and hence

$$\frac{1}{V}\frac{\partial}{\partial\alpha}e^{VL} = \alpha^+ (e^{\partial L/\partial \nu} - 1)\, e^{VL}\,. \qquad \text{(A.28)}$$

By comparing Eq. (A.28) with (A.26), we obtain the desired relation between L and σ',

$$\partial\sigma'(\bar{\nu}) \,/\, \partial\bar{\nu} = e^{\partial L/\partial \nu} - 1. \qquad \text{(A.29)}$$

Equations (A.29) and (A.25) imply Eq. (A.15), and by Eq. (A.24) this proves Eq. (A.14).

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ENERGY-SPECTRUM OF A NON-IDEAL BOSE GAS

S. T. BELIAEV

Academy of Sciences, U.S.S.R.

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The one-particle Green's function is calculated in a low-density approximation for a system of interacting bosons. The energy spectrum of states near to the ground state (quasi-particle spectrum is derived.

1. INTRODUCTION

In the preceding paper[1] the method of Green's functions was developed for a system consisting of a large number of bosons. The one-particle Green's function was expressed in terms of the effective potentials Σ_{ik} of pair interactions and the chemical potential μ of the system. Approximate methods must be used to determine Σ_{ik} and μ. In the present paper we study a "gaseous" approximation, in which the density n, or the ratio between the volume occupied by particles and the total volume, is treated as a small parameter. The interaction between particles is assumed to be central and short-range, but not necessarily weak. The first two orders of approximation involve only the scattering amplitude f of a two-particle system. But in the next order (proportional to $(\sqrt{nf^3})^2$) the effects of three-particle interaction amplitudes appear, which means that practical calculations to this order are hardly possible.

From the Green's function which we calculate, we derive the energy spectrum of excitations or quasi-particles, the energy of the ground state, and also the momentum distribution of particles in the ground state.

2. ESTIMATE OF THE GRAPHS CONTRIBUTING TO THE EFFECTIVE POTENTIALS

The definition of the potentials Σ_{ik}, and the rules for constructing Feynman graphs, were described in our earlier paper,[1] which we shall call I.

We shall estimate by perturbation theory the various graphs contributing to Σ_{ik} and μ. For the Fourier transform of the potential $U(\mathbf{p}) = U_{\mathbf{p}}$, we assume for simplicity* $U_p = U_0$ for $p < 1/a$, and $U_p = 0$ for $p > 1/a$. Then a is of the order of magnitude of the particle radius.

For definiteness we examine Σ_{20}. The graphs for Σ_{02}, Σ_{11} and μ are essentially similar. The first order of perturbation theory, as we saw in Sec. (I, 7), gives $\Sigma_{20}^{(1)} = n_0 U_p$; $\mu = n_0 U_0$.

In the estimate of any graph there may appear three parameters — U_0 and a, characterizing the interaction, and n_0, characterizing the density of particles in the condensed phase. The three parameters can be combined into two dimensionless ratios,

$$\xi = U_0/a; \qquad \beta = \sqrt{n_0 a^3}. \tag{2.1}$$

The quantity ξ is the usual parameter which appears in perturbation theory (in ordinary units $\xi \sim mU(r)a^2/\hbar^2$), while β is a parameter of gas-density.

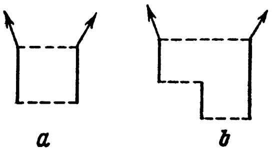

FIG. 1

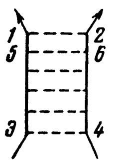

FIG. 2

The only non-vanishing graph in second order is the one shown in Fig. 1a. This gives a contribution

$$M_a \sim n_0 \int G^0(q+\mu)\, G^0(-q+\mu)\, U_0 U_{p+q} d^4 q$$

Substituting for G^0 from

$$G^0(p) = (p^0 - \varepsilon_p^0 + i\delta)^{-1}, \qquad \varepsilon_p^0 = p^2/2, \qquad \delta \to +0 \tag{2.2}$$

and carrying out the q^0-integration, we find

$$M_a \sim n_0 U_0^2 \int\limits_{qa<1} d\mathbf{q} \int \frac{dq^0}{(q^0+\mu-\varepsilon_q^0+i\delta)(-q^0+\mu-\varepsilon_q^0+i\delta)}$$

$$\sim n_0 U_0^2 \int\limits_{qa<1} d\mathbf{q} \frac{1}{\mu - \varepsilon_q^0 + i\delta}.$$

In the last integral the main contribution comes from $q \sim 1/a$, where $\mu/\epsilon^0 \sim n_0 U_0 a^2 = \xi\beta^2 \ll 1$. Therefore

$$M_a \sim n_0 U_0^2 / a = \Sigma_{20}^{(1)} \xi. \tag{2.3}$$

We consider next the third-order graph (1b). This gives a contribution

$$M_b \sim n_0^2 \int G^0(q+\mu)\,[G^0(-q+\mu)]^2\, U_0 U_q U_{p+q} d^4 q$$

$$\sim n_0^2 U_0^3 \int d\mathbf{q}/(\mu - \varepsilon_q^0 + i\delta)^2.$$

The last integral, unlike the previous one, converges at the upper limit, and the main contribution now comes from the range $q \sim \sqrt{\mu} = \sqrt{n_0 U_0}$. Therefore

$$M_b \sim n_0^2 U_0^3 / \sqrt{\mu} = \Sigma_{20}^{(1)} \xi^{3/2} \beta. \tag{2.4}$$

From Eqs. (2.3) and (2.4) we see that $M_b/M_a \sim \xi^{1/2}\beta$. This is a consequence of the fact that M_a contains an integral of a product of two factors G^0, formally diverging at the upper limit, while M_b contains an integral of a product of three factors G^0 and converges without any cut-off. In the graphs this difference is indicated by the number of continuous lines in the closed circuit formed by the continuous and dotted lines. The same result holds when the circuits form part of a more complicated graph.

Thus every circuit containing more than two continuous lines introduces the small parameter β, while circuits with two continuous lines do not involve β. In the lowest order we need consider only graphs whose circuits are all of the two-line type. All such graphs are of the "ladder" construction shown in Fig. 2. We denote by $-i\Gamma(12;34)$ the total contribution from all such graphs. The first-order approximation in β then differs from the first-order approximation in perturbation theory by changing the potential U (arising from a ladder with one rung) into Γ (arising from ladders of all lengths). Similar conclusions hold also for the higher approximations. A summation over a set of graphs, differing only by the insertion of ladder circuits into a fixed skeleton, produces a change of U into Γ. If we represent Γ by a rectangle, all graphs can be constructed by means of rectangles and continuous lines only. In this way the potential U is eliminated from the problem. The effective potential is Γ.

3.* EQUATION FOR THE EFFECTIVE POTENTIAL Γ

We can write down an integral equation

$$\Gamma(12;34) = U(1-2)\,\delta(1-3)\,\delta(2-4) + i\int U(1-2)\, G^0(1-5)\, G^0(2-6)\, \Gamma(56;34)\, d^4x_5 d^4x_6. \tag{3.1}$$

for the sum of contributions from all graphs of the ladder type (see Fig. 2). The notations are the

*The letters p, q, ... are used to denote the lengths of 3-vectors, or to denote 4-vectors. There can be no confusion, because they denote 4-vectors only when they appear as arguments in G(p), Σ(p), etc.

