adding the average value of the Hamiltonian of the non-interacting particles to (30), we get finally

$$E_0 = \frac{i}{(2\pi)^4} \int_C \left\{\varepsilon_p^0 - \frac{1}{2} \Sigma(p, \varepsilon)\right\} G(p, \varepsilon) d^4p. \tag{31}$$

Differentiation of E_0 with respect to the number of particles N gives the chemical potential, μ, of the system.

TWO-PARTICLE GREEN'S FUNCTION. KINETIC EQUATION

To study the energy spectrum and the behavior of the system in weak external fields, we must consider the two-particle Green's function. The two-particle Green's function $K(1, 2; 3, 4)$ is defined by

$$K(1,2; 3,4) = i \langle T \{\psi(1) \psi^+(2) \psi(3) \psi^+(4)\}\rangle, \tag{32}$$

where $1, 2; 3, 4$ stand for the sets of coordinates of the space-time points. If $t_1, t_2 > t_3, t_4$, K can be written in the form

$$K(1, 2; 3,4) = i \sum_s \chi_s(1,2) \tilde{\chi}_s(3,4), \tag{33}$$

where

$$\chi_s(1,2) = (T \{\psi(1) \psi^+(2)\})_{0s},$$
$$\chi_s(3,4) = (\tilde{T} \{\psi(4) \psi^+(3)\})_{0s}^*; \tag{34}$$

$\tilde{T}$ orders the operators in the reverse order from T. The functions $\chi_S(1, 2)$ for simultaneous times $t_1 = t_2$ have the physical meaning of wave functions describing the behavior of a particle and a hole in the state s. In the absence of external fields, the dependence on the coordinates of the "center of gravity" $X = (x_1 + x_2)/2$ can be separated off from χ:

$$\chi_s(x_1, x_2) = e^{ikX} f_{\mathbf{k},\omega}(x), \quad kX = \mathbf{kR} - \omega T,$$
$$x = x_1 - x_2, \quad \mathbf{k} = \mathbf{p}_s - \mathbf{p}_0, \quad \omega = E_s - E_0, \tag{35}$$

where $\mathbf{k}$ and ω are the momentum and energy of the excitation.

As will be shown later, the function $f_{\mathbf{k},\omega}(x)$ for $t_1 = t_2$ in momentum representation, i.e. $f_{\mathbf{k},\omega}(\mathbf{p})$, is the Fourier component of the distribution function $f(\mathbf{r}, \mathbf{p}, t)$. As a matter of fact, the density matrix, normalized to the total number of particles:

$$P(\mathbf{r}, \mathbf{r}', t)$$
$$= \sum_{i=1}^{N} \int \varphi^*(\mathbf{r}_1, \ldots, \mathbf{r}', \ldots, \mathbf{r}_N; t) \varphi(\mathbf{r}_1, \ldots, \mathbf{r}, \ldots, \mathbf{r}_N; t) \prod_{k \neq i} dV_k \tag{36}$$

(where $\varphi(\mathbf{r}_1, \ldots \mathbf{r}_N; t)$ is the wave function of the system in configuration space), can be written as the average value of an integral operator with the kernel

$$(\hat{P}(\mathbf{r}, \mathbf{r}', t))_{\mathbf{r}_i', \mathbf{r}_i} = \sum_i \delta(\mathbf{r}' - \mathbf{r}_i') \delta(\mathbf{r} - \mathbf{r}_i), \tag{37}$$

which we may call the density matrix operator. In the occupation number representation, this operator has the form

$$\hat{P}(\mathbf{r}, \mathbf{r}', t)$$
$$= \int \psi^+(\mathbf{r}_1') \delta(\mathbf{r}' - \mathbf{r}_1') \delta(\mathbf{r} - \mathbf{r}_1) \psi(\mathbf{r}_1) dV_1 dV_1' = \psi^+(\mathbf{r}') \psi(\mathbf{r}). \tag{37'}$$

