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FIELD GENERALIZATION OF ELLIPTIC CALOGERO – MOSER SYSTEM IN THE FORM OF HIGHER RANK LANDAU – LIFSHITZ MODEL

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We prove gauge equivalence between integrable field generalization of the elliptic Calogero – Moser model and the higher rank XYZ Landau – Lifshitz model of vector type on 1+1 dimensional space-time. Explicit formulae for the change of variables are derived, thus providing the Poisson map between these models.

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1. INTRODUCTION

We consider two types of integrable field theories on 1+1 dimensional space-time. The first one is the Landau – Lifshitz model [1], describing behaviour of the magnetization vector $\mathbf{S}(t,x) = (S_1, S_2, S_3)$ in the one-dimensional model of ferromagnet:

$$\partial_t \mathbf{S} = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J(\mathbf{S})\,, \quad \mathbf{S}_{xx} = \partial_x^2 \mathbf{S}\,, \qquad (1.1)$$

where $J(\mathbf{S}) = (J_1 S_1, J_2 S_2, J_3 S_3)$ with some constants J_1, J_2, J_3 describing the anisotropy. Here $t \in \mathbb{R}$ is the time variable, and x is the space variable. We assume x be a coordinate on a unit circle, and all the fields in this paper are periodic $\psi(t,x) = \psi(t, x+2\pi)$. We also imply that all the fields are $\mathbb{C}$-valued. Integrability of this model was proved in [2,3] through the classical inverse scattering method [4–7]. In particular, it was shown that the equation (1.1) is represented in the for of the Zakharov – Shabat equation (or, the Lax equation, or the zero curvature condition):

$$\partial_t U(z) - k\partial_x V(z) + [U(z), V(z)] = 0\,, \quad \forall z\,, \qquad (1.2)$$

where $U(z), V(z) \in \mathrm{Mat}_2$ are 2×2 matrices, z is a complex valued spectral parameter and $k \in \mathbb{R}$ is a parameter.

The second model is the field generalization of 2-body elliptic Calogero – Moser system [8–10]. At the level of classical finite-dimensional mechanics the 2-body system is described by the Hamiltonian

$$H^{\mathrm{CM}} = \frac{p^2}{2} - \frac{c^2}{8}\wp(q) \qquad (1.3)$$

and the canonical Poisson bracket $\{p, q\} = 1$ between the momentum p and the position of particle[1)] q. In (1.3) $c \in \mathbb{C}$ is a coupling constant and $\wp(q)$ is the elliptic Weierstrass $\wp$-function. It is an elliptic version for the inverse square function. All necessary definition of elliptic functions are given in the Appendix. In the field theory the Hamiltonian takes the form [8–10]

$$\mathcal{H}^{\mathrm{2dCM}} = \frac{1}{2}\oint dx \left(p^2\Big(1 - \frac{k^2 q_x^2}{c^2}\Big) + \right.$$
$$\left. + \frac{(3k^2 q_x^2 - c^2)}{4}\wp(q) - \frac{k^4 q_{xx}^2}{4(c^2 - k^2 q_x^2)} \right), \qquad (1.4)$$

where c is the coupling constant and $\zeta(u)$ the elliptic Weierstrass ζ-function, see Appendix. In this model we deal with the canonical fields $p(x)$ and $q(x)$ on a unit circle:

$$\{p(x), q(y)\} = \delta(x-y).$$

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1) In fact, there is a pair of particles. The Hamiltonian (1.3) is written in the center of mass frame, so that $q = q_1 - q_2$. For this reason the normalization of the Hamiltonian slightly differs in 2-body case compared to N-body case.

It was explained in [9, 10] and then computed in [11] that the Landau – Lifshitz model is gauge equivalent to the field Calogero – Moser system. This means that there exists a matrix $G(z) \in \mathrm{Mat}_2$, which relates U-matrices of both models through the gauge transformation:

$$U^{\mathrm{LL}}(z) = G(z)U^{\mathrm{2dCM}}(z)G^{-1}(z) + kG_x(z)G^{-1}(z)\,. \tag{1.5}$$

The matrix $G(z)$ depends on dynamical variables. It is a continuous version of the IRF-Vertex transformation introduced by R.J. Baxter [12] for the quantum statistical models. In this treatment the Landau – Lifshitz model is of the vertex type, while the Calogero – Moser system is on the IRF (interacting round a face) side.

Purpose of the paper. In this paper we generalize the above results to the higher rank models. The gl_N Calogero – Moser model [13–16] in classical mechanics is described by N-body Hamiltonian

$$H^{\mathrm{CM}} = \sum_{i=1}^{N} \frac{p_i^2}{2} - c^2 \sum_{i>j}^{N} \wp(q_i - q_j)\,, \tag{1.6}$$

Its field generalization was proposed in [17] using reduction from matrix KP equations, and the integrability was proved in [18] through the classical r-matrix structure. We describe this model in detail in Sect. 3. The higher rank generalization if the Landau – Lifshitz model was derived in [19] through the associative Yang-Baxter equation. It is also described in Sect. 3. Two models are related by the gauge transformation as given in (1.5). The matrix of the corresponding gauge transformation is a continuous version of the IRF-Vertex transformation for Belavin's R-matrix found in [20]. Moreover, in [18] the continuous version of the IRF-Vertex transformation was performed at the level of classical r-matrix structures for both field theories. Main purpose of this paper is to finish description of the gauge equivalence by evaluating explicit change of variables relating both integrable field theories. Similar results for the rational and trigonometric models were obtained in our previous papers [11, 21].