*The problems connected with Γ were solved in collaboration with V. M. Galitskii, who was working simultaneously on the analogous problems in Fermion systems.

same as in I. We next transform Eq. (3.1) into momentum representation. In order to relieve the equations of factors of 2π, we shall use the conventions

$$d^4p = (2\pi)^{-4} dp^1 dp^2 dp^3 dp^0;$$
$$\delta(p) = (2\pi)^4 \delta(p^1)\delta(p^2)\delta(p^3)\delta(p^0),$$

and similarly we understand $d\mathbf{p}$ and $\delta(\mathbf{p})$ to carry factors of $(2\pi)^3$. We write

$$\begin{aligned}\Gamma(p_1p_2;\ p_3p_4)\,\delta(p_1+p_2-p_3-p_4)\\ =\int \exp\{-ip_1x_1-ip_2x_2+ip_3x_3\\ +ip_4x_4\}\,\Gamma(12;34)\,d^4x_1d^4x_2d^4x_3d^4x_4,\end{aligned} \tag{3.2}$$

and introduce the relative and total momenta by

$$\begin{aligned}p_1+p_2=P';\ p_3+p_4=P;\\ p_1-p_2=2p',\ p_3-p_4=2p,\end{aligned} \tag{3.3}$$

Then, by Eq. (3.1), $\Gamma(p';p;P) \equiv \Gamma(p_1p_2;p_3p_4)$ satisfies the equation

$$\begin{aligned}\Gamma(p';p;P) = U(p'-p) + i\int d^4qU(p'-q)\,G^0(P/2+q)\\ \times G^0(P/2-q)\,\Gamma(q;p;P).\end{aligned} \tag{3.4}$$

Since the interaction U is instantaneous, $U(1-2) = U(\mathbf{x}_1-\mathbf{x}_2)\ \delta(t_1-t_2)$, and therefore the points 1, 2 and 3, 4 in $\Gamma(12;34)$ must be simultaneous. In momentum representation this means that $\Gamma(p_1p_2;p_3p_4)$ depends on the fourth components only in the combination $p_1^0+p_2^0=p_3^0+p_4^0=P^0$. Therefore $\Gamma(p';p;P)$ is independent of the fourth components of its first two arguments (the relative momenta). The q^0-integration in Eq. (3.4) can thus be carried out, giving

$$\begin{aligned}\int dq^0 G^0\left(\tfrac{1}{2}P+q\right)G^0\left(\tfrac{1}{2}P-q\right)\\ =-i\left(P^0-\tfrac{1}{4}\mathbf{P}^2-\mathbf{q}^2+i\delta\right)^{-1},\end{aligned} \tag{3.5}$$

and then Eq. (3.4) takes the form

$$\begin{aligned}\Gamma(\mathbf{p}';\mathbf{p};P) = U(\mathbf{p}'-\mathbf{p}) + \int d\mathbf{q}\,\frac{U(\mathbf{p}'-\mathbf{q})\,\Gamma(\mathbf{q};\mathbf{p};P)}{k_0^2-q^2+i\delta};\\ k_0^2 = P^0-\tfrac{1}{4}\mathbf{P}^2.\end{aligned} \tag{3.6}$$

Equation (3.6) cannot be solved explicitly, but its solution can be expressed in terms of the scattering amplitude of two particles in a vacuum. We write $\chi(\mathbf{q}) = (k_0^2-q^2+i\delta)^{-1}\ \Gamma(\mathbf{q};\mathbf{p};P)$. Then Eq. (3.6) becomes

$$(k_0^2-p'^2)\,\chi(\mathbf{p}') - \int U(\mathbf{p}'-\mathbf{q})\,\chi(\mathbf{q})\,d\mathbf{q} = U(\mathbf{p}'-\mathbf{p}). \tag{3.7}$$

Let $\Psi_{\mathbf{k}}(\mathbf{p}')$ be the normalized wave-function which satisfies the equation

$$(k^2-p'^2)\,\Psi_{\mathbf{k}}(\mathbf{p}') - \int U(\mathbf{p}'-\mathbf{q})\,\Psi_{\mathbf{k}}(\mathbf{q})\,d\mathbf{q} = 0, \tag{3.8}$$

Then the solution of Eq. (3.7) may be written

$$\chi(\mathbf{p}') = \int \frac{\Psi_{\mathbf{k}}(\mathbf{p}')\,\Psi_{\mathbf{k}}^*(\mathbf{q})}{k_0^2-k^2+i\delta}\,U(\mathbf{q}-\mathbf{p})\,d\mathbf{q},$$

and so $\Gamma(\mathbf{p}';\mathbf{p};P)$ becomes

$$\Gamma(\mathbf{p}';\mathbf{p};P) = (k_0^2-p'^2)\int \frac{\Psi_{\mathbf{k}}(\mathbf{p}')\,\Psi_{\mathbf{k}}^*(\mathbf{q})}{k_0^2-k^2+i\delta}\,U(\mathbf{q}-\mathbf{p})\,d\mathbf{q}. \tag{3.9}$$

We observe now that Eq. (3.8) is the Schrödinger equation in momentum representation. Thus $\Psi_{\mathbf{k}}(\mathbf{p})$ is the wave-function for a scattering problem with potential U. The scattering amplitude* $f(\mathbf{p}'\mathbf{p})$ is related to the Ψ-function by

$$f(\mathbf{p}'\mathbf{p}) = \int e^{-i\mathbf{p}'\mathbf{r}}\,U(\mathbf{r})\,\Psi_{\mathbf{p}}(\mathbf{r})\,d\mathbf{r} = \int U(\mathbf{p}'-\mathbf{q})\,\Psi_{\mathbf{p}}(\mathbf{q})\,d\mathbf{q}, \tag{3.10}$$

or by

$$\Psi_{\mathbf{p}}(\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}') + f(\mathbf{p}'\mathbf{p})/(p^2-p'^2+i\delta). \tag{3.11}$$

In the first Eq. (3.10), $\Psi_{\mathbf{p}}(\mathbf{r})$ is the wave-function in coordinate space which behaves at infinity like a plane wave with momentum $\mathbf{p}$ and an outgoing spherical wave. The usual scattering amplitude is the value of $f(\mathbf{p}'\mathbf{p})$ at $p' = p$. We consider arbitrary values of the arguments, so that $f(\mathbf{p}'\mathbf{p})$ is in general defined by Eq. (3.10).

Because $\Psi_{\mathbf{p}}(\mathbf{p}')$ satisfies orthogonality conditions in both its arguments, $f(\mathbf{p}'\mathbf{p})$ satisfies the unitarity conditions

$$\begin{aligned}f(\mathbf{p}'\mathbf{p}) - f^*(\mathbf{p}'\mathbf{p})\\ = \int d\mathbf{q}f(\mathbf{p}'\mathbf{q})\,f^*(\mathbf{p}\mathbf{q})\left[\frac{1}{q^2-p'^2+i\delta}-\frac{1}{q^2-p^2-i\delta}\right]\\ = \int d\mathbf{q}f^*(\mathbf{q}\mathbf{p}')\,f(\mathbf{q}\mathbf{p})\left[\frac{1}{q^2-p'^2+i\delta}-\frac{1}{q^2-p^2-i\delta}\right].\end{aligned} \tag{3.12}$$

When $\mathbf{p}' = \pm\mathbf{p}$, Eq. (3.12) gives the imaginary part of the forward and backward scattering amplitudes. Since $f(-\mathbf{p}'-\mathbf{p}) = f(\mathbf{p}'\mathbf{p})$, Eq. (3.12) implies

$$\operatorname{Im} f(\pm\mathbf{p}\mathbf{p}) = -i\pi\int d\mathbf{q}f(\mathbf{p}\mathbf{q})\,f^*(\pm\mathbf{p}\mathbf{q})\,\delta(q^2-p^2). \tag{3.13}$$

For the forward scattering amplitude, Eq. (3.13) gives just the well-known relation between the imaginary part of the amplitude and the total cross-section σ, $\operatorname{Im} f(\mathbf{p}\,\mathbf{p}) = -ip\sigma$.