$\chi_S(1, 2)$ for $t_1 = t_2$ is, to within a factor which is independent of points 1 and 2, the part of the density matrix which oscillates with frequency $\omega = E_S - E_0$. The Fourier component of the density matrix with respect to the coordinate $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2$ is related, as we know, to the distribution function (the density matrix in mixed representation). Therefore the function f, to within a normaliza-

tion factor, coincides with the Fourier component of the distribution function

$$f_{k,\omega}(p) = c\int f(r, p, t)\, e^{-i(kr-\omega t)}\, dvdt. \qquad (38)$$

From (34) we see that the two-particle Green's function K is suitable for studying excited states of a system of N particles, in which there are particles and holes, while the single particle Green's function G enables us to investigate states of a system of N + 1 particles which differ from the ground state of the N particle system by the presence (or absence) of one quasiparticle. The essential feature of the states described by the two-particle Green's function K is the interaction between particle and hole. If this interaction leads only to scattering of the particle by the hole, then the energy of the excitation is equal to the energy of the particle and hole at infinity, $E = \epsilon(p_1) - \epsilon(p_2)$, $p = p_1 - p_2$. In this case, the two-particle Green's function gives no new information concerning the energy spectrum of the system, beyond that from the single-particle function. In some cases the interaction can lead to the presence of excited states which can be interpreted as bound states of a particle and a hole. Such excited states were studied in the papers of Klimontovich and Silin[4,5] and Landau,[6] and were called zeroth sound.

As was shown in the papers of Schwinger,[7] and Gell-Mann and Low,[8] the equation for the function K has the form

$$K(x_1x_2;\ x_3x_4) = iG(x_1 - x_4)\,G(x_3 - x_2) - iG(x_1 - x_2)\,G(x_3 - x_4)$$
$$+ i\int G(x_1 - x_5)\,G(x_6 - x_2)\,\Gamma(x_5x_6,\ x_7x_8)\,K(x_7x_8;\ x_3x_4)\,d^4x_5d^4x_6d^4x_7d^4x_8, \qquad (39)$$

Γ is a compact four-pole diagram, i.e., a set of graphs which start and end with a pair of solid lines, while the graphs cannot be split into parts which are joined only by a pair of solid lines. The free term of this equation describes the propagation of non-interacting particle and hole, and does not contain the frequencies corresponding to bound states. Therefore, extraction of the function χ_s, which describes bound states, leads, as was shown in Ref. 8, to the following homogeneous equation for χ:

$$\chi_s(x_1x_2) = i\int G(x_1 - x_5)\,G(x_6 - x_2)\,\Gamma(x_5x_6,\ x_7x_8)\,\chi_s(x_7x_8)\,d^4x_5d^4x_6d^4x_7d^4x_8. \qquad (40)$$

Substituting χ_s in the form (35) into this equation, we get an eigenvalue problem whose solution gives us the frequency of zeroth sound and the functions χ_s. In the following, we shall limit ourselves to the case of a system of particles interacting with one another via a weak non-retarded potential V. To first order in the strength of the interaction, the zeroth order Green's functions should be used as the Green's function, while the compact four-pole Γ is given by the pair of graphs shown in Fig. 3. (The dotted lines on these graphs refer to the propagation function of the interaction, $-iV_q$. For our further work, we must include the spin of the particles. In the most usual case, of spin $\frac{1}{2}$, we can construct four functions χ_s:

$$\chi_s(1,2;\sigma_1,\sigma_2) = (T\{\psi_{\sigma_1}(1), \psi^+_{\sigma_2}(2)\})_{0s}. \qquad (41)$$

It is not hard to see that the functions $\chi_s(1,2;\frac{1}{2},-\frac{1}{2})$ and $\chi_s(1,2;-\frac{1}{2},\frac{1}{2})$ correspond to excitations with spin 1 and projections +1 and −1, respectively. As the compact four-pole in the equation for these χ_s, we can use the Γ which is described by only the first of the graphs in Fig. 3; because the potential is independent of the spin