The paper is organized as follows. In the next Section we recall main results for gl_2 models from [22]. The pair of models in the higher rank case are described in Sect. 3. In Sect. 4 we calculate explicit change of variables and argue that is provides the Poisson map between the models. Definitions and properties of elliptic functions are given in the Appendix.

2. AN OVERVIEW OF 2-BODY CASE

Here we briefly recall the result of [22]. Namely , we describe the field analogue of 2-body Calogero – Moser model and represent it in the form of the Landau – Lifshitz magnet.

Classical mechanics. The 2-body Calogero – Moser model is described by the Hamiltonian (1.3). Equations of motion take the form

$$\dot{p} = \frac{c^2}{8}\wp'(q), \quad \dot{q} = p\,. \tag{2.1}$$

They are represented in the Lax form

$$\dot{L}^{\mathrm{CM}}(z) \equiv \{H^{\mathrm{CM}}, L^{\mathrm{CM}}(z)\} = [L^{\mathrm{CM}}(z), M^{\mathrm{CM}}(z)] \tag{2.2}$$

with the Lax pair

$$L^{\mathrm{CM}}(z) = \begin{pmatrix} p & \frac{c}{2}\,\phi(-z,q) \\ \frac{c}{2}\,\phi(-z,-q) & -p \end{pmatrix}, \tag{2.3}$$

$$M^{\mathrm{CM}}(z) = \frac{1}{4}\begin{pmatrix} 0 & cf(-z,q) \\ cf(-z,-q) & 0 \end{pmatrix},$$

where the functions ϕ and f are given in the Appendix in (A.1) and (A.6) respectively.

1+1 field theory. In this case the momentum and coordinate become fields on a unit circle $\mathbb{S}^1$, and the canonical Poisson bracket turns into

$$\{p(x), q(y)\} = \delta(x-y)\,. \tag{2.4}$$

Equations of motion takes the form:

$$q_t = p\left(1 - \frac{k^2 q_x^2}{c^2}\right),$$

$$p_t = -\frac{k^2}{c^2}\partial_x\left(p^2 q_x\right) - \frac{(3k^2 q_x^2 - c^2)}{8}\wp'(q) + \tag{2.5}$$

$$+\frac{3k^2}{4}\partial_x\left(q_x\wp(q)\right) + \frac{k^4}{4}\partial_x\left(\frac{q_{xxx}\tilde{\nu} - \tilde{\nu}_x q_{xx}}{\tilde{\nu}^3}\right),$$

where

$$\tilde{\nu} = \sqrt{c^2 - k^2 q_x^2}\,, \quad c = \mathrm{const}\,, \tag{2.6}$$

$$U^{\mathrm{2dCM}}(z) = \frac{1}{2}\begin{pmatrix} 2p - kq_x E_1(z) & \tilde{\nu}\phi(-z,q) \\ \tilde{\nu}\phi(-z,-q) & -2p + kq_x E_1(z) \end{pmatrix}. \tag{2.7}$$

Landau – Lifshitz magnet. By inserting three components of $\mathbf{S}(t,x)$ into the traceless 2×2 matrix

$$S=\sum_{\alpha=1}^{3}\sigma_\alpha S_\alpha$$

in the Pauli matrices σ_α basis, the Landau – Lifshitz equation takes the form

$$\partial_t S=[J(S),S]-\alpha_0\,[S,S_{xx}]\,,\quad S_{xx}=\partial_x^2 S\,,\qquad (2.8)$$

where $J(S)$ is the linear map describing the anisotropy in the magnet

$$J(S)=\sum_{\alpha=1}^{3}S_\alpha J_\alpha\sigma_\alpha\,,\quad J_\alpha=-\frac{1}{2}\,\wp(\omega_\alpha),\qquad (2.9)$$

and

$$\alpha_0=k^2/(8\lambda^2)\qquad (2.10)$$

is a constant parameter. Here $\lambda\in\mathbb{C}$ (also constant) is an eigenvalue of the matrix S – the norm of the vector (S_1,S_2,S_3), and k is the constant coefficient behind ∂_x.

