

NON-SELF-AVERAGING IN THE CRITICAL POINT OF A RANDOM ISING FERROMAGNET

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1. INTRODUCTION

After intensive studies during last several decades, it is well established now that the presence of weak quenched disorder in a ferromagnetic system can essentially modify its critical behavior in the vicinity of the phase transition point such that new universal critical exponents may set [1–7]. According to the so called Harris criteria [1], weak disorder is relevant for the critical behavior only if the specific heat of the pure system is divergent, i. e., the corresponding critical exponent $\alpha > 0$. The critical behavior is then governed by a new, random renormalization-group fixed point, and the pure fixed point becomes unstable.

On the other hand, in recent years, it is argued that due to the presence of disorder, the statistical properties of some thermodynamical quantities at the critical point can become non-self-averaging [8–11], i. e., the behavior of a large sample with a specific realization of impurities will not be well described by the ensemble average normally calculated in an analytical or numerical approach. This clearly has profound consequences for the physical interpretation of the outcomes and the possibilities for comparing theoretical and experimental results.

Recently, an explicit expression for the probability distribution function of the critical free-energy fluctuations for a weakly disordered Ising ferromagnet was derived for $D < 4$ and its universal shape was obtained at $D = 3$ [12]. Away from the critical point, at scales much bigger than the correlation length R_c, the situation is sufficiently simple: here, the system could be considered as a set of essentially independent regions with the size R_c, and for that reason one could naively expect that the free energy distribution function must be Gaussian. In fact, besides the central Gaussian part (the "body"), this distribution has asymmetric and essentially non-Gaussian tails [11]. Approaching the critical point, one finds that the range of validity of the Gaussian body shrinks while the tails are getting of the same order as the body. Finally, when the correlation length becomes of the order of the system size (in the critical point) the free energy distribution function turns into a universal non-Gaussian curve.

A system of particular interest is the Ising model in two dimensions, where exact results are available both for the pure [13] and for the weakly disordered [6, 7] cases. Here, the Harris criterion is unable to decide the significance of weak disorder as $\alpha = 0$, and the system hence provides a marginal case. Still, it is now well established that such weak disorder "marginally" modifies the critical behavior of this system so that the logarithmic singularity of the specific heat is changed into a double logarithmic one. While a number of further aspects of this problem have also been investigated, the question of the disorder distribution of measurable quantities and their (lack of) self-averaging behavior was less studied. In the recent paper [14], it was shown that in the critical point the internal energy of this

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system is not self-averaging. Namely, both its typical sample-to-sample fluctuations and its average value scale with the system size L like $\sim L \ln\ln(L)$. In contrast, the specific heat is shown to be self-averaging with a distribution function that tends to a δ-peak in the thermodynamic limit $L \to \infty$.

2. REPLICA FORMALISM

Present investigation of the sample-to-sample fluctuations of the free energy will be performed in terms of the replica method. This approach was already successfully used for the study of the free energy distribution functions in mean-field spin-glasses and others strongly disordered systems [15, 16] (see also [17]). Here, it will be applied for the Ising systems containing weak disorder. As a matter of a general demonstration of how the replica method can be used for the study of the free energy probability distribution function, let us consider the Ising ferromagnet with quenched disorder in terms of the scalar field random temperature D-dimensional Ginzburg–Landau Hamiltonian:

$$H[\phi,\xi] = \int d^D\mathbf{x} \times \times \left[\frac{1}{2}\big(\nabla\phi(\mathbf{x})\big)^2 + \frac{1}{2}(\tau - \xi(\mathbf{x}))\phi^2(\mathbf{x}) + \frac{1}{4}\, g\phi^4(\mathbf{x})\right]. \quad (1)$$

Here, the independent random quenched parameters $\xi(\mathbf{x})$ are Gaussian distributed with $\overline{\xi^2} = 2u$, where u is the parameter which describes the strength of the disorder. In what follows, it will be supposed that the disorder is weak, namely, $u \ll \tau^2$ and $u \ll g$. For a given realization of the disorder, the partition function of the considered system is

$$Z[\xi] = \int \mathcal{D}\phi \exp\big(-H[\phi,\xi]\big) = \exp\big(-F[\xi]\big), \quad (2)$$

