

QUANTUM STATE OF THE ASYMPTOTIC GRAVITY FIELD FROM BRST QUANTIZATION

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The quantum state of the asymptotic linearized gravity field of a compact energy source is derived using BRST quantization. Barnich has previously derived the quantum state of the $U(1)$ field in QED using the Gauss law constraint. The role of Gauss's law is now played by the linearized Hamiltonian and momentum constraints in gravity. For a static mass source, only the scalar Hamiltonian constraint is shifted. The quantum state corresponding to the static Newton field originates in the non-radiative scalar constraint sector, not in the propagating transverse-traceless graviton sector. Implications for holography and for black hole information are discussed: 1) the exact quantum state of the compact source is encoded in the asymptotic quantum state of its gravity field; 2) Hawking amplitudes are dependent, via the gravity state outside the horizon, on the black hole internal state.

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1. INTRODUCTION

The aim of the paper is to derive the quantum state of the asymptotic linearized gravity field of a compact energy source, using BRST quantization. Barnich [1] (see [2] as well) has previously derived the quantum state of the $U(1)$ field in QED using the Gauss law constraint. We show that the role of Gauss's law is now played by the linearized Hamiltonian and momentum constraints in gravity. For a static mass source, only the scalar Hamiltonian constraint is shifted. The quantum state corresponding to the static Newton field originates in the non-radiative scalar constraint sector, not in the propagating transverse-traceless graviton sector. We discuss implications for holography and for black hole information: 1) the exact quantum state of the

compact source is encoded in the asymptotic quantum state of its gravity field; 2) Hawking amplitudes are dependent, via the gravity state outside the horizon, on the black hole internal state. This work is a non-perturbative (albeit still weak-field) generalization of work deriving quantum hair, i. e. corrections to the potential or metric, using perturbation theory [3–10]. We explicitly construct the quantum state of the gravity field, which was already discussed in general terms in [3].

In QED, Barnich showed that the Coulomb field can be understood as a coherent state of temporal and longitudinal photons. The crucial mechanism is that the external charge shifts Gauss's law, and therefore shifts the BRST (see e. g. [11] for a review) charge itself. The new BRST vacuum is annihilated by a shifted null annihilation operator, and that vacuum is a coherent state built from degrees of freedom in the unphysical sector of non-propagating $U(1)$ modes.

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The linearized gravity problem is closely analogous but not identical. The role of Gauss's law is now played by the linearized Hamiltonian and momentum constraints. For a static mass source, only the scalar Hamiltonian constraint is shifted. The static Newton field therefore lives in the *non-radiative scalar constraint sector*, not in the propagating transverse-traceless graviton sector. That is why the naive idea of shifting free on-shell graviton operators fails: free radiative modes are the wrong degrees of freedom for a time-independent $1/r$ field.

The goal of the paper is to derive the gravity state in the following sequence:

$$\begin{aligned} &\text{sourced constraints} \implies \\ &\implies \text{sourced BRST charge} \implies \\ &\implies \text{shifted null annihilator} \\ &\quad \text{in the scalar sector} \implies \\ &\implies \text{coherent BRST state} \\ &\quad \text{describing the Newton field.} \end{aligned}$$

Throughout, we work in linearized gravity around Minkowski spacetime, with an external conserved static source. We regulate the point mass at intermediate stages by a smooth profile $f_a(\mathbf{x})$ satisfying

$$\int d^3x f_a(\mathbf{x}) = 1, \quad \rho(\mathbf{x}) = M f_a(\mathbf{x}), \quad (1)$$

so that the point-source limit is recovered as $f_a(\mathbf{x}) \rightarrow \delta^{(3)}(\mathbf{x})$.

2. BARNICH'S QED CONSTRUCTION

Here we review the BRST construction in electrodynamics.

2.1. External charge and sourced Gauss law

Consider Maxwell theory coupled to a static point charge Q located at the origin. The action is

$$\begin{aligned} S_Q &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right], \\ j^\mu &= \delta_0^\mu Q \delta^{(3)}(\mathbf{x}). \end{aligned} \quad (2)$$

In the Hamiltonian formulation, the secondary constraint is Gauss's law. In the presence of the source it becomes

$$\varphi_2^Q(\mathbf{x}) = -\partial_i \pi^i(\mathbf{x}) + j^0(\mathbf{x}) = 0. \quad (3)$$

This is the first key structural point: the source does not first show up as a classical field configuration; it shows up *inside the constraint*. Barnich's BRST charge therefore becomes a sourced BRST charge.

2.2. Barnich's sourced BRST charge

Barnich writes the sourced BRST operator in momentum space as

$$\Omega_Q = \int d^3k \left[c^\dagger(\mathbf{k})(a(\mathbf{k}) - q(\mathbf{k})) + (a^\dagger(\mathbf{k}) - q(\mathbf{k}))c(\mathbf{k}) \right], \quad (4)$$

with

$$q(\mathbf{k}) = \frac{Q}{(2\pi)^{3/2} \sqrt{2} k^{3/2}}. \quad (5)$$

Here $a(\mathbf{k})$ is a null annihilation operator built from the longitudinal and temporal oscillators, and $c(\mathbf{k})$ is the ghost operator. The ordinary free BRST vacuum is no longer annihilated by (4), because the source has shifted the operator a to $a - q$.

Barnich therefore defines the shifted annihilator

$$a_Q(\mathbf{k}) := a(\mathbf{k}) - q(\mathbf{k}), \quad (6)$$

and the new vacuum by the condition

$$a_Q(\mathbf{k})|0\rangle_Q = 0. \quad (7)$$

The new vacuum is a coherent state built from the null creation operator $b^\dagger(\mathbf{k})$ conjugate to $a(\mathbf{k})$.

2.3. The null pair and the coherent-state solution

The unphysical sector contains one positive-norm and one negative-norm oscillator. Barnich packages them into a null pair a, b satisfying

$$\begin{aligned} [a(\mathbf{k}), b^\dagger(\mathbf{k}')] &= \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ [a, a^\dagger] &= [b, b^\dagger] = 0. \end{aligned} \quad (8)$$

Now consider the candidate coherent state

$$|0\rangle_Q = \exp \left[\int d^3k q(\mathbf{k}) b^\dagger(\mathbf{k}) \right] |0\rangle. \quad (9)$$

We check explicitly that it solves the shifted annihilation condition. Let

$$X := \int d^3k q(\mathbf{k}) b^\dagger(\mathbf{k}). \quad (10)$$

Then, using the Baker–Campbell–Hausdorff identity,

$$\begin{aligned} a(\mathbf{p})e^X &= e^X \left(a(\mathbf{p}) + [a(\mathbf{p}), X] + \right. \\ &\quad \left. + \frac{1}{2} [[a(\mathbf{p}), X], X] + \cdots \right). \end{aligned} \quad (11)$$

Because $[a(\mathbf{p}), b^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{p} - \mathbf{k})$, we have

$$[a(\mathbf{p}), X] = q(\mathbf{p}), \quad (12)$$

and the next commutator vanishes because $q(\mathbf{p})$ is a c -number. Therefore

$$a(\mathbf{p})e^X = e^X(a(\mathbf{p}) + q(\mathbf{p})). \quad (13)$$

Multiplying both sides by $|0\rangle$ and using $a(\mathbf{p})|0\rangle = 0$, we get

$$(a(\mathbf{p}) - q(\mathbf{p}))|0\rangle_Q = 0. \quad (14)$$

So (9) is indeed the shifted BRST vacuum.

2.4. Why this matters for gravity

The entire gravity construction below is an adaptation of this logic. We will need four ingredients:

- 1) a sourced first-class constraint;
- 2) a BRST charge in which that sourced constraint appears;
- 3) a null annihilation operator $A(\mathbf{k})$ adapted to the relevant sector of the constraint;
- 4) a conjugate null creator $B^\dagger(\mathbf{k})$ that generates the coherent shift.

The only nontrivial new feature in gravity is that the relevant constraint sector is the *scalar Hamiltonian constraint*, not a radiative spin-2 mode.