The Landau – Lifshitz equation (2.8) has the Hamiltonian formulation with the Poisson brackets

$$\{S_\alpha(x),S_\beta(y)\}=-\sqrt{-1}\varepsilon_{\alpha\beta\gamma}S_\gamma(x)\delta(x-y)\,,\qquad (2.11)$$

and the Hamiltonian

$$\mathcal{H}^{\rm LL}=\frac{1}{2}\oint dx\Big(\mathrm{tr}(SJ(S))-\alpha_0\mathrm{tr}(S_x^2)\Big)\,.\qquad (2.12)$$

The U-matrix (it is 2×2 matrix) entering the Zakharov – Shabat equation has the form [2–4, 23, 24]:

$$U^{\rm LL}(z)=\sum_{k=1}^{3}S_k\varphi_k(z)\sigma_k\,,\qquad (2.13)$$

where

$$\varphi_1(z)=e^{\pi\imath z}\phi\left(z,\frac{\tau}{2}\right),\quad \varphi_2(z)=e^{\pi\imath z}\phi\left(z,\frac{1+\tau}{2}\right),$$

$$\varphi_3(z)=\phi\left(z,\frac{1}{2}\right).$$

Gauge equivalence. The gauge transformation (1.5) with the matrix

$$G(z)=\frac{1}{\rho}\times$$

$$\times\left(\begin{array}{cc}\theta_{00}(z+q\,|\,2\tau)\tilde\nu & -\theta_{00}(q-z\,|\,2\tau)\,(c+kq_x)\\ -\theta_{10}(z+q\,|\,2\tau)\tilde\nu & \theta_{10}(q-z\,|\,2\tau)\,(c+kq_x)\end{array}\right),\qquad (2.14)$$

where $\tilde\nu$ is from (2.6) and

$$\rho=\sqrt{\tilde\nu(c+kq_x)\vartheta(z)\vartheta(q)}\,,\qquad (2.15)$$

leads to the following change of variables:

$$S_1(p,q,c)=\Big(p-\frac{c}{2}\,\frac{k^2q_{xx}}{c^2-k^2q_x^2}\Big)\frac{\theta_{01}(0)}{\vartheta'(0)}\frac{\theta_{01}(q)}{\vartheta(q)}+$$

$$+\frac{c}{2}\,\frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(q)\theta_{10}(q)}{\vartheta^2(q)}\,,$$

$$S_2(p,q,c)=\Big(p-\frac{c}{2}\,\frac{k^2q_{xx}}{c^2-k^2q_x^2}\Big)\frac{\sqrt{-1}\theta_{00}(0)}{\vartheta'(0)}\frac{\theta_{00}(q)}{\vartheta(q)}+$$

$$+\frac{c}{2}\,\frac{\sqrt{-1}\theta_{00}^2(0)}{\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(q)\theta_{01}(q)}{\vartheta^2(q)}\,,\qquad (2.16)$$

$$S_3(p,q,c)=\Big(p-\frac{c}{2}\,\frac{k^2q_{xx}}{c^2-k^2q_x^2}\Big)\frac{\theta_{10}(0)}{\vartheta'(0)}\frac{\theta_{10}(q)}{\vartheta(q)}+$$

$$+\frac{c}{2}\,\frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(q)\theta_{01}(q)}{\vartheta^2(q)}\,.$$

In the above formulae the notations (A.5) are used.

3. N-BODY CASE: DESCRIPTION OF MODELS

3.1. Field analogue of N-body elliptic Calogero – Moser model

Hamiltonian and equations of motion. The N-body Calogero – Moser model in classical mechanics is described by the Hamiltonian (1.6). Its field generalization was derived in [17]. The Hamiltonian takes the form

$$\mathcal{H}^{\rm 2dCM}=\oint dx\,\mathcal{H}^{\rm 2dCM}(x)\qquad (3.1)$$

with the density[2)]

$$\mathcal{H}^{\rm 2dCM}(x)=\sum_{i=1}^{N}p_i^2\,(c-kq_{ix})-\frac{1}{Nc}\left(\sum_{i=1}^{N}p_i\,(c-kq_{ix})\right)^2-$$

$$-\sum_{i=1}^{N}\frac{k^4q_{ixx}^2}{4\,(c-kq_{ix})}+\frac{k^3}{2}\sum_{i\neq j}^{N}\Big(q_{ix}q_{jxx}-q_{jx}q_{ixx}\Big)\zeta(q_i-q_j)-$$

$$-\frac{1}{2}\sum_{i\neq j}^{N}\Big((c-kq_{ix})^2\,(c-kq_{jx})+(c-kq_{ix})\,(c-kq_{jx})^2-$$

$$-ck^2\,(q_{ix}-q_{jx})^2\Big)\wp\,(q_i-q_j)\,.\qquad (3.2)$$

[2)] In the limit to the finite-dimensional mechanics all the fields become independent of x. This corresponds to the limit $k\to 0$. In this limit $\mathcal{H}^{\rm 2dCM}(x)\to 2cH^{\rm CM}$.