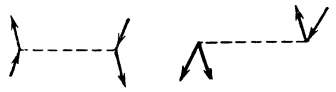

FIG. 3

variables, only a particle and hole with total spin equal to zero can participate in the second of the interactions. Thus the equation for these functions has the following form in momentum representation:

$$\chi_s(p_1, p_2;\sigma, -\sigma) = iG_0(p_1)\,G_0(p_2)\int V_q\chi_s(p_1 + q, p_2 + q;\sigma, -\sigma)\,\frac{dq}{(2\pi)^4}. \qquad (42)$$

In contrast to this case, the compact four-pole in the equation for the functions $\chi_s(1,2;\frac{1}{2},\frac{1}{2})$ and $\chi_s(1,2;-\frac{1}{2},-\frac{1}{2})$ is described by both of the

graphs in Fig. 3. The equation for these functions is

$$\chi_s(p_1, p_2; \sigma, \sigma) = iG_0(p_1)\,G_0(p_2)\left\{\int V_q \chi_s(p_1+q, p_2+q; \sigma, \sigma)\frac{dq}{(2\pi)^4} - V_{p_1-p_2}\sum_{\sigma'}\int \chi_s\left(p+\frac{p_1-p_2}{2}, p-\frac{p_1-p_2}{2}; \sigma', \sigma'\right)\frac{dp}{(2\pi)^4}\right\}. \quad (43)$$

We introduce functions $\chi_S^+(p_1, p_2)$ and $\chi_S^-(p_1, p_2)$:

$$\chi_s^+(p_1, p_2) = \frac{1}{\sqrt{2}}\left\{\chi_s\left(p_1, p_2; \frac{1}{2}, \frac{1}{2}\right) + \chi_s\left(p_1, p_2; -\frac{1}{2}, -\frac{1}{2}\right)\right\}, \quad (44)$$

$$\chi_s^-(p_1, p_2) = \frac{1}{\sqrt{2}}\left\{\chi_s\left(p_1, p_2; \frac{1}{2}, \frac{1}{2}\right) - \chi_s\left(p_1, p_2; -\frac{1}{2}, -\frac{1}{2}\right)\right\}. \quad (44')$$

The function $\chi_S^+(p_1, p_2)$ corresponds to excitation with spin zero, the function $\chi_S^-(p, p)$ to excitation with spin 1 and projection 0. The equations for these functions can be gotten by adding and subtracting Eq. (43) with spin values $\sigma = \frac{1}{2}$ and $\sigma = -\frac{1}{2}$. The equation for χ_S^- is the same as Eq. (42) for the function with spin 1 and projections ± 1. The equation for χ_S^+ is:

$$\chi_s^+(p_1, p_2) = iG_0(p_1)\,G_0(p_2)\left\{\int V_q \chi_s^+(p_1+q, p_2+q)\frac{dq}{(2\pi)^4} - 2V_{p_1-p_2}\int \chi_s^+\left(p+\frac{p_1-p_2}{2}, p-\frac{p_1-p_2}{2}\right)\frac{dp}{(2\pi)^4}\right\}. \quad (45)$$

Transforming to the "relative" momentum $k = p_1 - p_2$, which is equal to the momentum of the excited state, and the "total" momentum $p = (p_1 + p_2)/2$, we get the following equations for excitations with total spin zero and one, respectively:

$$\chi_{k\omega}^0(p) = iG_0\left(p+\frac{k}{2}\right)G_0\left(p-\frac{k}{2}\right) \times\left\{\int V_q \chi_{k\omega}^0(p+q)\frac{dq}{(2\pi)^4} - 2V_k\int \chi_{k\omega}^0(p)\frac{dp}{(2\pi)^4}\right\}, \quad (46)$$

$$\chi_{k\omega}^1(p) = iG_0\left(p+\frac{k}{2}\right)G_0\left(p-\frac{k}{2}\right)\int V_q \chi_{k\omega}^1(p+q)\frac{dq}{(2\pi)^4}. \quad (47)$$

The function $\chi_{k\omega}^0$ corresponds to excitation with total spin 0, the function $\chi_{k\omega}^1$ to excitation with total spin 1.