3. LINEARIZED GRAVITY WITH SOURCE, GAUGE FIXING, AND BRST SYMMETRY

3.1. Field variables and trace reversal

We write

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, & |h_{\mu\nu}| &\ll 1, \\ \eta_{\mu\nu} &= \text{diag}(-1, +1, +1, +1). \end{aligned} \quad (15)$$

Define the trace and trace-reversed field by

$$h = \eta^{\mu\nu} h_{\mu\nu}, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \quad (16)$$

Then

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h, \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \quad (17)$$

3.2. Source coupling

The linear coupling to an external conserved source is

$$S_{\text{int}} = \frac{1}{2} \int d^4x h_{\mu\nu} T^{\mu\nu}, \quad \partial_\mu T^{\mu\nu} = 0. \quad (18)$$

For the regulated static mass source used in this paper we take

$$T^{00}(t, \mathbf{x}) = \rho(\mathbf{x}) = M f_a(\mathbf{x}), \quad T^{0i} = T^{ij} = 0. \quad (19)$$

3.3. Gauge fixing and ghosts

The de Donder gauge condition is

$$C_\nu := \partial^\mu \bar{h}_{\mu\nu} = 0. \quad (20)$$

Introduce a Nakanishi–Lautrup auxiliary field B_μ and ghosts $c_\mu, \bar{c}_\mu$. A convenient gauge-fixed action is

$$\begin{aligned} S = S_{\text{FP}}^{(2)}[h] + \int d^4x & \left[B^\nu C_\nu + \frac{\alpha}{2} B^\nu B_\nu + \right. \\ & \left. + i\bar{c}^\nu \square c_\nu + \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right]. \end{aligned} \quad (21)$$

The linearized BRST transformations are

$$\begin{aligned} s h_{\mu\nu} &= \partial_\mu c_\nu + \partial_\nu c_\mu, & s c_\mu &= 0, \\ s \bar{c}_\mu &= i B_\mu, & s B_\mu &= 0, \end{aligned} \quad (22)$$

where s is the BRST differential. With these conventions the sign of the ghost term is fixed: since $sC_\nu = \square c_\nu$, the variation of $B^\nu C_\nu$ is canceled by the variation of $+i\bar{c}^\nu \square c_\nu$.

3.4. Explicit proof of BRST invariance of the source term

This proof is very short, but it is worth writing out explicitly because it explains why the source shifts the graviton sector while leaving the ghost vacuum untouched. Acting on the source term and using $T^{\mu\nu} = T^{\nu\mu}$, then integrating by parts and dropping the boundary term, we obtain

$$\begin{aligned} s S_{\text{int}} &= \frac{1}{2} \int d^4x (\partial_\mu c_\nu + \partial_\nu c_\mu) T^{\mu\nu} = \\ &= \int d^4x (\partial_\mu c_\nu) T^{\mu\nu} = \\ &= - \int d^4x c_\nu \partial_\mu T^{\mu\nu} = 0, \end{aligned} \quad (23)$$

because the source is conserved. So the source term is BRST invariant. In particular, the source does *not* act as a source for the ghost sector. That means the ghost part of the vacuum remains the same as in the source-free theory.

3.5. Classical sourced equation in de Donder gauge

In de Donder gauge the linearized Einstein equation becomes

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \quad (24)$$

with the sign convention appropriate to our metric signature and source normalization. For the static source

only the 00 component is nonzero. With the mostly-plus metric convention, $T_{00} = T^{00} = \rho$ and $\square = \nabla^2$ on a static field, hence

$$\nabla^2 \bar{h}_{00}^{\text{cl}}(\mathbf{x}) = -16\pi G \rho(\mathbf{x}), \quad \bar{h}_{0i}^{\text{cl}} = \bar{h}_{ij}^{\text{cl}} = 0. \quad (25)$$

Define the regulated Newton potential by

$$\Phi_N(\mathbf{x}) := -GM \int d^3y \frac{f_a(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \quad (26)$$

Using

$$\nabla^2 \left(-\frac{1}{|\mathbf{x} - \mathbf{y}|} \right) = 4\pi \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (27)$$

we obtain

$$\nabla^2 \Phi_N(\mathbf{x}) = 4\pi G \rho(\mathbf{x}). \quad (28)$$

Comparing with the 00 equation above,

$$\bar{h}_{00}^{\text{cl}}(\mathbf{x}) = -4\Phi_N(\mathbf{x}). \quad (29)$$

The corresponding $h_{\mu\nu}$ components are

$$h_{00}^{\text{cl}} = -2\Phi_N, \quad h_{ij}^{\text{cl}} = -2\Phi_N \delta_{ij}, \quad h_{0i}^{\text{cl}} = 0. \quad (30)$$

All of this is classical. The question of this paper is: *what is the corresponding quantum state, derived by BRST methods rather than by simply postulating the shift (30)?*

4. CANONICAL FORM AND THE SOURCED CONSTRAINTS

4.1. ADM split at linear order

To see where the source enters in the BRST charge, we need the canonical constraints. Write the spacetime perturbation in 3 + 1 form:

$$\gamma_{ij} := h_{ij}, \quad n_i := h_{0i}, \quad n := \frac{1}{2}h_{00}. \quad (31)$$

The variable γ_{ij} carries the spatial metric perturbation, while n and n_i play the roles of linearized lapse and shift.

The canonical action has the schematic form

$$S = \int dt d^3x \left[\pi^{ij} \dot{\gamma}_{ij} - p_n \dot{n} - p^i \dot{n}_i - H_{\text{FP}}^{(2)} - n \mathcal{H}_M - n_i \mathcal{H}_M^i \right] + S_{\text{gh}}, \quad (32)$$

where π^{ij} is the momentum conjugate to γ_{ij} , the momenta p_n, p^i enforce the primary constraints, and $\mathcal{H}_M, \mathcal{H}_M^i$ are the sourced secondary constraints.

4.2. The sourced Hamiltonian and momentum constraints

In the convention used in this paper, the unsourced scalar constraint is the linearized spatial Ricci scalar,

$$\mathcal{H} = \partial_i \partial_j \gamma_{ij} - \nabla^2 \gamma, \quad \gamma := \delta^{ij} \gamma_{ij}, \quad (33)$$

$$\mathcal{H}_i = -2\partial_j \pi_i^j. \quad (34)$$

The external mass density shifts only the scalar Hamiltonian constraint:

$$\mathcal{H}_M = \mathcal{H} - 16\pi G \rho, \quad \mathcal{H}_i^M = \mathcal{H}_i. \quad (35)$$

At this point the analogy with QED should be very clear. The charge density j^0 shifted Gauss's law. Here the mass density ρ shifts the Hamiltonian constraint.

4.3. Check that the Hamiltonian constraint reproduces the Newton equation

For the static isotropic Newton configuration compatible with (30),

$$\gamma_{ij} = -2\Phi \delta_{ij}. \quad (36)$$

Then

$$\gamma = \delta^{ij} \gamma_{ij} = -6\Phi, \quad (37)$$

and

$$\partial_i \partial_j \gamma_{ij} = \partial_i \partial_j (-2\Phi \delta_{ij}) = -2\nabla^2 \Phi. \quad (38)$$

Substituting into (33),

$$\mathcal{H} = -2\nabla^2 \Phi - \nabla^2 (-6\Phi) = 4\nabla^2 \Phi. \quad (39)$$

Now impose the sourced constraint (35):

$$0 = \mathcal{H}_M = 4\nabla^2 \Phi - 16\pi G \rho. \quad (40)$$

Dividing by 4 gives

$$\nabla^2 \Phi = 4\pi G \rho. \quad (41)$$

This is exactly (28). So the Newton potential is encoded directly in the sourced Hamiltonian constraint.

This is the central structural observation. The Newton field is not being put in by hand; it is the solution to the sourced scalar constraint.

5. THE BFV-BRST CHARGE FOR THE SOURCED LINEARIZED THEORY

5.1. Minimal and nonminimal variables

The lapse and shift variables n, n_i have primary constraints $p_n \approx 0$ and $p^i \approx 0$. Conservation of these

primary constraints gives the secondary constraints $\mathcal{H}_M \approx 0$ and $\mathcal{H}_i^M \approx 0$. In the linearized theory the first-class constraint algebra is abelian.

