ЖЭТФ, 2019, том 156, вып. 4 (10), стр. 707–710

QUANTIZATION OF HYDRODYNAMICS: ROTATING SUPERFLUID, AND GRAVITATIONAL ANOMALY

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Received May 27, 2019,
revised version June 1, 2019
Accepted for publication June 2, 2019

Contribution for the JETP special issue in honor of I. M. Khalatnikov's 100th anniversary

DOI: 10.1134/S0044451019100122

The problem of quantization of hydrodynamics beyond linear approximation is commonly considered intractable. Nevertheless, nature confronts us with beautiful quantum non-linear ideal fluids with experimentally accessible precise quantization. Among them, two quantum fluids stand out: superfluid helium and electronic fluid in the fractional quantum Hall state. In both cases, the precise quantization of vortex circulation in superfluid helium and the precise quantization of electric transport in FQHE leave no doubts of the quantum nature of these fluids.

The fundamental aspects of quantization of fluid dynamics most clearly appear in ideal flows. These are incompressible flows $\nabla\cdot\mathbf{u}=0$ of homogeneous inviscid fluids. The problem of quantization is further specialized in chiral two-dimensional ideal flows which we consider in the paper. These are 2D flows with extensive vorticity: the mean vorticity

$$2\Omega=\frac{1}{V}\int\omega\,dV,\quad \omega(r)=\nabla\times\mathbf{u}, \tag{1}$$

remains finite, as $V\to\infty$. The chiral flows are distinguished by the holomorphic character of the quantum states.

Two most perfect quantum fluids, rotating superfluid helium [1], and FQHE fall to the class of ideal chiral flows (see [2, 3] for the correspondence between FQHE and superfluid hydrodynamics). In the paper we had shown how to quantize chiral flows and describe some, not immediately obvious, consequences of quantization. The guidance for the quantization comes from the intersection between quantum chiral hydrodynamics and quantum two-dimensional gravity.

Classical ideal flows are characterized by the Hamiltonian and the Poisson structure

$$\begin{gathered}H=\frac{\rho_A}{2}\int\mathbf{u}^2dV,\\ \{\omega(r),\ \omega(r')\}=\rho_A^{-1}\left(\nabla_r\times\nabla_{r'}\right)\omega(r)\delta(r-r').\end{gathered} \tag{2}$$

It is well known, that the Poisson structure is the Lie–Poisson algebra of area preserving diffeomorphisms **SDiff**. Hence flows of an ideal fluid are the actions of area-preserving diffeomorphism **SDiff**, and should be studied from a geometric standpoint, see, e. g., [4].

Formally the quantization amounts supplanting the Poisson brackets by the commutator

$$\{\ ,\ \}\to\frac{1}{i\hbar}[\ ,\ ]$$

and identifying the Hilbert space with a representation space of **SDiff**. The latter, however, is not well understood. The difficulties appear in the regularization of short-distance divergencies. A regularization must be consistent with fundamental local symmetries of the theory. The local "symmetry" of fluids is the relabeling symmetry, or equivalently the invariance with respect to re-parametrization of flows. We had shown that the relabeling symmetry alone determines the universal quantum corrections to the Euler equation. Relabeling are diffeomorphisms in the manifold of Lagrangian coordinates. Invariance with respect to diffeomorphisms is also a guiding principle of quantum gravity. In the paper, we described the correspondence between chiral

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flows and 2D gravity, and explained how this correspondence yields to a unique short-distance regularization. The result of the regularization is quantum corrections to the Euler equation expressed in terms of the gravitational anomaly.

We start by the notion of quantum stress. Consider a traceless part of the momentum flux tensor

$$\Pi'_{ij} = \rho_A \left(u_i u_j - \frac{1}{2} u^2 \delta_{ij} \right).$$

In the quantum case, the velocity u_i is an operator. We will be interested in expectation value of Π'_{ij}. The cumulant of the bilinear product of velocities is the (minus) quantum stress

$$-T'_{ij} = \rho_A \left(\langle\langle u_i u_j \rangle\rangle - \frac{1}{2} \langle\langle u^2 \rangle\rangle \delta_{ij} \right).$$

The quantum stress corrects the Euler equation as

$$\rho_A D_t u_i + \nabla_i p = \nabla^j T'_{ij}. \tag{3}$$

In this equation, all entries are assumed to be expectation values $u_i \to \langle u_i \rangle$, hence the equation could be treated as classical. Also

$$D_t = \partial_t + \langle \mathbf{u} \rangle \cdot \nabla$$

is the material derivative with respect to the expectation value of velocity and ρ_A is the mass density of the fluid. The divergency of the quantum stress vanishes, it yields to the corrections to hydrodynamics.