where $\int \mathcal{D}\phi$ denotes the integration over all configurations of the fields $\phi(\mathbf{x})$ and $F\big[\xi\big]$ is the random (disorder dependent) free energy of the system. The distribution function of the random quantity $F\big[\xi\big]$ can be analyzed by studying the moments of the partition function $\overline{Z^N[\xi]} \equiv Z(N)$. Usually the replica partition function $Z(N)$ is studied for deriving the average value of the free energy. The heuristic (not well justified) procedure of the replica calculations requires performing the analytic continuation of the function $Z(N)$ from integer to arbitrary values of the replica parameter N. Then, taking the limit $N \to 0$, we formally get

$$\lim_{N\to 0}[Z(N) - 1]/N = \overline{F}.$$

In fact if we are interested not just in the average free energy of the system but in the properties of its distribution function, the situation even with weak disorder turns out to be rather nontrivial. Considering the system away from the critical point, one would naively expect that at least at scales greater than the correlation length (where the system is expected to split into a set of more or less independent "cells" of the size of the correlation length), "everything must be Gaussian distributed". One can easily prove that this is not true. First of all, it can be easily shown that all moments of the partition function $Z(N)$ with $N > g/u$ are divergent, and this automatically indicates that the free energy distribution function of the considered system cannot be Gaussian. Moreover, we can easily derive that the left asymptotic of the distribution function $P(F)$ has the following simple form:

$$P(F \to -\infty) \propto \exp\Big(-\frac{g}{u}|F|\Big).$$

It should be stressed that this phenomenon is quite general: it takes place for any dimension of the system independently of the value of the temperature parameter τ (except for the critical point) both in the paramagnetic and ferromagnetic phases.

3. TOY MODEL

To analyze the properties of the free energy distribution function of a random ferromagnet, one can start with very simple "toy" model [11] containing only one degree of freedom:

$$H(\phi,\xi) = \frac{1}{2}(\tau - \xi)\phi^2 + \frac{1}{4}\, g\phi^4, \quad (3)$$

where $\overline{\xi^2} = 2u$, and it is supposed that $u \ll \tau^2$ and $u \ll g$. It can be shown by sufficiently simple calculations that the free energy distribution function of this system is strongly non-symmetric and essentially non-Gaussian. Namely, the right tail of this distribution (at large positive values of the free energy),

$$P(F \to +\infty) \propto \exp\left[-\frac{1}{4u}\exp(4F)\right], \quad (4)$$

goes to zero much faster than the Gaussian curve, while its left tail (at large negative values of the free energy),

$$P(F \to -\infty) \propto \exp\Big(-\frac{g}{u}|F|\Big), \quad (5)$$

goes to zero much slower that the Gaussian one.

4. GINZBURG–LANDAU MODEL IN DIMENSIONS $D < 4$

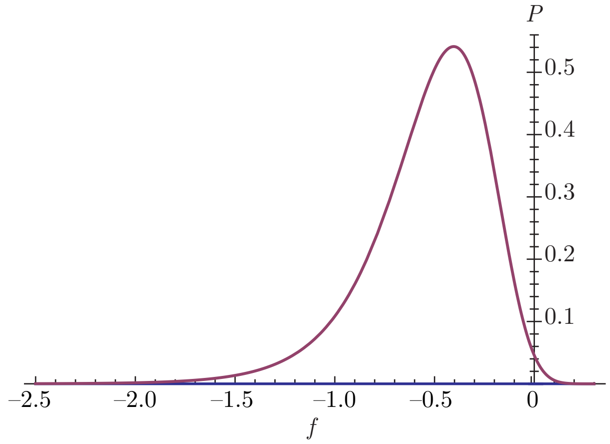
In this section, we consider the continuous version of the Ising ferromagnet in terms of the random temperature D-dimensional scalar field Ginzburg–Landau (GL) Hamiltonian, Eq. (1), where $\tau = (T - T_c)/T$ is the reduced temperature. Applying the renormalization group (RG) method to analyze the corresponding replica Hamiltonian in dimensions $D = 4 - \epsilon$, one does not encounter in the one-loop approximation the fixed point (FP) with both non-zero coordinates $u^* \neq 0$ and $g^* \neq 0$: this is because the system of equations for the fixed points is degenerate on the one-loop level [2,4,5]. This fixed point appears in the next, two-loop approximation.