Introduce minimal ghosts c, c_i for the secondary constraints, with conjugate ghost momenta $\mathcal{P}, \mathcal{P}_i$, and introduce independent minimal ghosts C, C_i for the primary constraints, with conjugate ghost momenta Π, Π_i . The ghost-number assignments are

$$\begin{aligned} \text{gh}(c) = \text{gh}(c_i) = \text{gh}(C) = \text{gh}(C_i) &= 1, \\ \text{gh}(\mathcal{P}) = \text{gh}(\mathcal{P}_i) = \text{gh}(\Pi) = \text{gh}(\Pi_i) &= -1. \end{aligned} \quad (42)$$

The nonminimal antighost/Nakanishi–Lautrup variables used for gauge fixing may be added in the usual way, but they are not needed to display the source-dependent part of the BRST charge.

5.2. The sourced BFV charge

The minimal BFV-BRST operator is

$$\Omega_M = \int d^3x \left[c \mathcal{H}_M + c_i \mathcal{H}_i^M + C p_n + C_i p^i \right]. \quad (43)$$

This has ghost number +1. Because the linearized constraint algebra is abelian and the source is fixed c -number data, the charge is nilpotent:

$$\Omega_M^2 = 0. \quad (44)$$

Equivalently, the graded Poisson bracket or graded commutator of (43) with itself vanishes. The important point is that the primary-constraint terms are multiplied by their own ghosts C, C_i , not by the ghost momenta conjugate to c, c_i .

The physical state condition is therefore

$$\Omega_M |\Psi_M\rangle = 0. \quad (45)$$

In a fully explicit canonical treatment one also chooses a gauge-fixing fermion Ψ and constructs the gauge-fixed Hamiltonian $H_\Psi = H + \{\Psi, \Omega_M\}$. For the present purpose, however, the essential piece is already visible in (43): the source enters by shifting the Hamiltonian constraint itself.

5.3. Why this already differs from the naive free-mode shift

Equation (43) tells us what must be shifted. The shifted object is not a free radiative graviton annihilation operator. It is the operator content of the *sourced*

scalar constraint. This distinction matters because the static Newton solution is time independent and obeys a Poisson equation, whereas free radiative modes obey wave equations and evolve with factors e^{-ikt} .

That is why the correct Barnich-style question in gravity is not «How do I shift a free graviton mode?», but rather «Which annihilation operator appears in the positive-frequency part of the scalar constraint?». Once that operator is identified, the rest of the Barnich logic goes through exactly.

6. ISOLATING THE SCALAR CONSTRAINT SECTOR

6.1. Scalar decomposition of the spatial metric perturbation

In Fourier space, a general scalar part of γ_{ij} can be written as

$$\gamma_{ij}^S(\mathbf{k}) = \alpha(\mathbf{k}) \delta_{ij} + \beta(\mathbf{k}) \left(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right), \quad (46)$$

$$\hat{k}_i := \frac{k_i}{k}.$$

The first term is isotropic. The second is the traceless scalar built from $\mathbf{k}$.

Let us compute the Fourier transform of the Hamiltonian constraint explicitly. In momentum space, every spatial derivative ∂_i becomes multiplication by ik_i , and therefore

$$\begin{aligned} \partial_i \partial_j \gamma_{ij}(\mathbf{x}) &\longrightarrow -k_i k_j \gamma_{ij}(\mathbf{k}), \\ \nabla^2 \gamma(\mathbf{x}) &\longrightarrow -k^2 \gamma(\mathbf{k}). \end{aligned} \quad (47)$$

From (46), the trace is

$$\gamma(\mathbf{k}) = \delta^{ij} \gamma_{ij}^S(\mathbf{k}) = 3\alpha(\mathbf{k}), \quad (48)$$

because the traceless term has zero trace. Also,

$$\begin{aligned} k_i k_j \gamma_{ij}^S(\mathbf{k}) &= k_i k_j \left[\alpha \delta_{ij} + \beta \left(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) \right] = \\ &= \alpha k_i k_i + \beta \left[k_i k_j \hat{k}_i \hat{k}_j - \frac{1}{3} k_i k_j \delta_{ij} \right] = \\ &= \alpha k^2 + \beta \left[k^2 - \frac{1}{3} k^2 \right] = \alpha k^2 + \frac{2}{3} \beta k^2. \end{aligned} \quad (49)$$

Substituting into (33),

$$\begin{aligned} \mathcal{H}(\mathbf{k}) &= (-k_i k_j \gamma_{ij}(\mathbf{k})) - (-k^2 \gamma(\mathbf{k})) = \\ &= -k_i k_j \gamma_{ij}(\mathbf{k}) + k^2 \gamma(\mathbf{k}) = \\ &= -\left(\alpha k^2 + \frac{2}{3} \beta k^2 \right) + 3\alpha k^2 = 2\alpha k^2 - \frac{2}{3} \beta k^2. \end{aligned} \quad (50)$$

It is convenient to package the specific scalar combination entering the constraint into a single variable,

$$\chi(\mathbf{k}) := -\alpha(\mathbf{k}) + \frac{1}{3}\beta(\mathbf{k}). \quad (51)$$

Then the constraint becomes

$$\mathcal{H}(\mathbf{k}) = -2k^2\chi(\mathbf{k}). \quad (52)$$

This is the unique scalar combination that the Hamiltonian constraint sees.

6.2. The isotropic Newton configuration as a special case

For the isotropic Newton field, $\beta = 0$ and $\alpha = -2\Phi$. Therefore

$$\chi = 2\Phi, \quad \mathcal{H}(\mathbf{k}) = -2k^2(2\Phi) = -4k^2\Phi(\mathbf{k}). \quad (53)$$

Since the sourced constraint says $\mathcal{H}(\mathbf{k}) = 16\pi G\rho(\mathbf{k})$, we recover

$$-4k^2\Phi(\mathbf{k}) = 16\pi G\rho(\mathbf{k}), \quad \text{so} \quad \Phi(\mathbf{k}) = -\frac{4\pi G\rho(\mathbf{k})}{k^2}. \quad (54)$$

This is the momentum-space Newton potential.

7. THE NULL SCALAR OSCILLATOR BASIS

7.1. Canonical normalization of the covariant graviton oscillators

The scalar null pair should not be introduced by fiat. It follows from the canonical oscillator algebra of the gauge-fixed linearized graviton. To make the dimensions explicit, write the dimensionless metric perturbation as

$$h_{\mu\nu} = \kappa\psi_{\mu\nu}, \quad \kappa^2 = 32\pi G, \quad (55)$$

where $\psi_{\mu\nu}$ is the canonically normalized spin-2 field. In de Donder gauge, the positive-frequency part of the spatial field may be expanded as

$$\psi_{ij}^{(+)}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{a_{ij}(\mathbf{k})}{\sqrt{2k}} e^{-ikt + i\mathbf{k}\cdot\mathbf{x}}. \quad (56)$$

The spatial oscillator algebra is

$$[a_{ij}(\mathbf{k}), a_{\ell m}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') P_{ij, \ell m}, \quad (57)$$

with

$$P_{ij, \ell m} = \frac{1}{2} (\delta_{i\ell} \delta_{jm} + \delta_{im} \delta_{j\ell} - \delta_{ij} \delta_{\ell m}). \quad (58)$$

Equivalently, for two symmetric spatial tensors s_{ij} and t_{ij} ,

$$[s^{ij} a_{ij}(\mathbf{k}), t^{\ell m} a_{\ell m}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \langle s, t \rangle_{\text{DW}}, \quad (59)$$

where

$$\langle s, t \rangle_{\text{DW}} = s^{ij} t_{ij} - \frac{1}{2} s^i_i t^j_j. \quad (60)$$

This indefinite DeWitt metric is the source of the scalar null pair. No additional assumption about positive- and negative-norm scalar oscillators is needed.