We substitute Eq. (3.11) into (3.9) and use Eq. (3.12). This gives two equivalent expressions for $\Gamma(\mathbf{p}';\mathbf{p};P)$,

$$\begin{aligned}\Gamma(\mathbf{p}';\mathbf{p};P) = f(\mathbf{p}'\mathbf{p})\\ +\int d\mathbf{q}f(\mathbf{p}'\mathbf{q})\,f^*(\mathbf{p}\mathbf{q})\left[\frac{1}{k_0^2-q^2+i\delta}+\frac{1}{q^2-p^2-i\delta}\right]\\ = f^*(\mathbf{p}\mathbf{p}') + \int d\mathbf{q}f(\mathbf{p}'\mathbf{q})\,f^*(\mathbf{p}\mathbf{q})\left[\frac{1}{k_0^2-q^2+i\delta}+\frac{1}{q^2-p'^2+i\delta}\right].\end{aligned} \tag{3.14}$$

*The quantity $f(\mathbf{p}'\,\mathbf{p})$ differs by a numerical factor from the usual amplitude $a(\mathbf{p}'\,\mathbf{p})$, in fact $f = -4\pi a$.

expressing the effective potential $\Gamma(\mathbf{p}'; \mathbf{p}; P)$ in terms of the scattering amplitudes of a two-particle system.

4. FIRST-ORDER GREEN'S FUNCTION

The effective potentials Σ_{ik} are determined by special values which Γ takes when two out of the four particles involved in a process belong to the condensed phase. Thus two of the four particles must have $\mathbf{p} = 0$, $p^0 = \mu$. Each particle of the condensed phase also carries a factor $\sqrt{n_0}$. Therefore we find

$$\Sigma_{20}(p+\mu) = n_0\Gamma(\mathbf{p}; 0; 2\mu); \quad \Sigma_{02}(p+\mu) = n_0\Gamma(0; \mathbf{p}; 2\mu),$$

$$\Sigma_{11}(p+\mu) = n_0\Gamma(\mathbf{p}/2; \mathbf{p}/2; p+2\mu) + n_0\Gamma(-\mathbf{p}/2; \mathbf{p}/2; p+2\mu). \tag{4.1}$$

To obtain the chemical potential we must let all four particles in Γ belong to the condensed phase, and divide by one power of n_0 [see Eq. (I, 3.20)]. We then have

$$\mu = n_0\Gamma(0; 0; 2\mu). \tag{4.2}$$

Substituting into Eqs. (4.1) and (4.2) the value of Γ from Eq. (3.14), we find

$$\mu = n_0 f(00) + n_0\int d\mathbf{q}\, |f(0\mathbf{q})|^2 \left[\frac{1}{2\mu - q^2 + i\delta} + \frac{1}{q^2}\right],$$

$$\Sigma_{20}(p+\mu) = n_0 f(\mathbf{p}0) + n_0\int d\mathbf{q} f(\mathbf{pq}) f^*(0\mathbf{q}) \left[\frac{1}{2\mu - q^2 + i\delta} + \frac{1}{q^2}\right],$$

$$\Sigma_{02}(p+\mu) = n_0 f^*(\mathbf{p}0) + n_0\int d\mathbf{q}\, f(0\mathbf{q}) f^*(\mathbf{pq}) \left[\frac{1}{2\mu - q^2 + i\delta} + \frac{1}{q^2}\right],$$

$$\Sigma_{11}(p+\mu) = 2n_0 f_s\left(\frac{\mathbf{p}}{2}\frac{\mathbf{p}}{2}\right) + 2n_0\int d\mathbf{q} \left|f_s\left(\frac{\mathbf{p}}{2}\mathbf{q}\right)\right|^2 \left[\frac{1}{p^0 + 2\mu - p^2/4 - q^2 + i\delta} + \frac{1}{q^2 - p^2/4 - i\delta}\right]. \tag{4.3}$$

In the last equation we have introduced the symmetrized amplitude

$$f_s(\mathbf{p}'\mathbf{p}) = [f(\mathbf{p}'\mathbf{p}) + f(-\mathbf{p}'\mathbf{p})]/2.$$

All the integrals in Eq. (4.3) converge at high momentum, even if the amplitudes are taken to be constant. For dimensional reasons these terms are of order $n_0 f^2\sqrt{\mu}$. Compared with the first terms in Eq. (4.3), these terms contain an extra factor $\sqrt{n_0 f^3}$, which is just the gas-density parameter (2.1) obtained by substituting the amplitude f for the particle radius. In first approximation we neglect the integral terms in Eq. (4.3) and obtain

$$\mu = n_0 f(00); \quad \Sigma_{20}(p+\mu) = \Sigma_{02}^*(p+\mu) = n_0 f(\mathbf{p}0);$$
$$\Sigma_{11}^{\pm} \equiv \Sigma_{11}(\pm p + \mu) = 2n_0 f_s\left(\frac{\mathbf{p}}{2}\frac{\mathbf{p}}{2}\right). \tag{4.4}$$

The Green's function G is given by Eq. (I, 5.6),

$$G(p+\mu) = \frac{p^0 + \varepsilon_{\mathbf{p}}^0 + \Sigma_{11}^- - \mu}{[p^0 - (\Sigma_{11}^+ - \Sigma_{11}^-)/2]^2 - [\varepsilon_{\mathbf{p}}^0 + (\Sigma_{11}^+ + \Sigma_{11}^-)/2 - \mu]^2 + \Sigma_{20}\Sigma_{02} + i\delta} \tag{4.5}$$

and after substituting from Eq. (4.4) this becomes

$$G(p+\mu) = \frac{p^0 + \varepsilon_{\mathbf{p}}^0 + 2n_0 f_s\left(\frac{\mathbf{p}}{2}\frac{\mathbf{p}}{2}\right) - n_0 f(00)}{p^{02} - \varepsilon_{\mathbf{p}}^2 + i\delta}, \tag{4.6}$$

with

$$\varepsilon_{\mathbf{p}} = \sqrt{\left[\varepsilon_{\mathbf{p}}^0 + 2n_0 f_s\left(\frac{\mathbf{p}}{2}\frac{\mathbf{p}}{2}\right) - n_0 f(00)\right]^2 - n_0^2 |f(\mathbf{p}0)|^2}. \tag{4.7}$$

The point $p_0(\mathbf{p})$, at which the Green's function $G(p+\mu)$ has a pole, determines the energy $\epsilon_{\mathbf{p}}$ of elementary excitations or quasi-particles[2] carrying momentum $\mathbf{p}$. To calculate $\epsilon_{\mathbf{p}}$ we must know three distinct amplitudes. $f_s(\mathbf{p}/2\ \mathbf{p}/2)$ is the ordinary symmetrized amplitude for forward scattering, and $f(0\ 0)$ is a special value of the same amplitude. However, $f(\mathbf{p}\ 0)$ does not have any obvious meaning in the two-particle problem, since it refers to a process which is forbidden for two particles in a vacuum.