Alternatively, RG equations have been analyzed directly at $D = 3$ using the minimal subtraction [18] and massive [19] RG schemes. To evaluate the divergent perturbation series in the renormalized couplings, appropriate resummation technique has been used. Results of the five loop calculations based on the minimal subtraction scheme at $D = 3$ are given in Ref. [20]. In the massive RG scheme, the most accurate results are obtained within accuracy of six loops in Ref. [21]. In particular, using two different resummation schemes based on the (i) conformal mapping and (ii) Padé approximants the following estimates for the FP values were obtained, respectively: (i) $u_* \simeq 2.14$, $g_* \simeq 6.28$ and (ii) $u_* \simeq 1.98$, $g_* \simeq 6.12$. For further calculations of the critical free energy distribution function, we take just the average of these two FP values: $u_* \simeq 2.06$ and $g_* \simeq 6.20$.

According to the general approach of the RG theory of critical phenomena in the vicinity of the phase transition point the total free energy F of the system can be decomposed into two essentially different contributions:

$$F = V f_0 + f,$$

where $V = L^D$ is the volume of the system (L is its linear size), f_0 is the regular (background) free energy density (which remains finite and non-singular at $T = T_c$). The second term, $f \sim L^D|\tau|^{2-\alpha}$ (where α is the specific heat critical exponent), represents the fluctuating part of the free energy which is singular at the critical point $\tau = 0$, and it is this part which is calculated in terms of the RG theory. Taking into account the standard relation among the critical exponents, $D\nu = 2 - \alpha$ (where ν is the critical exponent of the correlation length), one notes that at the critical point, when the correlation length becomes of the order of the system size, $R_c \sim |\tau|^{-\nu} \sim L$, the fluctuating part of the free energy $f \sim L^{(D\nu+\alpha-2)/\nu} \sim O(\ln L)$ is not proportional to the volume of the system. It is the distribution function of the random fluctuating part f of the free energy in the critical point which is derived in this section:



Critical free energy distribution function $P(f)$ of the disordered Ising ferromagnet in dimension $D = 3$

$$P(f) = \frac{1}{\sqrt{8\pi u_*}} \frac{\exp\left\{-f - \dfrac{1}{4u_*}\eta^2(-f)\right\}}{G'[\eta(-f)]}, \quad (6)$$

where the functional

$$G'(\eta(t)) = \frac{1}{2}\int_{-\infty}^{\infty} d\phi\, \phi^2 \exp\left\{\frac{1}{2}\eta(t)\,\phi^2 - \frac{1}{4}g_*\phi^4\right\} \quad (7)$$

and the function $\eta(t)$ is defined by the equation $\ln G(\eta) = t$. The universal curve for this distribution function is represented in the Figure. We see that, like in all the other systems, where the free energy probality distribution functions have been calculated [11, 22, 23], this function is essentially non-symmetric: the left tail is much more slow than the right one.

Note that both the left and the right tails of the probability distribution function $P(f)$ can be derived explicitly:

$$P(f \to -\infty) \propto \exp\left\{-\frac{g_*}{u_*}|f|\right\} \simeq \simeq \exp\{-3.01|f|\}, \quad (8)$$

$$P(f \to +\infty) \propto \exp\left\{2f - \frac{\pi^2}{u_*}\exp(4f)\right\} \simeq \simeq \exp\{2f - 4.79\exp(4f)\}. \quad (9)$$

We see that the left tail of the probability distribution function $P(f)$ is indeed much more slow than the right one. Note also that these asymptotics are quite similar to the ones of the toy model considered in the previous section, Eqs. (4) and (5).

5. TWO-DIMENSIONAL ISING MODEL

A system of particular interest is the Ising model in two dimensions, where exact results are available both for the pure [13] and weakly disordered [6, 7] cases. It is well known that the critical behavior of the two-dimensional ferromagnetic Ising model can be described in terms of free two-component Grassmann–Majorana spinor fields $\psi(\mathbf{r}) = \big(\psi_1(\mathbf{r}),\, \psi_2(\mathbf{r})\big)$, see, e. g., Ref. [24]. Correspondingly, the critical behavior of the weakly disordered two-dimensional Ising model can be described by the spinor Hamiltonian

$$H[\psi;\tau,\delta\tau] = \frac{1}{2}\int d^2r \times \times \Big[\overline{\psi}(\mathbf{r})\,\hat{\partial}\psi(\mathbf{r}) + \big(\tau+\delta\tau(\mathbf{r})\big)\,\overline{\psi}(\mathbf{r})\psi(\mathbf{r})\Big], \quad (10)$$