7.2. Scalar projectors and the null direction selected by the constraint

For fixed nonzero $\mathbf{k}$, define

$$\ell_{ij} := \hat{k}_i \hat{k}_j, \quad \theta_{ij} := \delta_{ij} - \hat{k}_i \hat{k}_j. \quad (61)$$

The traces and contractions are

$$\begin{aligned} \theta^i_i &= 2, & \ell^i_i &= 1, & \theta_{ij} \ell^{ij} &= 0, \\ \theta_{ij} \theta^{ij} &= 2, & \ell_{ij} \ell^{ij} &= 1. \end{aligned} \quad (62)$$

Using (60), one obtains

$$\begin{aligned} \langle \theta, \theta \rangle_{\text{DW}} &= 0, & \langle \ell, \ell \rangle_{\text{DW}} &= \frac{1}{2}, \\ \langle \theta, \ell \rangle_{\text{DW}} &= -1. \end{aligned} \quad (63)$$

Thus the transverse scalar tensor θ_{ij} is itself a null polarization of the gauge-fixed spatial graviton sector.

Now compute the positive-frequency part of the linearized Hamiltonian constraint. With the convention used in (33),

$$\mathcal{H} = \partial_i \partial_j h_{ij} - \nabla^2 h^i_i. \quad (64)$$

Since $h_{ij} = \kappa\psi_{ij}$, equations (56) and (64) give

$$\begin{aligned} \mathcal{H}^{(+)}(\mathbf{k}) &= \frac{\kappa}{\sqrt{2k}} (-k_i k_j + k^2 \delta_{ij}) a_{ij}(\mathbf{k}) = \\ &= \frac{\kappa k^{3/2}}{\sqrt{2}} \theta^{ij} a_{ij}(\mathbf{k}). \end{aligned} \quad (65)$$

It is therefore natural to define the active scalar annihilation operator by

$$A(\mathbf{k}) := \theta^{ij} a_{ij}(\mathbf{k}). \quad (66)$$

Then

$$\mathcal{H}^{(+)}(\mathbf{k}) = \frac{\kappa k^{3/2}}{\sqrt{2}} A(\mathbf{k}). \quad (67)$$

This replaces the earlier normalization-by-definition. The factor of κ is required by canonical normalization

and is the origin of the physically correct $\sqrt{G}$ scaling of the coherent-state amplitude.

Because θ is null in the DeWitt metric,

$$[A(\mathbf{k}), A^\dagger(\mathbf{k}')] = 0. \quad (68)$$

Thus the operator that appears in the positive-frequency part of the scalar Hamiltonian constraint is a null annihilation operator.

7.3. The conjugate null partner

We now construct a conjugate null oscillator $B(\mathbf{k})$ satisfying

$$\begin{aligned} [A(\mathbf{k}), B^\dagger(\mathbf{k}')] &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ [B(\mathbf{k}), B^\dagger(\mathbf{k}')] &= 0. \end{aligned} \quad (69)$$

Let

$$B(\mathbf{k}) = - \left(\frac{1}{4} \theta^{ij} + \ell^{ij} \right) a_{ij}(\mathbf{k}). \quad (70)$$

Then, using (66), (59) and (63),

$$\begin{aligned} [A(\mathbf{k}), B^\dagger(\mathbf{k}')] &= -(2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \times \\ &\times \left\langle \theta, \frac{1}{4} \theta + \ell \right\rangle_{\text{DW}} = \\ &= -(2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') (0 - 1) = \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \end{aligned} \quad (71)$$

Similarly,

$$\begin{aligned} \left\langle \frac{1}{4} \theta + \ell, \frac{1}{4} \theta + \ell \right\rangle_{\text{DW}} &= \\ &= \frac{1}{16} \langle \theta, \theta \rangle_{\text{DW}} + \frac{1}{2} \langle \theta, \ell \rangle_{\text{DW}} + \langle \ell, \ell \rangle_{\text{DW}} = \\ &= 0 - \frac{1}{2} + \frac{1}{2} = 0, \end{aligned} \quad (72)$$

so

$$[B(\mathbf{k}), B^\dagger(\mathbf{k}')] = 0. \quad (73)$$

Hence the scalar constraint sector contains the null pair

$$\begin{aligned} [A, A^\dagger] &= 0, \quad [B, B^\dagger] = 0, \\ [A(\mathbf{k}), B^\dagger(\mathbf{k}')] &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \end{aligned} \quad (74)$$

This is the gravity analogue of Barnich's longitudinal-temporal null pair in QED, now obtained directly from the gauge-fixed canonical graviton algebra.

7.4. Relation to positive- and negative-norm scalar oscillators

The previous algebra can be put into the more familiar Barnich form by defining

$$a_+(\mathbf{k}) := \frac{A(\mathbf{k}) + B(\mathbf{k})}{\sqrt{2}}, \quad a_-(\mathbf{k}) := \frac{A(\mathbf{k}) - B(\mathbf{k})}{\sqrt{2}}. \quad (75)$$

Then (74) implies

$$\begin{aligned} [a_+(\mathbf{k}), a_+^\dagger(\mathbf{k}')] &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ [a_-(\mathbf{k}), a_-^\dagger(\mathbf{k}')] &= -(2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \end{aligned} \quad (76)$$

with mixed commutators vanishing. Conversely,

$$A = \frac{a_+ + a_-}{\sqrt{2}}, \quad B = \frac{a_+ - a_-}{\sqrt{2}}. \quad (77)$$

Thus the positive- and negative-norm scalar oscillators are not an independent postulate. They are just the diagonal basis of the two-dimensional scalar subspace spanned by θ_{ij} and ℓ_{ij} under the DeWitt metric.

7.5. Hermitian constraint operator

With the conventions above, the full Hermitian scalar constraint operator is

$$\mathcal{H}(\mathbf{k}) = \frac{\kappa k^{3/2}}{\sqrt{2}} (A(\mathbf{k}) + A^\dagger(-\mathbf{k})). \quad (78)$$

For a static source the coherent amplitudes may be chosen real and even in $\mathbf{k}$, so the distinction between $\mathbf{k}$ and $-\mathbf{k}$ will not affect the following formulae.

8. SOLVING THE SOURCED BRST CONDITION

8.1. The scalar part of the sourced BRST charge

Once the scalar null pair has been derived, the scalar part of the sourced BRST charge takes the Barnich form

$$\begin{aligned} \Omega_M^{(S)} &= \int \frac{d^3 k}{(2\pi)^3} \left[c^\dagger(\mathbf{k}) (A(\mathbf{k}) - q_M(\mathbf{k})) + \right. \\ &\quad \left. + (A^\dagger(\mathbf{k}) - q_M(\mathbf{k})) c(\mathbf{k}) \right], \end{aligned} \quad (79)$$

where the c -number $q_M(\mathbf{k})$ is fixed by the sourced constraint. The full BRST state will be obtained by tensoring this scalar-sector state with the vacuum of the vector, transverse-traceless, ghost, and nonminimal sectors.

8.2. Determining $q_M(\mathbf{k})$ from the sourced constraint

By construction, the coherent BRST state should satisfy

$$\langle \mathcal{H}(\mathbf{k}) \rangle_{M,a} = 16\pi G \rho(\mathbf{k}) = 16\pi G M \tilde{f}_a(\mathbf{k}), \quad (80)$$

where the subscript reminds us that the source is regulated by f_a .

Using (78), the left-hand side is

$$\langle \mathcal{H}(\mathbf{k}) \rangle_{M,a} = \frac{\kappa k^{3/2}}{\sqrt{2}} \langle A(\mathbf{k}) + A^\dagger(-\mathbf{k}) \rangle_{M,a}. \quad (81)$$

In the coherent state constructed below, the expectation values of A and $A^\dagger$ are both equal to q_M for real even amplitudes. Therefore

$$\langle \mathcal{H}(\mathbf{k}) \rangle_{M,a} = \sqrt{2} \kappa k^{3/2} q_M(\mathbf{k}). \quad (82)$$

Comparing with (80), one obtains

$$q_M(\mathbf{k}) = \frac{16\pi GM \tilde{f}_a(\mathbf{k})}{\sqrt{2} \kappa k^{3/2}} = 2\sqrt{\pi G} \frac{M \tilde{f}_a(\mathbf{k})}{k^{3/2}}. \quad (83)$$

The exact sign of q_M follows the sign convention chosen for the constraint and for A . The important invariant point is the scaling $q_M \sim \sqrt{GM}/k^{3/2}$, not $GM/k^{3/2}$.

