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ANISOTROPIC STRING COSMOLOGICAL MODEL IN BRANS–DICKE THEORY OF GRAVITATION WITH TIME-DEPENDENT DECELERATION PARAMETER

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We discuss a spatially homogeneous and anisotropic string cosmological models in the Brans–Dicke theory of gravitation. For a spatially homogeneous metric, it is assumed that the expansion scalar θ is proportional to the shear scalar σ. This condition leads to $A = kB^m$, where k and m are constants. With these assumptions and also assuming a variable scale factor $a = a(t)$, we find solutions of the Brans–Dicke field equations. Various phenomena like the Big Bang, expanding universe, and shift from anisotropy to isotropy are observed in the model. It can also be seen that in early stage of the evolution of the universe, strings dominate over particles, whereas the universe is dominated by massive strings at the late time. Some physical and geometrical behaviors of the models are also discussed and observed to be in good agreement with the recent observations of SNe Ia supernovae.

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1. INTRODUCTION

In the original Newtonian theory of gravitation, the gravitational acceleration a generated by a body of mass M at a distance r is directly proportional to the mass M and inversely proportional to the square of the distance r, $a \approx GM/r^2$, where G is the proportionality coefficient called the gravitation constant. In the general theory of relativity, the gravitational field and hence the gravitational acceleration a was also assumed to be the same. But in the vicinity of several distant astronomical bodies, the real gravitational accelerations based on the observable mass M_{obs} was found to be much larger than what was predicted, $a \gg GM/r^2$. To explain this problem, it was traditionally assumed that, in addition to the observable masses, there also exist nonobservable masses. In this approach, to arrive at the observational data, it must be assumed that 95 % of the mass on the cosmological level is formed by hypothetical "dark matter" and "dark energy", which are not directly observable.

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Later on, Brans and Dicke [1], in their alternative theory of gravity, argued that instead of introducing such hypothetical types of matter, it is more reasonable to conclude that the parameter G describing the local strength of gravitational interactions may be taken as a local constant instead of a universal constant; hence, measurements of G at different points in space–time can lead, in general, to different results. In effect, they assumed a variable $\phi(x)$ instead of $1/G$, measured at different space–time points x, which forms a new scalar field. In such a theory, describing the gravitational field requires both the metric field g_{ij} and the scalar field ϕ. In terms of this new field, the Einstein term R/G in the Lagrangian takes the form ϕR. To obtain a full description of the scalar-tensor theory, we also need to add the term $\phi_{,i}\phi^{,i}/\phi$ to the Lagrangian, describing the effective energy density of the scalar field. As a result, we arrive at the Lagrangian

$$L_{BDT} = \phi\left(R - \frac{\phi_{,i}\phi^{,i}}{\phi^2}\right) + 16\pi L_{mat}.$$

Varying over g_{ij} and ϕ gives the field equations

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi}{\phi}T_{ij} + \frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) + \frac{1}{\phi}(\phi_{;ij} - g_{ij}\Box\phi)$$

and

$$\Box\phi = \phi^{;k}_{;k} = \frac{8\pi}{(2w+3)\phi},$$

where T is the trace T^i_i of the energy momentum tensor.

The above two expressions represents the simple form of the Brans–Dicke field equations, which we use in our study.

Earlier, work has been done in Einstein's theory of gravity. In [2], anisotropic Bianchi type-III string cosmological models were presented in the normal gauge for Lyra's manifold for Einstein's field equations with a variable deceleration parameter. The locally rationally symmetric (LRS) Bianchi type-II perfect fluid cosmological models in the normal gauge for Lyra's manifold were given in [3]. Anisotropic Bianchi type-I string cosmological models in the normal gauge for Lyra's manifold with a constant deceleration parameter were described in [4]. In [5], accelerating Bianchi type-V cosmology with perfect fluid and heat flow in the Sáez–Ballester theory were presented. String cosmological models in a scalar–tensor theory and in the Brans–Dicke theory of gravitation were discussed in [6,7]. String cosmological models in Bianchi type III and LRS Bianchi type I with time-dependent bulk viscosity were studied in [8,9].

Recently, solutions of the Brans–Dicke field equations for anisotropic string cosmological models with constant deceleration parameter were found in [10]. In this paper, we deal with a spatially homogeneous and anisotropic string cosmological models in the Brans–Dicke theory of gravitation with a time-dependent deceleration parameter. For a spatially homogeneous metric, we assume that the expansion scalar θ is proportional to the shear scalar σ. This condition leads to $A = kB^m$, where k and m are constants [11]. With these assumptions and also assuming a variable scale factor, solutions of the Brans–Dicke field equations have been found and observed to be consistent with three recent observations of SNe Ia supernovae [12–15]. It is also observed that in the early stage of the evolution of the universe, strings dominate over particles whereas the universe is dominated by massive strings at the late time.