Together with the canonical Poisson structure

$$\begin{aligned}\{q_i(x), p_j(y)\} = \delta(x-y)\,,\\ \{q_i(x), q_j(y)\} = \{p_i(x), p_j(y)\} = 0\end{aligned} \tag{3.3}$$

it provides equations of motion

$$\dot{q}_i = 2p_i\,(c - kq_{ix}) - \frac{2}{Nc}\sum_{l=1}^{N} p_l\,(c-kq_{lx})\,(c-kq_{ix}) \tag{3.4}$$

and

$$\begin{aligned}\dot{p}_i = -2kp_ip_{ix} + k\partial_x\Big(\frac{k^3q_{ixxx}}{2\,(c-kq_{ix})} + \frac{k^4q_{ixx}^2}{4\,(c-kq_{ix})^2}+\\ +\frac{2}{Nc}\sum_{l=1}^{N} p_ip_l(c-kq_{lx})\Big)+2\sum_{j:j\neq i}^{N}\Big(k^3q_{jxxx}\zeta\,(q_i-q_j)-\\ -3k^2(c-kq_{jx})\,q_{jxx}\wp\,(q_i-q_j)+(c-kq_{jx})^3\,\wp'\,(q_i-q_j)\Big).\end{aligned} \tag{3.5}$$

***U–V* pair.** The equations of motion (3.5) are represented in the Zakharov – Shabat form (1.2) with $N\times N$ matrices $U^{\rm 2dCM}(z)$ and $V^{\rm 2dCM}(z)$. The entries of the matrices are as follows:

$$\begin{aligned}U_{ij}^{\rm 2dCM}(z)=\delta_{ij}\Big(-p_i+\frac{1}{N}\sum_{k=1}^{N}p_k-\alpha_i^2E_1(z)-\frac{k\alpha_{ix}}{\alpha_i}\Big)-\\ -(1-\delta_{ij})\,\alpha_j^2\phi\,(q_j-q_i,z)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}V_{ij}^{\rm 2dCM}(z)=\delta_{ij}\Big(-q_{it}E_1\,(z)-Nc\alpha_i^2\wp\,(z)+\tilde{m}_i^0-\frac{\alpha_{it}}{\alpha_i}-\\ -\frac{1}{N}\sum_{i=1}^{N}\tilde{m}_i^0\Big)-(1-\delta_{ij})\,\alpha_j^2\Big(NcE_1(z)\phi\,(-q_i+q_j,z)-\\ -Nc\phi'\,(-q_i+q_j,Nz)-\widetilde{m}_{ij}\phi\,(-q_i+q_j,z)\Big).\end{aligned} \tag{3.7}$$

Here we use notations

$$\alpha_i^2 = kq_{ix} - c\,, \tag{3.8}$$

$$\begin{aligned}\tilde{m}_i^0 = p_i^2 + \frac{k^2\alpha_{ixx}}{\alpha_i} + 2\kappa p_i - \sum_{k:k\neq i}^{N}\Big(\,(2\alpha_k^4+\alpha_i^2\alpha_k^2)\times\\ \times\wp\,(q_i-q_k)+4k\alpha_k\alpha_{kx}\zeta\,(q_i-q_k)\Big),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\widetilde{m}_{ij} = p_i+p_j+2\kappa+\frac{k\alpha_{ix}}{\alpha_i}-\frac{k\alpha_{jx}}{\alpha_j}-\\ -\sum_{k:k\neq i,j}^{N}\alpha_k^2\eta\,(q_i,q_k,q_j)\;,\end{aligned} \tag{3.10}$$

$$\eta(\lambda,\nu,\mu) = \zeta(\lambda-\nu)+\zeta(\nu-\mu)-\zeta(\lambda-\mu) \tag{3.11}$$

and

$$\kappa = -\frac{1}{Nc}\sum_{k=1}^{N}p_k\,(c-kq_{kx})\;. \tag{3.12}$$

3.2. Higher rank Landau – Lifshitz model

Notations. Introduce the special matrix basis in $\mathrm{Mat}(N,\mathbb{C})$:

$$\begin{aligned}&T_a = T_{a_1a_2} = \exp\Big(\frac{\pi\imath}{N}\,a_1a_2\Big)\,Q^{a_1}\Lambda^{a_2}\,,\\ &a=(a_1,a_2)\in\mathbb{Z}_N\times\mathbb{Z}_N\,,\ \mathbb{Z}_N=\mathbb{Z}/N\mathbb{Z}\,,\\ &(Q)_{kl}=\delta_{kl}\exp(\frac{2\pi\imath}{N}k)\,,\\ &(\Lambda)_{kl}=\delta_{k-l+1=0 \bmod N}\,,\quad k,l=1,..,N\,.\end{aligned} \tag{3.13}$$

The basis matrices are numerated by a pair of numbers (a_1,a_2), $a_1,a_2 = 0,...,N-1$ defined modulo N. In particular, $T_{(0,0)} = 1_N$ – identity $N\times N$ matrix. Then

$$T_\alpha T_\beta = \kappa_{\alpha,\beta}T_{\alpha+\beta}\,,\quad \kappa_{\alpha,\beta} = \exp\Big(\frac{\pi\imath}{N}(\beta_1\alpha_2-\beta_2\alpha_1)\Big)\,. \tag{3.14}$$