We write the Euler equation in the Helmholtz form by taking the curl of (3)

$$\rho_A D_t \omega = \epsilon^{ik} \partial_k \nabla^j T'_{ij}. \tag{4}$$

We see that unless the rhs of (4) vanishes, the major property of classical hydrodynamics, the Helmholtz law does not hold for the expectation value of vorticity. We recall that the Helmholtz law $D_t \omega = 0$ states that vorticity is frozen in the flow. Departure from the Helmholtz law is the major result of the paper. Despite it the Kelvin theorem (conservation of vorticity of a fluid parcel) is intact. The Kelvin theorem holds when the divergence of the stress has no circulation along a liquid contour

$$\oint \nabla^j T_{ij} dx^i = 0$$

and it, indeed, vanishes.

In the paper, we expressed the stress in terms of the expectation values of the vorticity. We present the result in complex coordinates

$$T'_{ij} dx^i dx^j = \frac{1}{4} [T_{zz} (dz)^2 + T_{\bar{z}\bar{z}} (d\bar{z})^2].$$

If we assume that vorticity is a smooth function and is positive $\omega > 0$, then we had shown that

$$T_{zz} = \frac{\hbar\Omega}{12\pi} \left(\partial_z^2 \log \omega - \frac{1}{2} (\partial_z \log \omega)^2 \right). \tag{5}$$

Then the quantum Helmholtz equation reads

$$\rho_A D_t \omega = \frac{\hbar\Omega}{48\pi} \nabla \mathcal{R} \times \nabla \omega, \tag{6}$$

where

$$\mathcal{R} = -\omega^{-1} \Delta \log \omega. \tag{7}$$

The reader may recognize that T_{zz} is the Schwarzian of a Riemann surface with the metric

$$ds^2 = \omega |dz|^2, \tag{8}$$

and that $\mathcal{R}$ in (7) is the curvature of that surface. These relations are not accidental. We argued that the chiral flow may be understood as evolving Riemann surface. The surface which hosts the fluid is a complex manifold equipped with a closed vorticity 2-form,

$$\omega_{ij} dx^i \wedge dx^i, \quad \omega_{ij} = \partial_i u_j - \partial_j u_i.$$

Because vorticity of the chiral flow does not change the sign

$$\omega = \frac{1}{2} \epsilon^{ij} \omega_{ij} > 0,$$

the chiral flow gives the host surface a Kähler structure with the Kähler form

$$\omega\, dz \wedge d\bar{z},$$

and the Riemannian metric (8). The Kähler form, the volume element of the surface is the vorticity in the fluid volume. Adopting the language of quantum gravity we identify the manifold of Lagrangian coordinates with a target space and the host surface with a world-sheet.

The coordinates of the tangent space of the surface appear as Clebsch variables. We recall that Clebsch variables parameterize vorticity as

$$\omega = \nabla \lambda^1 \times \nabla \lambda^2. \tag{9}$$

It follows that the intersection of level lines of λ^1 and λ^2 are position of vortices. Hence, vorticity

$$\omega = \det \| \partial_i \lambda^a \|,$$

is the Jacobian of the map

$$(x^1, x^2) \to (\lambda^1, \lambda^2)$$

and

$$e_i^a dx^i = d\lambda^a$$

are the vielbeins.

In these terms the diffeomorphisms in the space of Clebsch variables which leave vorticity unchanged appear as relabeling of vortices. This is relabeling symmetry or a diffeomorphism invariance of hydrodynamics. In the literature the relabeling symmetry usually refers to fluid atoms. In our approach, it is relabeling of vortices. We want to keep this major symmetry intact in quantization. This amounts that the short distance cut-off must be kept uniform in Clebsch coordinates and corresponds to the interval ds of the auxiliary surface. The relation between Clebsch and Eulerian coordinates

$$d\lambda^i = dx^i/\ell[\omega],$$

where

$$\ell[\omega] = \omega^{-1/2}$$

is a mean distance between vortices suggests that the short distance cut-off should be $\ell[\omega]$. It is non-uniform. It depends on the flow and on the position within the flow.

Based on this principle we were able to obtain the diffeomorphism-invariant regularization of the bilinear of velocities $u_i u_j$. One way of doing this is to split points

$$u_i(r)u_j(r) \to u_i\left(r + \frac{\epsilon}{2}\right) u_j\left(r - \frac{\epsilon}{2}\right)$$

and to take into account that the short distance cut-off is the functional of the flow $\epsilon = \ell[\omega]$.