At small momenta, we may neglect the momentum dependence of $f_s(\mathbf{p}/2\ \mathbf{p}/2)$ and of $f(\mathbf{p}\ 0)$, setting $f_s(\mathbf{p}/2\ \mathbf{p}/2) \approx f(\mathbf{p}\ 0) \approx f(0\ 0) \equiv f_0$. This approximation is allowed when the wavelength is long compared with the characteristic size of the interaction region, which has an order of magnitude given by the scattering amplitude f_0. Therefore when $p < f_0^{-1}$ we may consider all the amplitudes in Eq. (4.6) and (4.7) to be constant. For higher excitations with $p \gtrsim f_0^{-1}$, the momentum dependence of the amplitude becomes important, and the problem cannot be treated in full generality. We shall examine the higher excitations (in Sec. 8) for the special example of a hard-sphere gas.

Confining ourselves to the case $pf_0 < 1$, we deduce from Eq. (4.6) and (4.7)

$$G(p+\mu) = (p^0 + \varepsilon_{\mathbf{p}}^0 + n_0 f_0)/(p^{02} - \varepsilon_{\mathbf{p}}^2 + i\delta), \tag{4.8}$$

with

$$\varepsilon_{\mathbf{p}} = \sqrt{\varepsilon_{\mathbf{p}}^{02} + 2n_0 f_0 \varepsilon_p^0}. \tag{4.9}$$

Equations (4.8) and (4.9) are formally identical with the results obtained from perturbation theory in Eq. (I, 7.3) and (I, 7.4). Only the scattering amplitude f_0 now appears instead of the Fourier transform $U_{\mathbf{p}}$ of the potential.

Equation (4.9) shows that quasi-particles with $p \ll \sqrt{n_0 f_0}$ have a sound-wave type of dispersion law $\epsilon_{\mathbf{p}} \approx p\sqrt{n_0 f_0}$. When $p \gg \sqrt{n_0 f_0}$ they go over into almost free particles with $\epsilon_{\mathbf{p}} \approx \epsilon_{\mathbf{p}}^0 + n_0 f_0$.

This sort of energy spectrum appears also when one considers particles moving in a continuous medium with a refractive index. The transition from phonon to free particle behavior occurs at $p \sim \sqrt{n_0 f_0} \ll 1/f_0$, so that the approximation of constant amplitudes is valid in both ranges.

The conditions $\sqrt{n_0 f_0^3} \ll 1$ and $pf_0 \ll 1$ are not independent. If we look at momenta p not greatly exceeding $\sqrt{n_0 f_0}$, then the second condition is a consequence of the first. If we are then neglecting quantities of order $\sqrt{n_0 f_0^3}$, we must also treat the amplitudes as constant.

In Sec. (I, 5) we introduced the quantity $\hat{G}(p+\mu)$, the analog of the Green's function G but constructed from graphs with two ingoing ends instead of one ingoing and one outgoing. The analogous quantity with two outgoing ends will be denoted by $\check{G}(p+\mu)$. It is obtained from $\hat{G}(p+\mu)$ when Σ_{02} is replaced by Σ_{20}. In the constant-amplitude approximation, Eq. (I, 5.7) and (4.4) give

$$\hat{G}(p+\mu) = \check{G}(p+\mu) = -n_0 f_0/(p^{02} - \varepsilon_p^2 + i\delta). \quad (4.10)$$

5. SECOND APPROXIMATION FOR THE GREEN'S FUNCTION

For the second approximation to Σ_{ik} and μ, we must retain quantities of order $\sqrt{n_0 f_0^3}$. As we saw at the end of the preceding section, we must then also retain terms of order pf_0 in the amplitudes. The real part of the amplitude involves only even powers of p, and the imaginary part only odd powers. Terms of order pf_0 arise only from the lowest approximation to the imaginary part of the amplitudes. The imaginary part of $f_s(\mathbf{p}/2\ \mathbf{p}/2)$ is given by Eq. (3.13), and from Eq. (3.10) we see that the amplitude $f(\mathbf{p}\ 0)$ is real [and anyway in this approximation we need only the square of the modulus of $f(\mathbf{p}\ 0)$].

The graphs of the first approximation give terms of order $\sqrt{n_0 f_0^3}$, namely the integral terms in Eq. (4.3). In these terms, as in Eq. (3.13), we may take the amplitudes to be constant. We have seen in Sec. 2 that graphs containing one circuit with three or more continuous lines give contributions of the same order. The summation over sets of graphs, which differ only in the number of continuous lines in a circuit, is automatically performed if one replaces the zero-order Green's function G^0 by the first-order functions G, $\hat{G}$ and $\check{G}$. We therefore consider immediately the circuits which can be built out of G, $\hat{G}$, $\check{G}$ and Γ. There are altogether ten essentially different circuits (see Fig. 3). A rectangle with a cross denotes a sum of two rectangles, one being a direct interaction and the other an exchange interaction. The two differ only by an

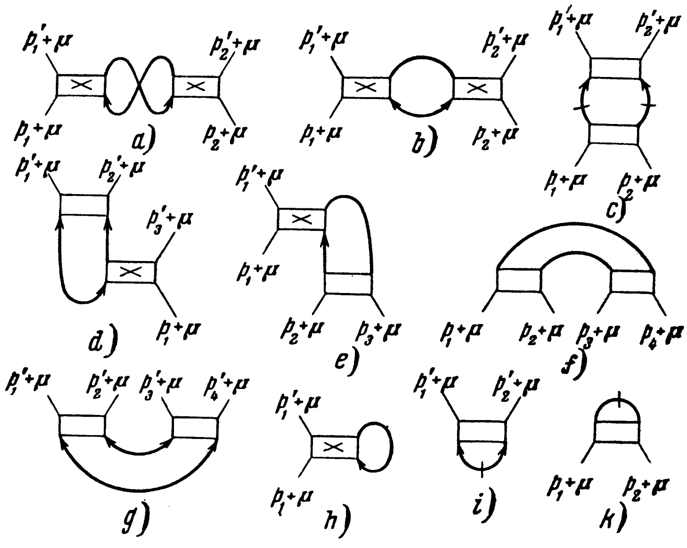

FIG. 3

interchange of the upper or the lower ends. The sum of the two rectangles introduces a factor $-i[\Gamma(12;34) + \Gamma(12;43)]$, or in momentum representation $-i[\Gamma(\mathbf{p}';\mathbf{p};P) + \Gamma(-\mathbf{p}';\mathbf{p};P)]$. If G, $\hat{G}$ and $\check{G}$ are expanded in powers of the effective potential Γ, then in the lowest approximation the graphs (3c, 3i, 3k) become circuits with two continuous lines. But all such circuits are already included in Γ and must therefore be omitted. This omission is represented in Fig. 3 by the strokes across the continuous lines. Let $-iF_{a,b\ldots}(p_1',\ldots,p_1\ldots)$ denote the contributions from the graphs of Fig. 3. In the constant-amplitude approximation these contributions are:

$$\begin{aligned}
F_a(p_1'p_2';\ p_1p_2) &= i4f_0^2\int G(q+\mu)\,G(p_1-p_1'+q+\mu)\,d^4q;\\
F_b &= i4f_0^2\int \hat{G}(q+\mu)\,\check{G}(p_1-p_1'+q+\mu)\,d^4q;\\
F_c &= if_0^2\int \{G(q+\mu)\,G(p_1+p_2-q+\mu)\\
&\quad - G^0(q+\mu)\,G^0(p_1+p_2-q+\mu)\}\,d^4q;\\
F_d &= i2f_0^2\int \hat{G}(q+\mu)\,G(p_2'+p_3'-q+\mu)\,d^4q,\\
F_e &= i2f_0^2\int \check{G}(q+\mu)\,G(p_1'+p_2'-q+\mu)\,d^4q;\\
F_f &= if_0^2\int \hat{G}(q+\mu)\,\hat{G}(p_1+p_2-q+\mu)\,d^4q;\\
F_g &= if_0^2\int \check{G}(q+\mu)\,\check{G}(p_1'+p_2'-q+\mu)\,d^4q;\\
F_h &= i2f_0\int G(q+\mu)\,d^4q;\\
F_i &= if_0\int\{\check{G}(q+\mu) - n_0f_0G^0(q+\mu)\,G^0(-q+\mu)\}\,d^4q;\\
F_k &= if_0\int\{\hat{G}(q+\mu) - n_0f_0G^0(q+\mu)\,G^0(-q+\mu)\}\,d^4q.
\end{aligned} \quad (5.1)$$

Momentum conservation $\Sigma p = \Sigma p'$ is assumed to hold everywhere. The q^0-integration in F_h is performed with a detour into the upper half-plane, since this contribution must vanish as $G \to G^0$. Everywhere on the right of Eq. (5.1) the first-approximation value $\mu^{(1)} = n_0 f_0$ should be substituted for μ.

The Σ_{ik} involve special values of the F, together with a factor $\sqrt{n_0}$ for each particle of the condensed phase:

$$\Sigma'_{20}(p+\mu) = n_0 F_a(p-p;\ 00) + n_0 F_b(p-p;\ 00) + n_0 F_e(p0-p;\ 0) + n_0 F_e(0p-p;\ 0) + n_0 F_e(-p0p;\ 0) + n_0 F_e(0-pp;\ 0) + n_0 F_g(p0-p0;) + n_0 F_g(p00-p;) + F_i(p-p;);$$

$$\Sigma'_{02}(p+\mu) = n_0 F_a(00;\ p-p) + n_0 F_b(00;\ p-p) + n_0 F_d(0;\ p-p0) + n_0 F_d(0;\ p0-p) + n_0 F_d(0;\ -pp0) + n_0 F_d(0;\ -p0p) + n_0 F_f(;\ p0-p0) + n_0 F_f(;\ p00-p) + F_k(;\ p-p); \tag{5.2}$$

$$\Sigma'_{11}(p+\mu) = n_0 F_a(p0;\ 0p) + n_0 F_b(p0;\ 0p) + n_0 F_c(p0;\ p0) + n_0 F_c(p0;\ 0p) + n_0 F_d(p;\ 0p0) + n_0 F_d(p;\ 00p) + n_0 F_e(p00;\ p) + n_0 F_e(0p0;\ p) + F_h(p;\ p).$$

To enumerate the vacuum loops which contribute to μ, we must first distinguish one incoming or outgoing particle of the condensed phase (see Section (I, 4)). After this we must sum the loops, counting separately all possible geometric structures and all possible positions of the distinguished particle. The vacuum loops include three types of rectangle, differing in the numbers of incoming and outgoing continuous lines, and corresponding to factors $\Sigma_{11}^{(1)}$, $\Sigma_{02}^{(1)}$ and $\Sigma_{20}^{(1)}$. The distinguished particle of the condensed phase may come out from $\Sigma_{11}^{(1)}$ or from $\Sigma_{02}^{(1)}$. The sums of contributions from graphs of these two types are respectively $-iF_h(0;0)$ and $-iF_i(0\,0;)$. The term in μ arising from all these vacuum loops is thus

$$\mu' = F_h(0;\ 0) + F_i(00;). \tag{5.3}$$

To carry out the q^0-integration in Eq. (5.1), it is convenient to represent G and $\hat{G} = \check{G}$ in the following form,

$$G(q+\mu) = \frac{A_{\mathbf{q}}}{q^0 - \varepsilon_{\mathbf{q}} + i\delta} - \frac{B_{\mathbf{q}}}{q^0 + \varepsilon_{\mathbf{q}} - i\delta};$$

$$\hat{G} = \check{G}(q+\mu) = -C_{\mathbf{q}}\left[\frac{1}{q^0 - \varepsilon_{\mathbf{q}} + i\delta} - \frac{1}{q^0 + \varepsilon_{\mathbf{q}} - i\delta}\right], \tag{5.4}$$

with

$$A_{\mathbf{q}} = (\varepsilon_{\mathbf{q}} + \varepsilon^0_{\mathbf{q}} + n_0 f_0)/2\varepsilon_{\mathbf{q}};$$
$$B_{\mathbf{q}} = (-\varepsilon_{\mathbf{q}} + \varepsilon^0_{\mathbf{q}} + n_0 f_0)/2\varepsilon_{\mathbf{q}} = n_0^2 f_0^2/2\varepsilon_{\mathbf{q}}(\varepsilon_{\mathbf{q}} + \varepsilon^0_{\mathbf{q}} + n_0 f_0);\quad C_{\mathbf{q}} = n_0 f_0/2\varepsilon_{\mathbf{q}} \tag{5.5}$$

depending only on $|\mathbf{q}|$. The q^0-integrations are now performed and the results substituted into Eq. (5.2) and (5.3). After some manipulations we obtain

$$\Sigma'_{02(20)}(p+\mu) = 2n_0 f_0^2 \int d\mathbf{q}\,[(A_{\mathbf{q}};\ B_{\mathbf{k}}) - (A_{\mathbf{q}} + B_{\mathbf{q}};\ C_{\mathbf{k}}) + 3C_{\mathbf{q}}C_{\mathbf{k}}] \times \left(\frac{1}{p^0 - \varepsilon_{\mathbf{q}} - \varepsilon_{\mathbf{k}} + i\delta} - \frac{1}{p^0 + \varepsilon_{\mathbf{q}} + \varepsilon_{\mathbf{k}} - i\delta}\right) - f_0 \int d\mathbf{q}\left\{C_{\mathbf{q}} + \frac{n_0 f_0}{2n_0 f_0 - 2\varepsilon^0_{\mathbf{q}} + i\delta}\right\},$$

$$\Sigma'_{11}(p+\mu) = 2n_0 f_0^2 \int d\mathbf{q}\left\{\frac{(A_{\mathbf{q}};\ B_{\mathbf{k}}) + 2C_{\mathbf{q}}C_{\mathbf{k}} + A_{\mathbf{q}}A_{\mathbf{k}} - 2(A_{\mathbf{q}};\ C_{\mathbf{k}})}{p^0 - \varepsilon_{\mathbf{q}} - \varepsilon_{\mathbf{k}} + i\delta} - \frac{(A_{\mathbf{q}};\ B_{\mathbf{k}}) + 2C_{\mathbf{q}}C_{\mathbf{k}} + B_{\mathbf{q}}B_{\mathbf{k}} - 2(B_{\mathbf{q}};\ C_{\mathbf{k}})}{p^0 + \varepsilon_{\mathbf{q}} + \varepsilon_{\mathbf{k}} - i\delta} - \frac{1}{p^0 + 2n_0 f_0 - \varepsilon^0_{\mathbf{q}} - \varepsilon^0_{\mathbf{k}} + i\delta}\right\} + 2f_0 \int d\mathbf{q}\, B_{\mathbf{q}}, \tag{5.6}$$

$$\mu' = 2f_0 \int d\mathbf{q}\, B_{\mathbf{q}} - f_0 \int d\mathbf{q}\left\{C_{\mathbf{q}} + \frac{n_0 f_0}{2n_0 f_0 - 2\varepsilon^0_{\mathbf{q}} + i\delta}\right\},$$