The basis has the property $\mathrm{tr}(T_\alpha T_\beta) = N\delta_{\alpha+\beta,(0,0)}$. See details in the Appendix of the paper [25]. Below we use the following set of functions:

$$\begin{aligned}\varphi_a(z,\omega_a+\hbar) = \exp(2\pi\imath\,\frac{a_2z}{N})\,\phi(z,\omega_a+\hbar) =\\ =\exp(2\pi\imath z\partial_\tau\omega_a)\,\phi(z,\omega_a+\hbar)\,,\quad a\in\mathbb{Z}_N\times\mathbb{Z}_N,\end{aligned} \tag{3.15}$$

where

$$\omega_a = \frac{a_1+a_2\tau}{N}\,,\quad a\in\mathbb{Z}_N\times\mathbb{Z}_N\,. \tag{3.16}$$

The classical elliptic Belavin-Drinfeld r-matrix [26] takes the following simple form:

$$\begin{aligned}r_{12}(z) = E_1(z)1_N\otimes 1_N+\\ +\sum_{a\neq(0,0)}T_a\otimes T_{-a}\exp(2\pi\imath\,\frac{a_2z}{N})\,\phi(z,\frac{a_1+a_2\tau}{N})\,,\end{aligned} \tag{3.17}$$

where the sum over a goes over all $a\in\mathbb{Z}_N\times\mathbb{Z}_N$, $a\neq(0,0)$. We use this notation in what follows.

Notice that the described above matrix basis reproduces (up to signs) the Pauli matrices in the $N=2$ case.

The transition between the standard matrix basis and the basis T_a is performed as follows. Let B be an arbitrary $N\times N$ matrix with entries B_{ij} in the standard basis and with $B_{(a_1,a_2)}$ components in the basis T_a. Then with the short notation $\mathbf{e}(x)=\exp(2\pi\imath x)$ we have

$$B_a = B_{(a_1,a_2)} = \frac{1}{N}\,\mathrm{tr}(BT_{-a}) =$$

$$= \frac{1}{N}\,\mathbf{e}(-\frac{a_1 a_2}{2N})\sum_{k=1}^{N} B_{k,k+a_2}\,\mathbf{e}(-\frac{a_1 k}{N}) \qquad (3.18)$$

and

$$B_{ij} = \begin{cases} \sum\limits_{a_1=0}^{N-1} B_{(a_1,j-i)}\mathbf{e}\Big(\dfrac{a_1(i+j)}{2N}\Big),\ j \geq i\,, \\ \sum\limits_{a_1=0}^{N-1} B_{(a_1,j-i+N)}\mathbf{e}\Big(\dfrac{a_1(i+j-N)}{2N}\Big),\ j < i\,. \end{cases} \qquad (3.19)$$

Equations of motion and Hamiltonian formulation. In gl_N case the Landau – Lifshitz equation (2.8) is generalized as [19]:

$$\partial_t S = 2c[S, J(S)] + \frac{k^2}{c}[S, S_{xx}] - 2k[S, E(S_x)]\,, \qquad (3.20)$$

where $S = S(t,x) \in \mathrm{Mat}(N, \mathbb{C})$ is a matrix of dynamical variables (fields). Here we assume this matrix is not an arbitrary but the one, which has a special set of eigenvalues $0, .., 0, c$:

$$\mathrm{Spec}(S) = (0, ..., 0, c), \quad \mathrm{tr}(S) = c\,. \qquad (3.21)$$

The latter means that S is a rank one matrix. It satisfies the property

$$S^2 = cS\,. \qquad (3.22)$$

The linear maps $J(S)$ and $E(S)$ entering equation (3.20) are of the following form:

$$J(S) = \frac{N\vartheta'''(0)}{3\vartheta'(0)}1_N S_{(0,0)} - \\ -N\sum_{a\neq(0,0)} T_a S_a E_2\Big(\frac{a_1+a_2\tau}{N}\Big) = \\ = \frac{N\vartheta'''(0)}{3\vartheta'(0)}\,S - N\sum_{a\neq(0,0)} T_a S_a \wp\Big(\frac{a_1+a_2\tau}{N}\Big) \qquad (3.23)$$

and

$$E(S) = N\sum_{a\neq(0,0)} T_a S_a \Big(2\pi\imath\,\frac{a_2}{N} + E_1\Big(\frac{a_1+a_2\tau}{N}\Big)\Big)\,. \qquad (3.24)$$

Notice that in the $N = 2$ case $E(S) = 0$ for any matrix S and the original Landau – Lifshitz equation (2.8) is reproduced.