We illustrate this idea in terms of the path integral approach to quantization. In this approach, one typically integrates over pathlines of fluid parcels. Instead, we choose to integrate over vorticity and for this purpose we need to know the measure on the space of vorticity. In order to determine it, we invoke the relation of the chiral flow to 2D quantum gravity described above. Since vorticity is a metric we effectively integrate over metrics. The measure on the space of metrics has been established in quantum gravity [5]. It consists of the Fadeev–Popov determinant restoring the re-parametrization invariance of the surface. In quantum gravity, it is a source of the Liouville action. In quantum hydrodynamics, it is a source of quantum stress.

In order to prove that the measure on vorticity configuration is the same as a measure of metrics, we employ the following procedure. We first assume that the flow consists of a large, albeit a finite number of vortices. Then the vorticity operator reads

$$\begin{aligned} \omega_k &= \int e^{-i\mathbf{k}\cdot\mathbf{r}} \omega(r)\, dV, \\ \omega_k &= \Gamma \sum_{i \le N_\text{v}} e^{-\frac{i}{2}\mathrm{k} z_i^\dagger} e^{-\frac{i}{2}\bar{\mathrm{k}} z_i}, \end{aligned} \qquad (10)$$

where Γ is a circulation of each vortex and z_i is a complex coordinate of a vortex. The operators acts in the Bargmann state [6, 7] of holomorphic polynomials of z_i and the conjugate coordinate $z_i^\dagger$ obey the Heisenberg algebra

$$[z_i, \bar{z}_j] = (\pi n_A)^{-1} \delta_{ij},$$

where

$$n_A = (\Gamma/h)\rho_A$$

is the number of helium atoms. Then it follows that vorticity operator obeys the sine-algebra [8]

$$\begin{aligned} [\omega_k,\ \omega_{k'}] &= i e^{\left(\frac{\mathbf{k}\cdot\mathbf{k}'}{4\pi n_A}\right)} e_{kk'} \omega_{k+k'}, \\ e_{kk'} &= 2\Gamma \sin\left(\frac{\mathbf{k}\times\mathbf{k}'}{4\pi n_A}\right). \end{aligned} \qquad (11)$$

This algebra is a finite-dimensional "approximation" of the Lie algebra **SDiff**. The latter is obtained by supplanting the Poisson brackets by the Lie brackets. The sine-algebra depends on two deformation parameters $\hbar$ and n_A. The limit $\hbar \to 0$ and $n_A \to \infty$ which keeps the mass density $\rho_A = h n_A/\Gamma$ and is taken at a fixed $\mathbf{k}$ brings back the Poisson structure (2). In order to obtain the quantum corrections we need a different limit when $\mathbf{k} \times \mathbf{k}'$ increases with the same rate as n_A. A proper execution of this limit yields the result for the quantum stress (5). The formula (5) essentially means that the stress obeys the conformal Ward identity and its moments are generators of the Virasoro algebra. Let us represent the stress by the Laurent series about the origin (the fixed point of the rotation)

$$T_{zz}(z) = -\sum_n z^{-n-2} L_n$$

and set $\omega = 2\Omega$. Then L_n generate the Virasoro algebra (with the central charge $c = 1$)

$$[L_n,\ L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.$$

Summing up, in the paper we present a consistent scheme of quantization of chiral flows of the ideal 2D fluid. The quantization is based on a geometric relation of chiral flows to two-dimensional quantum gravity and is implemented by the gravitational anomaly. The effect of the gravitational anomaly violates the major property of classical hydrodynamics, the Helmholtz law: vortices are no longer frozen into the flow. We show that quantum stress generates the Virasoro algebra, the centrally extended algebra of holomorphic diffeomorphisms. The result follows as the limit of the finite-dimensional approximation of Lie algebra of area-preserving diffeomorphisms **SDiff** yielding diffeomorphism invariant regularization of the advection term in Euler equation. The main applications of this theory are rotating superfluid and electronic systems in the magnetic field in the regime of a fractional Hall effect.

The full text of this paper is published in the English version of JETP.

REFERENCES

1. I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, Vol. 23, Perseus Books (1989).

2. P. B. Wiegmann, Phys. Rev. B **88**, 241305 (2013).

3. P. Wiegmann, Zh. Eksp. Teor. Fiz. **144**, 617 (2013) [J. Exp. Theor. Phys. **117**, 538 (2013)].

4. V. I. Arnold and B. A Khesin, *Topological Methods in Hydrodynamics*, Vol. 125, Springer (1999).

5. A. M. Polyakov, *Gauge Fields and Strings* (*Contemporary Concepts in Physics*), Harwood Academic Publishers, Switzerland (1987).

6. S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. B **33**, 2481 (1986).

7. V. Bargmann, Comm. on Pure and Appl. Math. **14**, 187 (1961).

8. D. B. Fairlie and C. K. Zachos, Phys. Lett. B **224**, 101 (1989).