Here $\mathbf{k} = \mathbf{p} - \mathbf{q}$, and the symbol (;) denotes a symmetrized product, $(A_{\mathbf{q}};\ B_{\mathbf{k}}) = A_{\mathbf{q}}B_{\mathbf{k}} + B_{\mathbf{q}}A_{\mathbf{k}}$. The integrands are all symmetrical in $\mathbf{q}$ and $\mathbf{k}$.

Before we add to Eq. (5.6) the second-order terms from Eq. (4.2), we transform the expression (4.3) for Σ_{11}. Remembering that

$$q^2 + p^2/4 = \varepsilon^0_{\mathbf{p}/2+\mathbf{q}} + \varepsilon^0_{\mathbf{p}/2-\mathbf{q}}$$

and introducing the new integration variable $\mathbf{q}' = \mathbf{q} + \mathbf{p}/2$, we find that Eq. (4.3) gives to the required approximation

$$\Sigma_{11}(p+\mu) = 2n_0 f_0 + 2n_0 \operatorname{Im} f_s\left(\frac{\mathbf{p}}{2}\,\frac{\mathbf{p}}{2}\right) + 2n_0 f_0^2 \int d\mathbf{q}\left[\frac{1}{p^0 + 2n_0 f_0 - \varepsilon^0_{\mathbf{q}} - \varepsilon^0_{\mathbf{k}} + i\delta} - \frac{1}{\varepsilon^0_p - \varepsilon^0_{\mathbf{q}} - \varepsilon^0_{\mathbf{k}} + i\delta}\right]. \tag{5.7}$$

The total of all second-order terms in now obtained from Eq. (5.6), (4.3), and (5.7), and after some algebra becomes

$$\mu^{(2)} = 2f_0\int d\mathbf{q}\, B_\mathbf{q} + \frac{1}{2}\, n_0 f_0^2 \int d\mathbf{q}\left(\frac{1}{\varepsilon_\mathbf{q}^0} - \frac{1}{\varepsilon_\mathbf{q}}\right),$$

$$\Sigma_{20(02)}^{(2)}(p+\mu) = 2n_0 f_0^2 \int d\mathbf{q}\, [(A_\mathbf{q};\ B_\mathbf{k}) - (A_\mathbf{q} + B_\mathbf{q};\ C_\mathbf{k}) + 3C_\mathbf{q}C_\mathbf{k}] \times \left(\frac{1}{p^0 - \varepsilon_\mathbf{q} - \varepsilon_\mathbf{k} + i\delta} - \frac{1}{p^0 + \varepsilon_\mathbf{q} + \varepsilon_\mathbf{k} - i\delta}\right) + \frac{1}{2}\, n_0 f_0^2 \int d\mathbf{q}\left(\frac{1}{\varepsilon_\mathbf{q}^0} - \frac{1}{\varepsilon_\mathbf{q}}\right);$$

$$\Sigma_{11}^{(2)}(p+\mu) = 2n_0 f_0^2 \int d\mathbf{q} \left\{ \frac{(A_\mathbf{q};\ B_\mathbf{k}) + 2C_\mathbf{q}C_\mathbf{k} + A_\mathbf{q}A_\mathbf{k} - 2(A_\mathbf{q};\ C_\mathbf{k})}{p^0 - \varepsilon_\mathbf{q} - \varepsilon_\mathbf{k} + i\delta} - \frac{(A_\mathbf{q};\ B_\mathbf{k}) + 2C_\mathbf{q}C_\mathbf{k} + B_\mathbf{q}B_\mathbf{k} - 2(B_\mathbf{q};\ C_\mathbf{k})}{p^0 + \varepsilon_\mathbf{q} + \varepsilon_\mathbf{k} - i\delta} + \frac{1}{4}\left(\frac{1}{\varepsilon_\mathbf{q}} + \frac{1}{\varepsilon_\mathbf{k}}\right)\right\} + 2f_0 \int d\mathbf{q}\, B_\mathbf{q} + 2n_0 \operatorname{Im} f_s\left(\frac{\mathbf{p}}{2}\ \frac{\mathbf{p}}{2}\right) - 2n_0 f_0^2 \int d\mathbf{q} \left[\frac{1}{\varepsilon_p^0 - \varepsilon_\mathbf{q}^0 - \varepsilon_\mathbf{k}^0 + i\delta} + \frac{1}{4\varepsilon_\mathbf{q}} + \frac{1}{4\varepsilon_\mathbf{k}}\right]. \tag{5.8}$$

The value of $\operatorname{Im} f_s(\mathbf{p}/2\ \mathbf{p}/2)$ can be obtained from Eq. (3.13), and the integrals not involving p^0 can be carried out exactly. In this way Eq. (5.8) becomes

$$\Sigma_{20(02)}^{(2)}(p+\mu) = \frac{1}{2}\, n_0 f_0^2 \int \frac{d\mathbf{q}}{\varepsilon_\mathbf{q}\varepsilon_\mathbf{k}} R(qk)\left[\frac{1}{p^0 - \varepsilon_\mathbf{q} - \varepsilon_\mathbf{k} + i\delta} - \frac{1}{p^0 + \varepsilon_\mathbf{q} + \varepsilon_\mathbf{k} - i\delta}\right] + \frac{1}{\pi^2}\sqrt{n_0 f_0^3}\, n_0 f_0, \tag{5.9}$$

$$\Sigma_{11}^{(2)}(p+\mu) = \frac{1}{2}\, n_0 f_0^2 \int \frac{d\mathbf{q}}{\varepsilon_\mathbf{q}\varepsilon_\mathbf{k}} \left[\frac{Q^-(qk)}{p^0 - \varepsilon_\mathbf{q} - \varepsilon_\mathbf{k} + i\delta} - \frac{Q^+(qk)}{p^0 + \varepsilon_\mathbf{q} + \varepsilon_\mathbf{k} - i\delta} + \varepsilon_\mathbf{q} + \varepsilon_\mathbf{k}\right] + \frac{8}{3\pi^2}\sqrt{n_0 f_0^3}\, n_0 f_0, \tag{5.10}$$

$$\mu^{(2)} = (5/3\pi^2)\sqrt{n_0 f_0^3}\cdot n_0 f_0, \tag{5.11}$$

with

$$R(qk) = 2\varepsilon_\mathbf{q}^0\varepsilon_\mathbf{k}^0 - 2\varepsilon_\mathbf{q}\varepsilon_\mathbf{k} + n_0^2 f_0^2,$$
$$Q^\mp(qk) = 3\varepsilon_\mathbf{q}^0\varepsilon_\mathbf{k}^0 - \varepsilon_\mathbf{q}\varepsilon_\mathbf{k} + n_0 f_0(\varepsilon_\mathbf{q}^0 + \varepsilon_\mathbf{k}^0) + n_0^2 f_0^2 \mp [n_0 f_0(\varepsilon_\mathbf{q} + \varepsilon_\mathbf{k}) - \varepsilon_\mathbf{q}\varepsilon_\mathbf{k}^0 - \varepsilon_\mathbf{k}\varepsilon_\mathbf{q}^0]. \tag{5.12}$$