The equation (3.20) has the Hamiltonian description, that is

$$\partial_t S(t,x) = \{S(t,x), \mathcal{H}^{\mathrm{LL}}\} \qquad (3.25)$$

with the following Hamiltonian

$$\mathcal{H}^{\mathrm{LL}} = \oint dx \mathcal{H}^{\mathrm{LL}}(x)\,, \qquad (3.26)$$

$$\mathcal{H}^{\mathrm{LL}}(x) = Nc\mathrm{tr}(SJ(S)) - \frac{Nk^2}{2c}\mathrm{tr}\,(\partial_x S\partial_x S) + \\ + Nk\mathrm{tr}\,(\partial_x S E(S)) \qquad (3.27)$$

and the Poisson brackets

$$\{S_{ij}(x), S_{kl}(y)\} = \frac{1}{N}\,(S_{il}(x)\delta_{kj} - S_{kj}(x)\delta_{il})\,\delta(x-y)\,. \qquad (3.28)$$

***U*–*V* pair.** It is helpful to use the following notation

$$L(S,z) = \mathrm{tr}_2\Big(r_{12}(z)(1_N \otimes S)\Big) = NS_{0,0}E_1(z)1_N + \\ + N\sum_{a\neq(0,0)} T_a S_a \varphi_a(z, \omega_a)\,, \qquad (3.29)$$

where tr_2 is the trace over the second tensor component and $r_{12}(z)$ is the classical elliptic r-matrix (3.17). The above expression, in fact, is the Lax matrix of the elliptic Euler – Arnold top [9,10,27,28] in finite-dimensional classical mechanics. Equations of motion for this model $\dot{S} = 2c[S, J(S)]$ are obtained from (3.20) in the limit $k \to 0$. It was also explained in [9,10] that the U-matrix in the field theory case has the same form

$$U^{\mathrm{LL}}(z) = L(S,z) \qquad (3.30)$$

although here we imply $S = S(t,x)$, while in mechanics $S = S(t)$. For V-matrix we have

$$V^{\mathrm{LL}}(z) = V_1(z) - cV_2(z) + \frac{Nc^2\vartheta'''(0)}{3\vartheta'(0)}1_N\,, \qquad (3.31)$$

where

$$V_1(z) = -Nc\partial_z L(S,z) + L(E(S)S, z) \qquad (3.32)$$

and

$$V_2(z) = L\,(W, z)\,, \qquad W = -\frac{k}{c^2}\,[S, S_x]\,. \qquad (3.33)$$

The matrix W is a solution of the equation $-k\partial_x S = [S, W]$.

4. IRF-VERTEX RELATION AND CHANGE OF VARIABLES

Gauge equivalence. Let us introduce the following matrix:

$$G(z) = b(x,t)\Xi(z,q)D^{-1}(q) \in \mathrm{Mat}(N, \mathbb{C})\,, \qquad (4.1)$$

where the matrix $\Xi(z)$ and the diagonal matrix $D(q)$ are defined as follows:

$$\Xi_{ij}(z,q) = \theta\left[\begin{array}{c} i/N-1/2 \\ N/2 \end{array}\right](z+N\bar{q}_j \mid N\tau)\,, \quad (4.2)$$

$$\bar{q}_j = q_j - \frac{1}{N}\sum_{k=1}^{N} q_k\,,$$
$$D_{ij}(q) = \delta_{ij}\prod_{k\neq i}\vartheta(q_i-q_k)\,. \quad (4.3)$$

The coefficient function $b(x,t)$ in (4.1) has the form

$$b(x,t) = \prod_{k<l}^{N}\vartheta(q_l-q_k)^{\frac{1}{N}}\prod_{m=1}^{N}\Big(N(kq_{m,x}-c)\Big)^{\frac{1}{2N}}\,. \quad (4.4)$$

In the above formulae the condition (which is by definition of $\bar{q}_j$)

$$\sum_{k=1}^{N}\bar{q}_k = 0 \quad (4.5)$$

is necessary. The defined above matrix $G(z)$ is (up to the function $b(x,t)$) the matrix of the IRF-Vertex transformation [9, 10, 18, 20].

The reason why we use this matrix is as follows. The U-matrices for both models have certain quasi-periodic behaviour in spectral parameter:

$$U^{\rm 2dCM}(z+1) = U^{\rm 2dCM}(z)\,,$$
$$U^{\rm 2dCM}(z+\tau) = \exp(2\pi\imath\,{\rm diag}(q_1,...,q_N))U^{\rm 2dCM}(z)\times$$
$$\times\exp(-2\pi\imath\,{\rm diag}(q_1,...,q_N) - 2\pi\imath c 1_N +$$
$$+2\pi\imath k\partial_x{\rm diag}(q_1,...,q_N) \quad (4.6)$$

and

$$U^{\rm LL}(z+1) = Q^{-1}U^{\rm LL}(z)Q\,,$$
$$U^{\rm LL}(z+\tau) = \Lambda^{-1}U^{\rm LL}(z)\Lambda - 2\pi\imath c 1_N\,, \quad (4.7)$$

where Q,Λ are the matrices from (3.13). The matrix $G(z)$ has very special structure. The action by $G(z)$ as in the gauge transformation

$$U^{\rm LL}(z) = G(z)U^{\rm 2dCM}(z)G^{-1}(z) + kG_x(z)G^{-1}(z) \quad (4.8)$$

maps the quasi-periodic properties (4.6) into (4.7). On the one hand, the matrix $G(z)$ is degenerated at $z=0$: $\det G(0)=0$, that is $G^{-1}(z)$ has simple pole at $z=0$. On the other hand, the conjugation of $U^{\rm 2dCM}(z)$ by $G(z)$ does not provide the second order pole in $U^{\rm LL}(z)$. Details can be found in [9, 10] for a similar relation at the level of classical finite-dimensional mechanics. See also [29], where different aspects of the transformation matrix $G(z)$ are discussed.