In the absence of retardation, the Fourier component of the potential does not depend on the fourth component of q: $V_q \equiv V(q)$, so that Eqs. (46) and (47) can be integrated with respect to ϵ (the fourth component of p). Integration of the function χ with respect to ϵ corresponds in coordinate representation to equating the times t_1 and t_2, so that as a result of the integration we get an equation for $f_{k\omega}(p)$, the Fourier components of the distribution function:

$$f_{k\omega}^0(p) = \frac{n_0(p+k/2) - n_0(p-k/2)}{\omega - kp - i\delta[n_0(p+k/2) - n_0(p-k/2)]}\left\{\int V(q) f_{k\omega}^0(p+q)\frac{dq}{(2\pi)^3} - 2V(k)\int f_{k\omega}^0(p)\frac{dp}{(2\pi)^3}\right\}, \quad (46')$$

$$f_{k\omega}^1(p) = \frac{n_0(p+k/2) - n_0(p-k/2)}{\omega - kp - i\delta[n_0(p+k/2) - n_0(p-k/2)]}\int V(q) f_{k\omega}^1(p+q)\frac{dq}{(2\pi)^3}, \quad (47')$$

where $n_0(p)$ are the occupation numbers for non-interacting particles.

Let us consider the case of short range forces ($ap_0 \ll 1$, where a is the range of the potential). For excitations with low momentum k, the function $f_{k\omega}(p)$ differs from zero over a narrow range of momenta near p_0. Because of this, the potential $V(q)$ can be taken out from under the integral sign and, like $V(k)$, replaced by $V(0)$. Writing the difference $n_0(p + k/2) - n_0(p - k/2)$ in the form $\frac{1}{2}k\partial f_0/\partial p$ ($f_0 = 2n_0(p)$ is the distribution function of the non-interacting particles in the ground state), we find

$$f_{k\omega}^0(p) = -\frac{1}{2}\frac{k\,\partial f_0/\partial p}{\omega - kp - i\delta[n_0(p+k/2) - n_0(p-k/2)]}V(0)\int f_{k\omega}^0(p)\,dp/(2\pi)^3, \quad (46'')$$

$$f_{k\omega}^1(p) = \frac{1}{2}\frac{k\,\partial f_0/\partial p}{\omega - kp - i\delta[n_0(p+k/2) - n_0(p-k/2)]}V(0)\int f_{k\omega}^1(p)\,dp/(2\pi)^3 \quad (47'')$$

Equation (46") coincides with the kinetic equation in the self-consistent field approximation, but with the number of particles reduced by a factor of two. This equation has a solution only for $V(0) > 0$, i.e., in the case of repulsion between the particles.

The formal solution of equation (47") for the case of attraction is not justified, because of the readjustment of the Fermi sphere caused by the formation of correlated pairs. Thus for a repulsive short-range potential, propagation of spinless zeroth sound is possible.

In the case of long-range repulsive forces, $V(k)$ has a pole for $k \to 0$, so that the second term in (46′) is much greater than the first, in which the integration makes the pole in V unimportant (we note that this result remains true in all approximations). Neglecting the first term, we write (46′) as

$$f^0_{k\omega} = \frac{-k\,\partial f_0/\partial p}{\omega - kp - i\delta\,[n_0(p+k/2) - n_0(p-k/2)]} \times V(k)\int f^0_{k\omega}(p)\,\frac{dp}{(2\pi)^3}. \tag{48}$$

This equation coincides with the kinetic equation for the k, ω Fourier components of the distribution function in the self-consistent field approximation.

Let us treat the behavior of the system in a weak electromagnetic field $A(r, t)$. The Hamiltonian for the interaction of the system with the field is

$$H' = \frac{1}{c}\int j^\alpha(r, t)\,A^\alpha(r, t)\,dv. \tag{49}$$

The summation over α extends from 1 to 4.