It is convenient to express the Green's function (4.5) in a form analogous to Eq. (5.4). In this approximation we find

$$G(p+\mu) = \frac{A_p + \alpha_p}{p^0 - \varepsilon_p - \Lambda_p^-} - \frac{B_p + \alpha_p}{p^0 + \varepsilon_p + \Lambda_p^+}, \tag{5.13}$$

where $\alpha_\mathbf{p}$ and $\Lambda_\mathbf{p}^\mp$ are the second-order corrections

$$\alpha_p = \frac{n_0 f_0}{4\varepsilon_\mathbf{p}^3}\{2\varepsilon_\mathbf{p}^0\Sigma_{20}^{(2)} - n_0 f_0(\Sigma_{11}^+ + \Sigma_{11}^- - 2\mu - 2\Sigma_{20})^{(2)}\};$$

$$\Lambda_p^\mp = \frac{\varepsilon_\mathbf{p}^0}{2\varepsilon_\mathbf{p}}(\Sigma_{11}^+ + \Sigma_{11}^- - 2\mu)^{(2)} + \frac{n_0 f_0}{2\varepsilon_p}(\Sigma_{11}^+ + \Sigma_{11}^- - 2\mu - 2\Sigma_{20})^{(2)} \pm \frac{1}{2}(\Sigma_{11}^+ - \Sigma_{11}^-)^{(2)}. \tag{5.14}$$

These $\alpha_\mathbf{p}$ and $\Lambda_\mathbf{p}^\mp$ are combinations of the integrals (5.9) and (5.10). In the limits of small and large momentum (compared with $\sqrt{n_0 f_0}$), explicit expressions can be obtained for the functions $\alpha_\mathbf{p} = \alpha(p^0;\mathbf{p})$ and $\Lambda_\mathbf{p}^\mp = \Lambda^\mp(p^0;\mathbf{p})$. When these are examined it is found that there are no new poles of the Green's function. We here exhibit the behavior of the Green's function near to the poles $p_0 \approx \pm\epsilon_\mathbf{p}$. In this region we may write $|p^0| = \epsilon_\mathbf{p}$ in $\alpha_\mathbf{p}$, and we need retain only terms of first order in the difference $(\epsilon_\mathbf{p} \mp p^0)$ in $\Lambda_\mathbf{p}^\mp$. For small momenta $(p \ll \sqrt{n_0 f_0})$ we then find

$$\alpha_p = \sqrt{n_0 f_0^3}\left(\frac{2}{3\pi^2}\frac{n_0 f_0}{\varepsilon_p} + i\frac{1}{64\pi}\frac{\varepsilon_p}{n_0 f_0}\right)$$
$$(p \ll \sqrt{n_0 f_0};\ \ \varepsilon_p \approx p\sqrt{n_0 f_0}),$$

$$\Lambda_p^\mp \equiv \Omega_p + \lambda_p(\varepsilon_p \mp p^0) = \sqrt{n_0 f_0^3}\left(\frac{7}{6\pi^2}\varepsilon_p - i\frac{3}{640\pi}\frac{\varepsilon_p^5}{n_0^4 f_0^4}\right) + (\varepsilon_p \mp p^0)\sqrt{n_0 f_0^3}\left(\frac{1}{2\pi^2} + i\frac{1}{32\pi}\frac{\varepsilon_p^2}{n_0^2 f_0^2}\right). \tag{5.15}$$

For large momenta only the imaginary part of $\Lambda_\mathbf{p}^\mp$ is important,

$$\Lambda_p^\mp = \Omega_p = -\frac{i}{4\pi}\, p f_0 n_0 f_0 \quad (p \gg \sqrt{n_0 f_0}). \tag{5.16}$$

For small momenta, in virtue of Eq. (5.13) and (5.15), the Green's function near to the poles may be written in the form

$$G(p+\mu) = (1 - \lambda_\mathbf{p})\left[\frac{A_p + \alpha_p}{p^0 - \varepsilon_\mathbf{p} - \Omega_p} - \frac{B_p + \alpha_p}{p^0 + \varepsilon_p + \Omega_p}\right]. \tag{5.17}$$

For large momenta, $\alpha_\mathbf{p}$ and $\lambda_\mathbf{p}$ may be neglected in Eq. (5.17).

6. QUASI-PARTICLE SPECTRUM AND GROUND-STATE ENERGY

We have already mentioned that the energy of a quasi particle is determined by the value of $p^0(\mathbf{p})$ at a pole of $G(p+\mu)$. Only those poles are to be considered for which the imaginary part of the energy is negative, so that the damping is positive. In the range $p \ll \sqrt{n_0 f_0}$, Eq. (5.15) and (5.17) give

$$\varepsilon = p\sqrt{n_0 f_0}\left(1 + \frac{7}{6\pi^2}\sqrt{n_0 f_0^3}\right) - i\frac{3}{640\pi}\sqrt{n_0 f_0^3}\,\frac{p^5}{(n_0 f_0)^{3/2}} \quad (p \ll \sqrt{n_0 f_0}), \tag{6.1}$$

In the high-momentum range, according to Eq. (5.16), we have

$$\varepsilon = \varepsilon_{\mathbf{p}}^{0} + n_0 f_0 \left(1 - \frac{i}{4\pi} p f_0\right) \approx \varepsilon_p^0 + n_0 f(\mathbf{pp}) \quad (p \gg \sqrt{n_0 f_0}). \tag{6.2}$$

Equation (6.1) shows that for small p the quasi particles are phonons. The second approximation gives a correction to the sound velocity, and a damping proportional to p^5 which is connected with a process of decay of one phonon into two. In the high-momentum range, the second approximation gives a damping which is related to the imaginary part of the forward scattering amplitude, and so to the total cross section.

In Sec. (I, 7) we found connections between the Green's function and various physical properties of the system. The mean number of particles $\overline{N}_{\mathbf{p}}$ with a given momentum $\mathbf{p}$ in the ground state of the system is related to the residue of the Green's function at its upper pole,

$$\overline{N}_{\mathbf{p}} = i \int G\, dp^0 / 2\pi = (B_{\mathbf{p}} + \alpha_p)(1 - \lambda_p). \tag{6.3}$$

When $p \ll \sqrt{n_0 f_0}$, Eqs. (5.15) and (5.5) give

$$\overline{N}_p = \frac{n_0 f_0}{2\varepsilon_p}\left(1 + \frac{5}{6\pi^2}\sqrt{n_0 f_0^3}\right) \tag{6.4}$$

The imaginary parts of α_p and λ_p here cancel, as they should. To find the total number of particles with $\mathbf{p} \neq 0$, we need to know $\overline{N}_{\mathbf{p}}$ for all momenta. We therefore use only the first approximation formula for $\overline{N}_{\mathbf{p}}$, namely $\overline{N}_{\mathbf{p}} = B_{\mathbf{p}}$. For the density of particles with $\mathbf{p} \neq 0$ we find

$$n - n_0 = i \int G(p + \mu)\, d^4 p = \int B_{\mathbf{p}}\, d\mathbf{p} = \sqrt{n_0 f_0^3}\, n_0 / 3\pi^2. \tag{6.5}$$

Equation (6.5) gives the relation between the density n_0 of particles in the condensed phase, which appeared as a parameter in all our equations, and the total number of particles in the system.