Change of variables. The gauge transformation (4.8) relates both U-matrices. It allows to compute explicit change of variables between the models. For any $a=(a_1,a_2)\in\mathbb{Z}_N\times\mathbb{Z}_N$ we have

$$S_a(p,q,c) = \frac{c}{N}\,\delta_{a,(0,0)} + (-1)^{a_1+a_2}\frac{e^{\pi\imath a_2\omega_a}}{N^2}\left(\frac{\vartheta(\omega_\alpha)}{\vartheta'(0)}\right)^N\times$$
$$\times\sum_{m=1}^{N}\Big(P_m + c\sum_{k:k\neq m}E_1(q_{mk}+\omega_a)\Big)\times$$
$$\times\prod_{l:l\neq m}^{N}\varphi_a(q_m-q_l,\omega_a)\,, \quad (4.9)$$

where ω_a is from (3.16) and

$$P_m = -p_m - \frac{k\alpha_{m,x}}{\alpha_m} + \sum_{l:l\neq m}^{N}\alpha_l^2E_1(q_m-q_l) =$$
$$= -p_m - \frac{1}{2}\frac{k^2q_{m,xx}}{kq_{m,x}-c} + \sum_{l:l\neq m}^{N}(kq_{l,x}-c)E_1(q_m-q_l)\,. \quad (4.10)$$

Using (3.19) one can write the formulae (4.9) in the standard matrix basis as well.

Poisson map. The statement that the obtained change of variables provides the Poisson map between two models means that the canonical Poisson brackets (3.3) are mapped into the Lie-Poisson brackets (3.28). Put it differently, the Poisson brackets for $S_{ij}(p,q,c)$ computed by means of (3.3) should reproduce (3.28).

In fact, this statement was implicitly obtained in [18]. It was proved that the field analogue of the elliptic Calogero – Moser model is described by the non-ultralocal Maillet type r-matrix structure

$$\{U_1^{\rm 2dCM}(z,x),U_2^{\rm 2dCM}(w,y)\} = \Big(-k\partial_x\mathbf{r}_{12}(z,w|x)+$$
$$+[U_1^{\rm 2dCM}(z,x),\mathbf{r}_{12}^{\rm 2dCM}(z,w|x)]-[U_2^{\rm 2dCM}(w,y),\mathbf{r}_{21}^{\rm 2dCM}(w,z|x)]\Big)\times$$
$$\times\delta(x-y) - \Big(\mathbf{r}_{12}^{\rm 2dCM}(z,w|x)+\mathbf{r}_{21}^{\rm 2dCM}(w,z|x)\Big)k\delta'(x-y)\,, \quad (4.11)$$

where the classical r-matrix $\mathbf{r}_{12}^{\rm 2dCM}(z,w|x)$ is very similar to its finite-dimensional version[3]. At the same time the classical r-matrix structure for the Landau–Lifshitz model is

$$\{U_1^{\rm LL}(z,x),U_2^{\rm LL}(w,y)\}=\Big([U_1^{\rm LL}(z,x),\mathbf{r}_{12}^{\rm LL}(z,w|x)]-$$
$$-[U_2^{\rm LL}(w,y),\mathbf{r}_{21}^{\rm LL}(w,z|x)]\Big)\delta(x-y)\,, \qquad (4.12)$$

where $\mathbf{r}_{12}^{\rm LL}(z,w|x)$ is the elliptic non-dynamical r-matrix (3.17). It was shown in [18] that the gauge transformation (4.8) transforms (4.11) into (4.12). This is exactly what we need since (3.28) follows from (4.12).

Existence of the classical r-matrix structures for both models means the Poisson commutativity

$$\{{\rm tr}(T^k(z,2\pi)),{\rm tr}(T^m(w,2\pi))\}=0$$

for the corresponding monodromy matrices

$$T(z,x)={\rm Pexp}\left(\frac{1}{k}\int_0^x dy\,U(z,y)\right). \qquad (4.13)$$

Due to the gauge equivalence the monodromies of both models are equal to each other. This provides relation between the Hamiltonians. Exact relation is as follows:

$$H^{\rm 2dCM}(x)=H^{\rm LL}(x)-\frac{N^2c^3\vartheta'''(0)}{3\vartheta'(0)}\,. \qquad (4.14)$$

This relation was verified numerically. Its direct proof will be given elsewhere.