where $\tau = (T-T_c)/T_c \ll 1$ and the random function $\delta\tau(\mathbf{r})$ is characterized as a spatially uncorrelated Gaussian distribution with zero mean and the variance

$$\overline{\delta\tau(\mathbf{r})\delta\tau(\mathbf{r}')} = 2u_0\,\delta(\mathbf{r}-\mathbf{r}'),$$

where the parameter $u_0 \ll 1$ defines the disorder strength. For a given realization of the quenched function $\delta\tau(\mathbf{r})$, the partition function of the considered system is

$$Z[\tau;\delta\tau] = \int \mathcal{D}\psi \exp\big\{-H[\psi;\tau,\delta\tau]\big\} = = \exp\big\{-F[\tau;\delta\tau]\big\}, \quad (11)$$

where $F[\tau;\delta\tau]$ is a random free-energy function. The internal energy of a given realization is the first derivative of this free energy with respect to the temperature parameter:

$$E[\tau;\delta\tau] = \frac{\partial}{\partial\tau}F[\tau;\delta\tau].$$

It is clear that $E[\tau;\delta\tau]$ must be a singular function of τ in the limit $\tau\to 0$ (in the pure system $E_0(\tau) \propto \propto \tau\ln(1/|\tau|)$). Additionally, $E[\tau;\delta\tau]$ also must be a random function exhibiting sample-to-sample fluctuations. In this section, it is shown that in close vicinity of the critical point, at $\tau \ll \exp(-\pi/2u_0)$, and at sufficiently large system size, $L \gg \exp(\pi/2u_0)$, the critical internal energy E can be written as a sum of its mean value and a fluctuating part:

$$E \sim -\frac{1}{u_0}L\ln\ln(L) + \frac{1}{\sqrt{u_0}}\,L\,\ln\ln(L)\cdot f, \qquad (12)$$

where the random quantity f does not scale with L, $f\sim 1$, and is described by a standard normal distribution

$$P_c(f) = \frac{1}{\sqrt{2\pi}}\,\exp\left(-\frac{1}{2}f^2\right).$$

This result demonstrates that at criticality, the internal energy of the two-dimensional random-bond Ising ferromagnet is not self-averaging as the typical value of the sample-to-sample fluctuations,

$$E_c^*(L) \sim u_0^{-1/2}L\ln\ln(L),$$

scales with the system size in the same way as its average value,

$$E_c(L) \sim u_0^{-1}L\ln\ln(L).$$

On the other hand, unlike the singular part of the internal energy, the specific heat in the vicinity of the critical point is a self-averaging quantity, as its distribution function turns out to be the δ-function:

$$\mathcal{P}_\tau(C) = \delta\Big(C - C(\tau,L)\Big),$$

where

$$C(\tau,L) = \frac{1}{2g_0}\,L^2\ln\left(1+\frac{2}{\pi}u_0\ln\frac{1}{|\tau|}\right).$$

In particular, at large but finite value of the system size $L \gg L_* \propto \exp(2/\pi u_0)$ at the critical point at $\tau_c \propto 1/L$, the critical specific heat $C(L)$ scales with the system size as

$$C(L) \propto \frac{1}{2u_0}\,L^2\,\ln\ln(L).$$

6. CONCLUSIONS

In this brief review, we have presented recent studies of non-self-averaging phenomena in the critical point of weakly disordered ferromagnetic Ising model. In the case of the three-dimensional Ising system, we have derived the explicit expression for the probability distribution function of the critical free energy fluctuations. It should be stressed that the mere existence of such non-trivial distribution function (which is not the δ-function) in the thermodynamic limit means that the critical free energy fluctuations in the considered system are non-self-averaging. This, of course, is not surprising as the values of these critical fluctuations are not extensive with volume of the system. In more specific case of weakly disordered two-dimensional Ising model, we have presented the derivation of an explicit expression for the probability distribution function of the sample-to-sample fluctuations of the internal energy. This result shows that the internal energy of this system is not self-averaging. In the thermodynamic limit, as the system size L tends to infinity, the typical value of the internal energy sample-to-sample fluctuations scales in the same way as its average, proportional to $L \ln\ln(L)$. On the other hand, the specific heat was shown here to exhibit self-averaging with a distribution function that converges to a δ-function.

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