After the field is switched on at time t_0, the wave function of the system varies in time according to the law

$$\Phi(t) = T\left\{\exp\left(-i\int_{t_0}^{t} H'dt'\right)\right\}\Phi_0 = T\left\{\exp\left[-\frac{i}{c}\int_{t_0}^{t}\int j^\alpha(r, t')A^\alpha(r, t')\,dvdt'\right]\right\}\Phi_0, \tag{50}$$

or, in first approximation in powers of the external field,

$$\Phi(t) = \left\{1 - \frac{i}{c}\int_{t_0}^{t}\int j^\alpha(r, t')\,A^\alpha(r, t')\,dvdt'\right\}\Phi_0. \tag{50′}$$

In (50) and (50′) the current operator is taken in the Heisenberg representation for the unperturbed Hamiltonian of the system. The current of the system at time t is determined by the average value of the operator $j(r, t)$ over the function $\Phi(t)$. Making use of the fact that the current of the system is equal to zero in the unperturbed state Φ_0, we easily find

$$j^\beta(r, t) = -\frac{i}{c}\int_{t_0}^{t}\int dt'dv'\,\langle[j^\beta(r, t), j^\alpha(r', t')]\rangle\,A^\alpha(r', t'), \tag{51}$$

where $\langle\ \rangle$ denotes an average over the ground state of the system. For the k-Fourier components, the relation (51) takes the form:

$$j^\beta_k(t) = -\frac{i}{c}\int_{t}^{t} dt'\,\langle[j^\beta_k(t), j^\alpha_{-k}(t')]\rangle\,A^\alpha_k(t')$$

$$= -\frac{i}{c}\int_{t_0}^{t} dt'\iint dpdp'j^\beta(p)\,j^\alpha(p') \times \langle[a^+_{p-k/2}(t)\,a_{p+k/2}(t),\ a^+_{p'+k/2}(t')\,a_{p'-k/2}(t')]\rangle\,A^\alpha_k(t'), \tag{51′}$$

where $j^\alpha(p) = p^\alpha$ for $\alpha = 1, 2, 3$, and $j^4(p) = 1$. The average value of the commutator under the integral sign in (51) can be expressed in terms of the functions $f_{k(t)}(p)$

$$\langle[a^+_{p-k/2}(t)\,a_{p+k/2}(t),\ a^+_{p'+k/2}(t')\,a_{p'-k/2}(t')]\rangle = \sum_s\{e^{-i\omega_s(t-t')}f_{k,\omega_s}(p)\,f^*_{k,\omega_s}(p') - e^{i\omega_s(t-t')}f_{-k,\omega_s}(p')\,f^*_{-k,\omega_s}(p)\}. \tag{52}$$

Thus the knowledge of this system of functions is sufficient for determining the current of the system. On the other hand, this commutator can be expressed directly in terms of the two-particle Green's function K. Denoting by $\tilde{K}$ the two-particle Green's function in momentum representation for $t_1 = t_2 = t$ and $t_3 = t_4 = t'$ $(t - t' = \tau)$:

$$K\left(p + \frac{k}{2},\ t, p - \frac{k}{2}, t;\ p' - \frac{k}{2}, t', p' + \frac{k}{2},\ t'\right) \equiv \tilde{K}(p, p', k; \tau), \tag{53}$$

we easily find

$$\langle[a^+_{p-k/2}(t)\,a_{p+k/2}(t),\ a^+_{p'+k/2}(t')\,a_{p'-k/2}(t')]\rangle = -i\tilde{K}(p, p', k; \tau) + i\tilde{K}^*(p, p', -k; \tau). \tag{54}$$

Substituting (54) in (51′), we have

$$j^\alpha_k(t) = -\frac{1}{c}\int_{t}^{t} dt'\iint dpdp'j^\alpha(p)\,j^\beta(p')\,\{\tilde{K}(p, p', k; \tau) - \tilde{K}^*(p, p', -k; \tau)\}\,A^\beta_k(t'). \tag{55}$$