8.3. The coherent-state solution

Define the shifted annihilator

$$A_M(\mathbf{k}) := A(\mathbf{k}) - q_M(\mathbf{k}). \quad (84)$$

We now solve the condition

$$A_M(\mathbf{k})|M; a\rangle_S = 0 \quad \text{for all } \mathbf{k}. \quad (85)$$

Take the scalar vacuum $|0\rangle_S$ annihilated by $A(\mathbf{k})$ and $B(\mathbf{k})$, and define

$$|M; a\rangle_S := \exp\left[\int \frac{d^3k}{(2\pi)^3} q_M(\mathbf{k}) B^\dagger(\mathbf{k})\right] |0\rangle_S. \quad (86)$$

Exactly as in the QED warm-up,

$$\begin{aligned} A(\mathbf{p})e^X &= e^X(A(\mathbf{p}) + q_M(\mathbf{p})), \\ X &:= \int \frac{d^3k}{(2\pi)^3} q_M(\mathbf{k}) B^\dagger(\mathbf{k}), \end{aligned} \quad (87)$$

because (74) gives $[A(\mathbf{p}), X] = q_M(\mathbf{p})$ and the second commutator vanishes. Therefore

$$(A(\mathbf{p}) - q_M(\mathbf{p}))|M; a\rangle_S = 0. \quad (88)$$

So (86) is the desired scalar-sector BRST coherent state.

The full sourced BRST state is then

$$\begin{aligned} |M; a\rangle_{\text{BRST}} &= |M; a\rangle_S \otimes |0\rangle_V \otimes \\ &\quad \otimes |0\rangle_{\text{TT}} \otimes |0\rangle_{\text{gh}} \otimes |0\rangle_{\text{nonmin}}. \end{aligned} \quad (89)$$

This makes the physical interpretation precise: the source shifts the scalar constraint sector while the propagating transverse-traceless gravitons remain in their vacuum.

9. COMPUTING THE NEWTON POTENTIAL FROM THE BRST STATE

9.1. Expectation value of the constraint

By construction,

$$\langle \mathcal{H}(\mathbf{k}) \rangle_{M,a} = 16\pi GM \tilde{f}_a(\mathbf{k}). \quad (90)$$

The Fourier-space relation (53) for the isotropic Newton sector gives

$$\mathcal{H}(\mathbf{k}) = -4k^2 \Phi(\mathbf{k}). \quad (91)$$

Therefore

$$-4k^2 \langle \Phi(\mathbf{k}) \rangle_{M,a} = 16\pi GM \tilde{f}_a(\mathbf{k}), \quad (92)$$

and hence

$$\langle \Phi(\mathbf{k}) \rangle_{M,a} = -\frac{4\pi GM \tilde{f}_a(\mathbf{k})}{k^2}. \quad (93)$$

This is exactly the regulated momentum-space Newton potential.

9.2. Transforming back to position space

Using the Fourier convention

$$\begin{aligned} f(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k}), \\ \tilde{f}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}), \end{aligned} \quad (94)$$

we have the standard identity

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}. \quad (95)$$

Applying this to (93),

$$\begin{aligned} \langle \Phi(\mathbf{x}) \rangle_{M,a} &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \langle \Phi(\mathbf{k}) \rangle_{M,a} = \\ &= -4\pi GM \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\tilde{f}_a(\mathbf{k})}{k^2} = \\ &= -4\pi GM \int d^3y f_a(\mathbf{y}) \times \\ &\quad \times \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} = \\ &= -GM \int d^3y \frac{f_a(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \end{aligned} \quad (96)$$

This is exactly (26).

In the point-source limit $f_a \rightarrow \delta^{(3)}$, we obtain

$$\langle \Phi(\mathbf{x}) \rangle_M \rightarrow -\frac{GM}{r}, \quad r := |\mathbf{x}|. \quad (97)$$

So the BRST coherent state (89) carries the Newton field.

9.3. Metric expectation values

Using (30), we therefore have

$$\begin{aligned}\langle h_{00}(\mathbf{x}) \rangle_{M,a} &= -2\langle \Phi(\mathbf{x}) \rangle_{M,a}, \\ \langle h_{ij}(\mathbf{x}) \rangle_{M,a} &= -2\langle \Phi(\mathbf{x}) \rangle_{M,a} \delta_{ij}, \\ \langle h_{0i}(\mathbf{x}) \rangle_{M,a} &= 0.\end{aligned}\quad (98)$$

Meanwhile,

$$\langle h_{ij}^{\text{TT}}(\mathbf{x}) \rangle_{M,a} = 0, \quad (99)$$

because the TT sector was not shifted at all. This is an operator-theoretic way of stating that Newton's law comes from the non-radiative scalar constraint sector, not from a cloud of TT gravitons.

10. WHY THIS IS NOT JUST A SHIFT OF THE CLASSICAL BACKGROUND

At first sight the final answer may look similar to the familiar displaced-vacuum picture of a quadratic field theory. Indeed, once the classical sourced solution $\bar{h}_{\mu\nu}^{\text{cl}}$ is known, one may write a formal translation operator

$$U_M = \exp \left[-i \int d^3x \bar{h}_{\mu\nu}^{\text{cl}}(\mathbf{x}; M) \Pi_h^{\mu\nu}(\mathbf{x}) \right], \quad (100)$$

and then the source-dependent vacuum can be written as

$$|\Omega_M\rangle = U_M |0\rangle. \quad (101)$$

However, that is not how we derived the state in this paper. Our logic was:

- 1) identify the sourced Hamiltonian constraint;
- 2) build the sourced BRST charge;
- 3) solve the sourced BRST annihilation condition;
- 4) obtain the coherent state in the scalar null sector;
- 5) and only then recognize that the result is equivalent to (101).

This order matters conceptually. The displaced-vacuum formula (101) is a useful repackaging of the answer, but it does not explain *which sector* has been shifted or *why* the Newton field is not made of radiative gravitons. The BRST derivation does explain that, because it reveals that the source sits directly in the scalar Hamiltonian constraint.

11. POINT-SOURCE LIMIT AND GENERALIZED STATES

For smooth regulator f_a , the state (89) is a perfectly good Gaussian-coherent state of the linearized theory.

In the ideal point-source limit, however, two familiar singular features appear:

- 1) a short-distance singularity at $r = 0$, coming from the behavior $\Phi \sim -GM/r$;
- 2) a long-range infrared tail, because the field falls only as $1/r$.

These features mean that the exact point-mass state is best regarded as a generalized coherent state or algebraic state rather than as a normalizable vector in the original free Fock space. This is the gravity analogue of the infrared subtleties already familiar from QED.

12. SUMMARY OF RESULTS

Let us collect the central formulae in one place.

Step 1: sourced constraint

For a static regulated mass density $\rho(\mathbf{x}) = M f_a(\mathbf{x})$,

$$\begin{aligned}\mathcal{H}_M(\mathbf{x}) &= \mathcal{H}(\mathbf{x}) - 16\pi G \rho(\mathbf{x}), \\ \mathcal{H}(\mathbf{x}) &= \partial_i \partial_j \gamma_{ij} - \nabla^2 \gamma.\end{aligned}\quad (102)$$

Step 2: sourced BRST charge

The corresponding BFM charge is

$$\begin{aligned}\Omega_M &= \int d^3x \left[c \mathcal{H}_M + c_i \mathcal{H}_i^M + \right. \\ &\quad \left. + C p_n + C_i p^i \right].\end{aligned}\quad (103)$$

Step 3: scalar null mode

The scalar null pair is derived from the gauge-fixed canonical graviton algebra. With $h_{ij} = \kappa \psi_{ij}$ and $\kappa^2 = 32\pi G$,

$$\begin{aligned}A(\mathbf{k}) &= \theta^{ij} a_{ij}(\mathbf{k}), \\ B(\mathbf{k}) &= - \left(\frac{1}{4} \theta^{ij} + \ell^{ij} \right) a_{ij}(\mathbf{k}),\end{aligned}\quad (104)$$

where $\theta_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j$ and $\ell_{ij} = \hat{k}_i \hat{k}_j$. These satisfy

$$\begin{aligned}[A(\mathbf{k}), B^\dagger(\mathbf{k}')] &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ [A, A^\dagger] &= [B, B^\dagger] = 0,\end{aligned}\quad (105)$$

and the constraint operator is

$$\mathcal{H}^{(+)}(\mathbf{k}) = \frac{\kappa k^{3/2}}{\sqrt{2}} A(\mathbf{k}). \quad (106)$$

Step 4: coherent amplitude

The sourced BRST condition fixes

$$q_M(\mathbf{k}) = 2\sqrt{\pi G} \frac{M \tilde{f}_a(\mathbf{k})}{k^{3/2}}, \quad (107)$$

up to the overall sign convention.