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APPENDIX. ELLIPTIC FUNCTIONS

We actively use the elliptic Kronecker function:

$$\phi(z,u)=\frac{\vartheta'(0)\vartheta(z+u)}{\vartheta(z)\vartheta(u)}=\phi(u,z)\,,$$
$$\operatorname*{Res}_{z=0}\phi(z,u)=1\,,\quad \phi(-z,-u)=-\phi(z,u)\,, \qquad (A.1)$$

where $\vartheta(z)$ is theta-function:

$$\vartheta(z)=\vartheta(z,\tau)\equiv-\theta\left[\begin{array}{c}1/2\\1/2\end{array}\right](z|\,\tau)\,, \qquad (A.2)$$

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](z|\,\tau)=\sum_{j\in\mathbb{Z}}\exp\left(2\pi\imath(j+a)^2\frac{\tau}{2}+\right.$$
$$\left.+2\pi\imath(j+a)(z+b)\right),\quad {\rm Im}(\tau)>0\,. \qquad (A.3)$$

By definition, $\vartheta(z)$ in (A.2) is the first Jacobi theta function:

$$\theta_1(u|\tau)=\vartheta(u,\tau)=-i\sum_{k\in\mathbb{Z}}(-1)^kq^{(k+\frac{1}{2})^2}e^{\pi i(2k+1)u}\,,$$
$$q=e^{\pi i\tau}\,. \qquad (A.4)$$

Relation to the standard Riemann and Jacobi notations is as follows:

$$\theta\left[\begin{array}{c}1/2\\0\end{array}\right](z,\tau)=\theta_{10}(z)=\theta_2(z)\,,$$
$$\theta\left[\begin{array}{c}0\\0\end{array}\right](z,\tau)=\theta_{00}(z)=\theta_3(z)\,, \qquad (A.5)$$
$$\theta\left[\begin{array}{c}0\\1/2\end{array}\right](z,\tau)=\theta_{01}(z)=\theta_4(z)\,.$$

The derivative $f(z,u)=\partial_u\phi(z,u)$ is given by

$$f(z,u)=\phi(z,u)(E_1(z+u)-E_1(u)),$$
$$f(-z,-u)=f(z,u) \qquad (A.6)$$

in terms of the first Eisenstein function:

$$E_1(z)=\frac{\vartheta'(z)}{\vartheta(z)}=\zeta(z)+\frac{z}{3}\frac{\vartheta'''(0)}{\vartheta'(0)}\,,$$
$$E_2(z)=-\partial_zE_1(z)=\wp(z)-\frac{\vartheta'''(0)}{3\vartheta'(0)}\,, \qquad (A.7)$$

$$E_1(-z)=-E_1(z)\,,\quad E_2(-z)=E_2(z)\,, \qquad (A.8)$$

where $\wp(z)$ and $\zeta(z)$ are the Weierstrass functions. The second order derivative $f'(z,u)=\partial_u^2\phi(z,u)$ is

$$f'(z,u)=\phi(z,u)\Big(\wp(z)-E_1^2(z)+2\wp(u)-$$
$$-2E_1(z)E_1(u)+2E_1(z+u)E_1(z)\Big)=$$
$$=2\Big(\wp(u)-\rho(z)\Big)\phi(z,u)+2E_1(z)f(z,u)\,, \qquad (A.9)$$

where

$$\rho(z)=\frac{E_1^2(z)-\wp(z)}{2}\,. \qquad (A.10)$$

[3] Similar results are also known for the field analogue of the spin generalization of the Calogero–Moser systems and other 2d models [30–32].

The defined above functions satisfy the widely known addition formulae:

$$\phi(z_1,u_1)\phi(z_2,u_2)=\phi(z_1,u_1+u_2)\phi(z_2-z_1,u_2)+$$

$$+\phi(z_2,u_1+u_2)\phi(z_1-z_2,u_1)\,, \qquad (A.11)$$

$$\phi(z,u_1)\phi(z,u_2)=\phi(z,u_1+u_2)\Big(E_1(z)+E_1(u_1)+$$

$$+E_1(u_2)-E_1(z+u_1+u_2)\Big)\,, \qquad (A.12)$$

$$\phi(z,u_1)f(z,u_2)-\phi(z,u_2)f(z,u_1)=$$

$$=\phi(z,u_1+u_2)\Big(\wp(u_1)-\wp(u_2)\Big)\,, \qquad (A.13)$$

$$\phi(z,u)\phi(z,-u)=\wp(z)-\wp(u)=E_2(z)-E_2(u)\,, \qquad (A.14)$$

$$\phi(z,u)f(z,-u)-\phi(z,-u)f(z,u)=\wp'(u)\,. \qquad (A.15)$$

Also, the following two identities are useful:

$$\frac{1}{2}\,\frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}=\zeta(z+w)-\zeta(z)-\zeta(w)=$$

$$=E_1(z+w)-E_1(z)-E_1(w) \qquad (A.16)$$

and

$$\Big(\zeta(z+w)-\zeta(z)-\zeta(w)\Big)^2=\wp(z)+\wp(w)+\wp(z+w)\,. \qquad (A.17)$$

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