Going to the limit of $t_0 \to -\infty$, we get the relation between the time Fourier components of $j(t)$ and $A(t)$:

$$j^\alpha_{k,\omega} = \varkappa_{\alpha,\beta}(k, \omega)\,A^{\ \beta}_{k,\omega},$$
$$\varkappa_{\alpha,\beta}(k, \omega) = \frac{i}{2\pi c}\int dpdp'\,\frac{d\omega'}{\omega - \omega' + i\delta}\{\tilde{K}(p, p'; k, \omega') - \tilde{K}^*(p, p'; -k, -\omega')\}, \tag{56}$$

where $\tilde{K}(p, p'; k, \omega)$ is the time Fourier component of the function $\tilde{K}(p, p'; k, \tau)$.

In conclusion, the authors express their thanks to L. D. Landau and S. T. Beliaev for interesting discussions.

[1] H. Lehmann, Nuovo cimento 11, 342 (1954).

[2] M. Ia. Lavrent'ev and B. V. Shabat, Методы теории функчий комплексного переменного

(Methods of the Theory of Functions of a Complex Variable), GITTL, 1951, p. 257.

[3] A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 399 (1957), Soviet Phys. JETP **5**, 333 (1957).

[4] Iu. L. Klimontovich and V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **23**, 151 (1952).

[5] V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **23**, 641 (1952).

[6] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 59 (1957), Soviet Phys. JETP **5**, 101 (1957).

[7] J. Schwinger, Proc. Nat. Acad. Sci. **37**, 452 (1951).

[8] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

Translated by M. Hamermesh
22

SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 1 JULY, 1958

THE ENERGY SPECTRUM OF A NON-IDEAL FERMI GAS

V. M. GALITSKII

Moscow Engineering-Physics Institute

Submitted to JETP editor July 12, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 151-162 (January, 1958)

We have evaluated the energy spectrum and ground state energy of a non-ideal Fermi gas with repulsive interactions, using an expansion in powers of the ratio of the range of the potential to the mean distance apart of the particles (gas approximation). We have obtained the first two terms of the expansion.

INTRODUCTION

IT is well known that in many cases one can consider the excited states of a system of interacting Fermi particles as a gas of elementary excitations — quasiparticles. The energy of a quasiparticle is determined by its momentum in such a way that the energy of the excitation of the system ϵ_s is equal to $\epsilon(p_1) - \epsilon(p_2)$, where $p_1 > p_0 > p_2$ with p_0 the momentum at the Fermi surface. Such a spectrum is called a spectrum of the "Fermi type." A description of a system by means of the method of quasiparticles is exact only in the case of an ideal gas. If there are interactions between the particles, the excited states of the "Fermi type" do not represent the exact stationary states of the systems. This leads to the damping of the quasiparticles.

It was shown in Ref. 1 that it is convenient to apply the methods of quantum field theory to determine the energy spectrum of a system. The energy $\epsilon(p)$ and attenuation $\gamma(p)$ of the quasiparticles can be found as the poles of the analytical continuation of the single-particle Green function $G(p)$. In the present paper we shall apply the methods of quantum field theory to the problem of a non-ideal Fermi gas in which the interaction between the particles is short range $na^3 \ll 1$ (n is the density of the particles in the system and a the range of the potential), but not necessarily weak. We assume that the radially symmetrical potential $V(r)$ is positive and that the interaction between the particles is not retarded. We expand in powers of the parameter $p_0 f_0$, where f_0 is the real part of the scattering amplitude for small momenta. We shall find the energy spectrum of the system and the ground state energy up to quadratic terms in this parameter. Terms corresponding to higher powers than the cubic can not be expressed by means of two-particle parameters which makes it difficult to obtain them in a general form.* This fact was first remarked on in Ref. 2 in connection with the evaluation of the ground state energy.

1. SINGLE PARTICLE GREEN FUNCTION. THE METHOD OF GRAPHS

It is well known that the single particle Green

*The author is obliged to E. M. Lifshitz for this comment.