Step 5: Newton state

The scalar-sector state is

$$|M; a\rangle_S = \exp \left[\int \frac{d^3 k}{(2\pi)^3} q_M(\mathbf{k}) B^\dagger(\mathbf{k}) \right] |0\rangle_S, \quad (108)$$

and the full source-dependent BRST state is

$$|M; a\rangle_{\text{BRST}} = |M; a\rangle_S \otimes |0\rangle_V \otimes |0\rangle_{\text{TT}} \otimes |0\rangle_{\text{gh}} \otimes |0\rangle_{\text{nonmin}}. \quad (109)$$

Step 6: Newton potential

This state satisfies

$$\langle \Phi(\mathbf{k}) \rangle_{M,a} = -\frac{4\pi G M \tilde{f}_a(\mathbf{k})}{k^2} \quad (110)$$

and therefore

$$\begin{aligned} \langle \Phi(\mathbf{x}) \rangle_{M,a} &= -GM \int d^3 y \frac{f_a(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \rightarrow \\ &\rightarrow -\frac{GM}{r} \quad (f_a \rightarrow \delta^{(3)}). \end{aligned} \quad (111)$$

This is the Barnich-style BRST derivation of the Newton state in linearized gravity.

13. DRESSING NORM AND CORPUSCULAR SCALING

The coherent state constructed above describes the static Newton field as a displacement in the non-radiative scalar constraint sector of the unreduced BRST Hilbert space. It is therefore important not to confuse this state with a coherent state of propagating transverse-traceless gravitons. The transverse-traceless sector remains in its vacuum,

$$\langle h_{ij}^{\text{TT}}(x) \rangle_{M,a} = 0, \quad (112)$$

and the Newtonian $1/r$ field is carried instead by the scalar constraint-sector dressing.

Nevertheless, one may ask whether the coherent dressing has a useful quantitative size. In ordinary positive-norm coherent states, the squared coherent

amplitude is naturally interpreted as a particle occupation number. In the present case the relevant operators are null operators in an indefinite-metric space, so no gauge-invariant graviton-number interpretation is available. However, the integral of the squared displacement amplitude still provides a useful diagnostic of the size of the unreduced dressing. We shall call this quantity the dressing norm, or more cautiously the dressing mode-count,

$$\mathcal{N}_{\text{dress}} := \int \frac{d^3 k}{(2\pi)^3} |q_M(k)|^2. \quad (113)$$

This is not an observable by itself. It depends on the choice of BRST representative, the infrared regulator, and the ultraviolet smearing of the source. Its role is only to measure the size of the coherent displacement needed to reproduce the classical Newton field.

Using the amplitude derived above,

$$q_M(k) = \frac{2\sqrt{\pi G} M \tilde{f}_a(k)}{k^{3/2}}, \quad (114)$$

we find

$$|q_M(k)|^2 = \frac{4\pi G M^2 |\tilde{f}_a(k)|^2}{k^3}. \quad (115)$$

Therefore

$$\mathcal{N}_{\text{dress}} = 4\pi G M^2 \int \frac{d^3 k}{(2\pi)^3} \frac{|\tilde{f}_a(k)|^2}{k^3}. \quad (116)$$

For a compact source whose profile suppresses momenta above $\Lambda \sim a^{-1}$, and for an infrared cutoff $\mu \sim L^{-1}$, the leading behaviour is

$$\mathcal{N}_{\text{dress}} \simeq 4\pi G M^2 \frac{4\pi}{(2\pi)^3} \int_{\mu}^{\Lambda} \frac{dk}{k}. \quad (117)$$

Thus

$$\mathcal{N}_{\text{dress}} \simeq \frac{2GM^2}{\pi} \log \left(\frac{\Lambda}{\mu} \right), \quad (118)$$

up to convention-dependent numerical factors. The robust statement is therefore the scaling

$$\mathcal{N}_{\text{dress}} \sim GM^2 \log \left(\frac{\Lambda}{\mu} \right) = \left(\frac{M}{M_{\text{P}}} \right)^2 \log \left(\frac{\Lambda}{\mu} \right), \quad (119)$$

where $M_{\text{P}}^{-2} = G$ in the convention used here.

This logarithmic behaviour has a simple origin. The Newton potential behaves as $1/r$, so in momentum space the field scales as $1/k^2$. The corresponding coherent displacement scales as

$$q_M(k) \sim \frac{\sqrt{G} M}{k^{3/2}}, \quad (120)$$

and the measure d^3k then produces a logarithmic integral,

$$d^3k |q_M(k)|^2 \sim k^2 dk \frac{GM^2}{k^3} \sim GM^2 \frac{dk}{k}. \quad (121)$$

The logarithmic infrared sensitivity is therefore a direct consequence of the long-range nature of the Newton field.

For a compact object of size R , a natural infrared cutoff is of order $\mu \sim R_{\text{IR}}^{-1}$, where R_{IR} is the scale at which the asymptotic field is effectively measured or enclosed. The ultraviolet cutoff is set either by the inverse size of the source, $\Lambda \sim a^{-1}$, or by the scale at which the linearized description breaks down. For a black hole one might formally take the characteristic gravitational scale to be $R_S = 2GM$, but then one should not interpret the above linearized calculation as a controlled derivation of a black-hole condensate. At most, it shows that the exterior Newtonian dressing has the same parametric mass dependence often emphasized in corpuscular descriptions [12–17],

$$\mathcal{N}_{\text{dress}} \sim \left(\frac{M}{M_{\text{P}}} \right)^2, \quad (122)$$

modulo the logarithmic dependence on the chosen infrared and ultraviolet regulators.

This result should be distinguished from the occupation number of physical gravitons. A static Newton field does not correspond to a collection of on-shell transverse-traceless quanta. The TT number operator annihilates the unshifted TT vacuum sector of the state constructed in this paper. Thus any literal statement such as

$$N_{\text{phys}} = \int \frac{d^3k}{(2\pi)^3} a_{ij}^{\text{TT}\dagger}(k) a_{ij}^{\text{TT}}(k) \quad (123)$$

would give no coherent TT population for the static field. The nonzero quantity above is instead a measure of the scalar constraint-sector dressing in the unreduced BRST description.

This distinction is essential. The coherent state derived in this work is physical only through its BRST-invariant content and through gauge-invariant observables such as the Newton potential, curvature components, and boundary charges. The dressing mode-count $\mathcal{N}_{\text{dress}}$ is therefore best understood as a useful diagnostic of semiclassicality: large GM^2 means that the classical Newton field is represented by a large coherent displacement in the constrained sector. It is not, by itself, a gauge-invariant particle number.

The connection with corpuscular gravity should therefore be stated in the following limited sense. The

BRST coherent-state construction reproduces the same parametric scaling,

$$\mathcal{N}_{\text{dress}} \sim \left(\frac{M}{M_{\text{P}}} \right)^2, \quad (124)$$

which is often associated with the number of soft gravitational quanta in a classical field. However, in the present derivation these modes are not propagating TT gravitons; they are the non-radiative scalar modes required by the gravitational constraints. The result supports the broader intuition that a macroscopic gravitational field corresponds to a large quantum displacement, but it does not by itself establish a gauge-invariant graviton condensate picture.