We note here one important point. It can be seen from the way the calculations were done that the validity of the "gaseous" approximation requires that n_0 be small. It is not directly required that the total density n be small, since n does not appear explicitly in the problem. But Eq. (6.5) shows that when n_0 is small n is necessarily small, too. This means that it is not possible to decrease significantly the density of the condensed phase by increasing the interaction or the total density, so long as $n_0 \ll f_0^{-3}$. This result confirms and strengthens the assertion made in I that the condensed phase does not disappear when interactions are introduced.

We can calculate the ground-state energy from the chemical potential μ. By Eq. (4.4) and (5.11),

$$\mu = n_0 f_0 \left(1 + \frac{5}{3\pi^2}\sqrt{n_0 f_0^3}\right), \tag{6.6}$$

Expressing n_0 in terms of n by means of Eq. (6.5), we have in the same approximation

$$\mu = n f_0 \left(1 + \frac{4}{3\pi^2}\sqrt{n f_0^3}\right). \tag{6.7}$$

By definition we have $\mu = \frac{\partial}{\partial n}\left(\frac{E_0}{V}\right)$. Therefore, integrating Eq. (6.7) with respect to n, we obtain the ground-state energy

$$\frac{E_0}{V} = \frac{1}{2} n^2 f_0 \left(1 + \frac{16}{15\pi^2}\sqrt{n f_0^3}\right), \tag{6.8}$$

This coincides with the result of Lee and Yang[3] for the hard-sphere gas, if we remember that in that case $f_0 = 4\pi a$.

The condition for the system to be thermodynamically stable is $\partial P/\partial V = -\partial^2 E/\partial V^2 < 0$. This condition reduces to $f_0 > 0$. Our results are only meaningful when this condition is satisfied.

7. POSSIBILITY OF HIGHER APPROXIMATIONS

In the first two approximations, all the results can be expressed in terms of the amplitudes f. Thus the problem of many interacting particles is reducible to the problem of two particles.

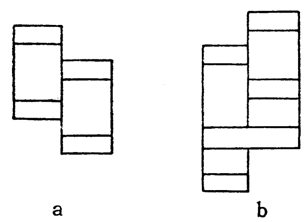

FIG. 4

In the next approximation we must consider contributions to Σ_{ik} proportional to $n_0 f_0^3$. Among other graphs, we must include the "triple ladders" illustrated in Fig. 4. The integrals arising from graphs of this type diverge at high momenta and become finite only when the momentum dependence of f is taken into account. For an estimate we may cut the integrals off at a momentum $p \sim f_0^{-1}$. We see then that an increase in the number of "rungs" does not change the order of magnitude of the integral. In fact, each rung adds a factor $f_0 G^2$ and an integration over one momentum 4-vector. For a rough estimate we take $q^0 \sim q^2$, $G \sim q^{-2}$, and find

$$\int f_0 G^2 d^4q \sim f_0 \int dq \sim 1,$$

Therefore we have to consider simultaneously all such graphs with any number of rungs. The totality of these triple ladders describes completely the interaction of three particles. Therefore the sum of contributions from such graphs can be expressed only by means of three-particle amplitudes.

In the third approximation (terms proportional to $n_0 f_0^3$) we thus require a solution of the three-particle problem (see also Ref. 4). Since the problem of three strongly interacting particles is in general insoluble, the higher approximations to the many-particle problem are physically meaningless.

8. HIGH EXCITATIONS ($pf_0 \sim 1$) IN A HARD-SPHERE GAS

For the high-energy excitations, the momentum dependence of the amplitudes becomes important. We therefore consider as an example the case of a gas of hard spheres of radius $(a/2)$. We also consider only the first approximation in the density expansion, i.e., we use Eq. (4.6). The amplitude $f(\mathbf{p}\,0)$ can be computed exactly from Eq. (3.10). For $f_s(\mathbf{p}/2\ \mathbf{p}/2)$ we consider only s-waves. The higher waves (the symmetrized amplitude involves only even values of ℓ) add a numerically unimportant contribution. For example the d-waves at $pa \sim 1$ contribute about 10 per cent. We substitute into Eq. (4.7) the values of the amplitudes

$$f(\mathrm{p}0) = 4\pi \frac{\sin pa}{p}; \quad f_s\left(\frac{\mathrm{p}}{2}\frac{\mathrm{p}}{2}\right) = \frac{8\pi}{p} \sin \frac{pa}{2}\, e^{-ipa/2}, \tag{8.1}$$

and obtain for the quasi-particle energy

$$\varepsilon = \left[\left(\frac{p^2}{2} + 8\pi n_0 \frac{\sin pa}{p} - 4\pi n_0 a\right)^2 - 16\pi^2 n_0^2 \frac{\sin^2 pa}{p^2}\right]^{1/2}, \tag{8.2}$$

At high momenta this becomes

$$\varepsilon \approx \frac{p^2}{2} + 4\pi n_0 a \left(2\frac{\sin pa}{pa} - 1\right). \tag{8.3}$$

The second term in Eq. (8.3) changes sign at $pa \approx 1.9$. An oscillating component is superimposed on the usual parabolic dependence. This oscillation will not be important since the magnitude of the term is small; when $pa \sim 1$ it is of relative order $n_0 a^3$. However, if one formally allows the parameter $n_0 a^3$ to become larger in Eqs. (8.3) or (8.2), the second term of Eq. (8.3) produces an increasing departure of the dispersion law from the parabolic form, until at sufficiently high densities there appears first a point of inflection and finally a maximum and a minimum in the curve. The spectrum then resembles qualitatively the spectrum postulated by L. D. Landau[5] to explain the properties of liquid helium II. This extrapolation is certainly unwarranted. But it allows one to suppose that the difference between liquid helium and a non-ideal Bose gas is only a quantitative one, and that no qualitatively new phenomena arise in the transition from gas to liquid.

9. CONCLUSION

We summarize the main features of the approximation which we have studied.

(1) The interaction between particles is specified not by a potential but by an exact scattering amplitude. This allows us to deal with strong interactions. After the potential has been replaced by the amplitude, it is possible to make a perturbation expansion in powers of the amplitude, or more precisely in powers of $\sqrt{n_0 f_0^3}$.

(2) We make a series expansion not of the quasi-particle energy (this appears as the denominator of the Green's function), but of the effective interaction potentials Σ_{ik} and the chemical potential μ. The formula giving the Green's function in terms of Σ_{ik} and μ is exact.

From Eq. (4.7) and (4.9) we see that $\epsilon_\mathbf{p}$ can be expanded in powers of f only for high-momentum excitations with $p \gg \sqrt{n_0 f_0}$. The low-lying excitations of the system are in principle impossible to obtain by perturbation theory. For this reason, the expression obtained by Huang and Yang[4] for the energy of the low excitations of a Bose hard-sphere gas is incorrect. They used perturbation theory with a "pseudopotential," and their result agrees with a formal expansion of Eq. (4.7) in powers of f_0.

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