In summary, the static Newton field carries a large coherent BRST dressing whose size scales as

$$\mathcal{N}_{\text{dress}} \sim GM^2 \log \left(\frac{\Lambda}{\mu} \right). \quad (125)$$

This scaling is physically meaningful as a measure of the semiclassical size of the gravitational dressing, but not as a literal count of physical on-shell gravitons. The central conclusion of the paper is therefore preserved: the Newton field is encoded in the scalar constraint sector, while the propagating transverse-traceless graviton sector remains unexcited.

14. CONCLUSION: ASYMPTOTIC GRAVITY STATES, HOLOGRAPHY, AND BLACK HOLE INFORMATION

We have derived the quantum state of the asymptotic linearized gravitational field of a compact static source by imposing the sourced BRST condition. The central result is that the static Newton field is not a coherent state of propagating transverse-traceless gravitons. It is a coherent state in the non-radiative scalar constraint sector of the unreduced BRST Hilbert space. The relevant source appears in the Hamiltonian constraint,

$$\mathcal{H}_M(x) = \mathcal{H}(x) - 16\pi G\rho(x), \quad (126)$$

and therefore shifts the scalar null oscillator $A(\mathbf{k})$ appearing in the positive-frequency part of the constraint:

$$A(\mathbf{k}) \longrightarrow A(\mathbf{k}) - q_M(\mathbf{k}). \quad (127)$$

For a regulated compact source of total energy E , the resulting BRST state has the schematic form

$$|g(E); a\rangle_{\text{BRST}} = \exp \left[\int \frac{d^3k}{(2\pi)^3} q_E(\mathbf{k}) B^\dagger(\mathbf{k}) \right] |0\rangle_{\text{BRST}}, \quad (128)$$

with

$$q_E(\mathbf{k}) = 2\sqrt{\pi G} \frac{E \tilde{f}_a(\mathbf{k})}{k^{3/2}}, \quad (129)$$

up to the sign and normalization conventions fixed in the main text. This state satisfies

$$\langle g(E); a | \mathcal{H}(\mathbf{k}) | g(E); a \rangle = 16\pi G E \tilde{f}_a(\mathbf{k}), \quad (130)$$

and hence

$$\langle \Phi(\mathbf{k}) \rangle_E = -\frac{4\pi G E \tilde{f}_a(\mathbf{k})}{k^2}. \quad (131)$$

In the point-source limit this becomes the ordinary Newton potential,

$$\langle \Phi(\mathbf{x}) \rangle_E \longrightarrow -\frac{GE}{r}. \quad (132)$$

Now consider a compact quantum source prepared in a superposition of energy eigenstates, with some spread in energies concentrated around a semiclassical value:

$$|\psi\rangle_{\text{src}} = \sum_n c_n |E_n\rangle_{\text{src}}. \quad (133)$$

The BRST construction associates to each energy eigenvalue E_n a corresponding asymptotic scalar gravity state,

$$|E_n\rangle_{\text{src}} \longmapsto |g(E_n); a\rangle_{\text{BRST}}. \quad (134)$$

Thus the joint source-gravity state is

$$|\Psi\rangle = \sum_n c_n |E_n\rangle_{\text{src}} \otimes |g(E_n); a\rangle_{\text{BRST}}. \quad (135)$$

If we focus on the asymptotic gravitational state itself, suppressing the source label, we obtain

$$|\Psi\rangle_{\text{grav}} = \sum_n c_n |g(E_n); a\rangle_{\text{BRST}}. \quad (136)$$

This is the gravity analogue of the exterior-geometry state written by Calmet and Hsu [4] as

$$|\Psi\rangle = \sum_n c_n |g(E_n)\rangle. \quad (137)$$

The map from source energies to asymptotic gravity states is already visible at the semiclassical level:

$$E_n \longmapsto \langle \Phi(\mathbf{x}) \rangle_{E_n} = -GE_n \int d^3y \frac{f_a(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \quad (138)$$

Equivalently, the ADM energy can be read from the $1/r$ coefficient of the asymptotic potential:

$$E_n = -\frac{1}{G} \lim_{r \rightarrow \infty} r \langle \Phi(r) \rangle_{E_n}. \quad (139)$$

In operator language, the boundary Hamiltonian is the surface term associated with the gravitational constraint. At spatial infinity it gives

$$H_\partial = -\frac{1}{4\pi G} \lim_{r \rightarrow \infty} \int_{S_r^2} d\Omega r^2 \partial_r \Phi, \quad (140)$$

so that

$$H_\partial |g(E_n); a\rangle_{\text{BRST}} = E_n |g(E_n); a\rangle_{\text{BRST}}. \quad (141)$$

Thus the energy decomposition of the compact source is encoded directly in the asymptotic gravity state.

More generally, suppose the distinct asymptotic gravity states

$$|g_n\rangle \equiv |g(E_n); a\rangle_{\text{BRST}} \quad (142)$$

are linearly independent. Define their Gram matrix

$$G_{mn} = \langle g_m | g_n \rangle. \quad (143)$$

On the subspace spanned by these states, the dual basis is

$$\langle \tilde{g}^n | = \sum_m (G^{-1})^{nm} \langle g_m |. \quad (144)$$

Then the coefficients of the original source state can be reconstructed from the asymptotic gravity state by

$$c_n = \langle \tilde{g}^n | \Psi \rangle_{\text{grav}}. \quad (145)$$

At leading order in the static Newton field, this reconstruction distinguishes energy eigenvalues. If there are exact degeneracies among compact-source states with the same E_n , the leading $1/r$ field alone is insufficient. However, the full quantum gravitational dressing is not exhausted by the leading Newton potential. Quantum corrections to the exterior gravitational field, including loop-induced long-distance terms and higher multipole/gravitational dressing data, can depend on the internal quantum state of the compact source. In that sense the complete asymptotic gravitational state can carry more information than the classical no-hair data.

This conclusion is suggestive of holography [18]. The information about the compact source is not localized purely in the interior matter degrees of freedom. The gravitational constraints tie the bulk state to asymptotic boundary data. In the present linearized calculation this statement is concrete: the Hamiltonian constraint fixes the scalar BRST coherent state at infinity, and the coefficient of that state determines the source energy. In a complete theory of quantum gravity, this is the mechanism by which boundary observables can encode bulk information. The result is not yet

a full derivation of holography, but it exhibits the same structural feature: physical information about the bulk is redundantly available in the gravitational field at the boundary.

The same logic has direct implications for black hole evaporation. Let the initial black hole state be a superposition of energy eigenstates,

$$|\psi\rangle_{\text{BH}} = \sum_n c_n |E_n\rangle_{\text{BH}}. \quad (146)$$

The exterior gravitational field is then

$$|\Psi\rangle_{\text{ext}} = \sum_n c_n |g(E_n)\rangle, \quad (147)$$

where $|g(E_n)\rangle$ now denotes the quantum state of the exterior geometry, including the scalar constraint-sector dressing derived in this paper. The emission amplitude for a Hawking quantum depends on the exterior geometry state. Following the notation of Calmet and Hsu [4], denote the full set of quantum numbers of the i -th emitted particle by

$$r_i \sim \{\omega_i, \mathbf{p}_i, s_i, q_i, \dots\}, \quad (148)$$

and write the amplitude for emission of r_i from an exterior geometry of energy E as $\alpha(E, r_i)$. After one emission, the state evolves as

$$\begin{aligned} \sum_n c_n |g(E_n)\rangle &\longrightarrow \\ &\longrightarrow \sum_n \sum_{r_1} c_n \alpha(E_n, r_1) |g(E_n - \omega_1), r_1\rangle. \end{aligned} \quad (149)$$

After a second emission,

$$\begin{aligned} \sum_n \sum_{r_1, r_2} c_n \alpha(E_n, r_1) \alpha(E_n - \omega_1, r_2) \times \\ \times |g(E_n - \omega_1 - \omega_2), r_1, r_2\rangle. \end{aligned} \quad (150)$$

Iterating, the final radiation state takes the form

$$\begin{aligned} |\Psi\rangle_{\text{rad}} = \sum_n \sum_{r_1, \dots, r_N} c_n \alpha(E_n, r_1) \times \\ \times \alpha(E_n - \omega_1, r_2) \alpha(E_n - \omega_1 - \omega_2, r_3) \cdots \\ \cdots \times |r_1 r_2 \cdots r_N\rangle, \end{aligned} \quad (151)$$

where the final geometry label has been omitted because the black hole has completely evaporated. The important point is that the amplitudes in this expression are not universal thermal amplitudes defined on a fixed classical background. They depend on the quantum exterior geometry, and therefore on the branch of the black hole state.

This is the quantum-hair mechanism in the language of the present BRST construction. The black hole state determines the asymptotic gravitational state. The asymptotic gravitational state affects the Hawking amplitudes. Therefore the coefficients c_n of the initial black hole state can influence the detailed final radiation state. The final state is a coherent superposition over radiation histories, not a tensor product of an unchanged black hole interior with independent thermal Hawking pairs. In particular, the factorized nice-slice form [10]

$$|\psi\rangle_{\text{BH}} \otimes \prod_i \left(\frac{1}{\sqrt{2}} |0\rangle_{e_i} |0\rangle_{b_i} + \frac{1}{\sqrt{2}} |1\rangle_{e_i} |1\rangle_{b_i} \right) \quad (152)$$

does not describe the full gravitational Hilbert space. It freezes the geometry and omits the branch-dependent exterior gravity states $|g(E_n)\rangle$.

The picture that emerges is conservative but powerful. The BRST constraints imply that the quantum state of a compact mass is inseparable from its asymptotic gravitational field. For ordinary compact sources this gives a precise quantum version of the Newton field. For black holes it implies that the exterior quantum geometry carries information about the black hole state, and that this information can enter the Hawking radiation amplitudes during evaporation. The calculation in this paper supplies the explicit linearized BRST origin of that exterior state: it is a scalar constraint-sector coherent state, not a cloud of propagating gravitons. This provides a concrete mechanism by which gravitational constraints, boundary data, and quantum hair are tied together.

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APPENDIX A FOURIER CONVENTIONS AND USEFUL IDENTITIES

For convenience, let us list the Fourier conventions used throughout:

$$\begin{aligned} f(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k}), \\ \tilde{f}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}). \end{aligned} \quad (153)$$

Under these conventions,

$$\begin{aligned} \partial_i f(\mathbf{x}) &\longleftrightarrow ik_i \tilde{f}(\mathbf{k}), \\ \nabla^2 f(\mathbf{x}) &\longleftrightarrow -k^2 \tilde{f}(\mathbf{k}). \end{aligned} \quad (154)$$

The Green's-function identity used repeatedly in the paper is

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2} = \frac{1}{4\pi|\mathbf{r}|}. \quad (155)$$

Equivalently,

$$\nabla^2 \left(-\frac{1}{r} \right) = 4\pi\delta^{(3)}(\mathbf{r}). \quad (156)$$

APPENDIX B EXPLICIT CHECK OF THE COHERENT-STATE EXPECTATION VALUE

It is useful to check directly that the state (86) gives the expected one-point function of A . Write

$$|M; a\rangle_S = e^X |0\rangle_S, \quad X = \int \frac{d^3k}{(2\pi)^3} q_M(\mathbf{k}) B^\dagger(\mathbf{k}). \quad (157)$$

Then

$$A(\mathbf{p})e^X = e^X (A(\mathbf{p}) + q_M(\mathbf{p})). \quad (158)$$

So

$$A(\mathbf{p})|M; a\rangle_S = q_M(\mathbf{p})|M; a\rangle_S. \quad (159)$$

The state is therefore an eigenstate of $A(\mathbf{p})$ with eigenvalue $q_M(\mathbf{p})$. Taking the Hermitian adjoint and using real q_M , we similarly get

$${}_S\langle M; a|A^\dagger(\mathbf{p}) = {}_S\langle M; a|q_M(\mathbf{p}). \quad (160)$$

Hence

$$\begin{aligned} {}_S\langle M; a|\mathcal{H}(\mathbf{k})|M; a\rangle_S &= \\ &= \frac{\kappa k^{3/2}}{\sqrt{2}} (q_M(\mathbf{k}) + q_M(\mathbf{k})) = \\ &= \sqrt{2} \kappa k^{3/2} q_M(\mathbf{k}), \end{aligned} \quad (161)$$

as used in the main text.

APPENDIX C DERIVING THE FORM OF $B^\dagger(\mathbf{k})$ IN THE COHERENT STATE

The construction in the main text gives an explicit canonical form for the null partner:

$$\begin{aligned} B(\mathbf{k}) &= -\left(\frac{1}{4}\theta^{ij} + \ell^{ij} \right) a_{ij}(\mathbf{k}), \\ B^\dagger(\mathbf{k}) &= -\left(\frac{1}{4}\theta^{ij} + \ell^{ij} \right) a_{ij}^\dagger(\mathbf{k}), \end{aligned} \quad (162)$$

where

$$\theta_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j, \quad \ell_{ij} = \hat{k}_i \hat{k}_j. \quad (163)$$

This is the operator that appears in

$$|M; a\rangle_S = \exp \left[\int \frac{d^3k}{(2\pi)^3} q_M(\mathbf{k}) B^\dagger(\mathbf{k}) \right] |0\rangle_S. \quad (164)$$

Let us verify the two required commutators directly. From the spatial graviton oscillator algebra,

$$\begin{aligned} [s^{ij} a_{ij}(\mathbf{k}), t^{\ell m} a_{\ell m}^\dagger(\mathbf{k}')] &= \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \left(s^{ij} t_{ij} - \frac{1}{2} s^i_i t^j_j \right). \end{aligned} \quad (165)$$

The active constrained mode is

$$A(\mathbf{k}) = \theta^{ij} a_{ij}(\mathbf{k}). \quad (166)$$

Using

$$\begin{aligned} \langle \theta, \theta \rangle_{\text{DW}} &= 0, \quad \langle \theta, \ell \rangle_{\text{DW}} = -1, \\ \langle \ell, \ell \rangle_{\text{DW}} &= \frac{1}{2}, \end{aligned} \quad (167)$$

one finds

$$\begin{aligned} [A(\mathbf{k}), B^\dagger(\mathbf{k}')] &= \\ &= -(2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \left\langle \theta, \frac{1}{4}\theta + \ell \right\rangle_{\text{DW}} = \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \end{aligned} \quad (168)$$

and

$$\begin{aligned} [B(\mathbf{k}), B^\dagger(\mathbf{k}')] &= \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \left\langle \frac{1}{4}\theta + \ell, \frac{1}{4}\theta + \ell \right\rangle_{\text{DW}} = \\ &= 0. \end{aligned} \quad (169)$$

Thus $B^\dagger$ is the null creation operator conjugate to the active constrained annihilator A .

Finally, define

$$X := \int \frac{d^3k}{(2\pi)^3} q_M(\mathbf{k}) B^\dagger(\mathbf{k}). \quad (170)$$

Then

$$\begin{aligned} [A(\mathbf{p}), X] &= \int \frac{d^3 k}{(2\pi)^3} q_M(\mathbf{k}) [A(\mathbf{p}), B^\dagger(\mathbf{k})] = \\ &= q_M(\mathbf{p}), \end{aligned} \quad (171)$$

so all higher nested commutators vanish and

$$(A(\mathbf{p}) - q_M(\mathbf{p}))e^X = e^X A(\mathbf{p}). \quad (172)$$

Acting on the scalar vacuum gives

$$(A(\mathbf{p}) - q_M(\mathbf{p}))|M; a\rangle_S = 0. \quad (173)$$

The choice of $B^\dagger$ is not unique as a representative in the unreduced indefinite-metric state space. Any replacement

$$B^\dagger \longrightarrow B^\dagger + Y^\dagger, \quad [A, Y^\dagger] = 0, \quad (174)$$

produces the same displacement of A . If one also demands that the new partner remain a null oscillator, then $Y^\dagger$ must obey the corresponding additional null condition. The source fixes the shifted constrained mode and hence the Newton field; it does not fix a unique unreduced Fock-space dressing.

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