The outline of the paper is as follows. In Sec. 2, we define the metric, the field equations and various parameters used in what follows. Solutions of the field equations are derived in Sec. 3. In Sec. 4, we interprete the solutions obtained and present some of them graphically. Finally, the conclusions are given in Sec. 5.

2. METRIC, FIELD EQUATIONS, AND PARAMETERS

We consider the LRS metrics

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)\left[dy^2 + dz^2\right], \qquad (1)$$

where $A(t)$ and $B(t)$ are functions of the cosmic time t. We consider the simple Brans–Dicke field equations

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi}{\phi}T_{ij} + \frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) + \\ + \frac{1}{\phi}(\phi_{;ij} - g_{ij}\Box\phi), \qquad (2)$$

where

$$\Box\phi = \phi^{;k}_{;k} = \frac{8\pi}{(2w+3)\phi} \qquad (3)$$

and w represents a constant of coupling, which is dimensionless.

The energy–momentum tensor for an anisotropic dark energy fluid is taken as

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j, \qquad (4)$$

$$T^{ij}_{,i} = 0. \qquad (5)$$

Here, ρ is the proper energy density and λ is the string tension density. These two are related by

$$\rho = \rho_p + \lambda, \qquad (6)$$

where ρ_p is the particle density of the configuration.

The four velocity u^i has components $(0,0,0,1)$ for a cloud of particles, and x^i represents the direction of the string, which satisfy

$$u^i u_i = -x^i x_i = 1, \quad u^i x_i = 0. \qquad (7)$$

We define the following parameters to be used in solving the Brans–Dicke field equations for the LRS metric. The mean scale factor a of the model is defined as

$$a(t) = (AB^2)^{1/3}. \qquad (8)$$

The volume scale factor V is given by

$$V = a(t)^3 = AB^2. \qquad (9)$$

We also define the mean Hubble parameter H as

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right). \qquad (10)$$

Here, and in what follows, a dot indicates ordinary differentiation with respect to t.

Further, the deceleration parameter q is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{11}$$

We introduce the kinematical quantities such as the expansion scalar θ, the shear scalar σ^2, and an anisotropy parameter A_m, defined as

$$\theta = u^i_{;i}, \tag{12}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}, \tag{13}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \tag{14}$$

where $u^i = (0,0,0,1)$ is the matter 4-velocity vector and

$$\sigma_{ij} = \frac{1}{2}\left(u_{i;\alpha}P^\alpha_j + u_{j;\alpha}P^\alpha_i\right) - \frac{1}{3}\theta P_{ij}. \tag{15}$$

The projection tensor P_{ij} has the form

$$P_{ij} = g_{ij} - u_i u_j. \tag{16}$$

These dynamical scalars, in the assumed metric, have the forms

$$\theta = 3H = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}, \tag{17}$$

$$2\sigma^2 = \left[\left(\frac{\dot{A}}{A}\right)^2 + 2\left(\frac{\dot{B}}{B}\right)^2\right] - \frac{\theta^2}{3}, \tag{18}$$

$$A_m = \frac{1}{3}\left[\left(\frac{H_1 - H}{H}\right)^2 + 2\left(\frac{H_2 - H}{H}\right)^2\right], \tag{19}$$

where $H_1 = \dot{A}/A$ and $H_2 = \dot{B}/B$ are directional Hubble parameters.

3. SOLUTIONS OF THE FIELD EQUATIONS

Brans–Dicke field equations (2) for metric (1) with the energy–momentum tensor as in Eq. (4) yield the differential equations

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = \frac{8\pi\lambda}{\phi} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \left(\frac{\ddot{\phi}}{\phi}\right) + 2\frac{\dot{\phi}\dot{B}}{\phi B}, \tag{20}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right), \tag{21}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} = \frac{8\pi\rho}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right), \tag{22}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) = \frac{8\pi(\lambda + \rho)}{\phi(3 + 2\omega)}, \tag{23}$$

$$\dot{\rho} + (\rho - \lambda)\frac{\dot{A}}{A} + \rho\frac{2\dot{B}}{B} = 0. \tag{24}$$

Here, we have four independent equations for five unknowns A, B, ϕ, ρ, and λ. Therefore, we need more relations to find the sought solutions of these equations. Any one quantity may be chosen freely to solve the system of equations. Hence, without any loss of generality, we assume that the Brans–Dicke scalar field ϕ is some power of the mean scale factor; this has already been used in [16] in the context of Robertson–Walker Brans–Dicke models. Then

$$\phi \propto a^\gamma, \tag{25}$$

where γ is any number, which implies that $\phi = ca^\gamma$, where c is a constant.

For a spatially homogeneous metric, the normal congruence to homogeneous expansion implies that the expansion scalar θ is proportional to shear scalar σ, i.e., $\sigma/\theta = n$ where n is a constant. This condition leads to [11]

$$A = kB^m, \tag{26}$$

where k is a constant.

We define the time-dependent deceleration parameter $q(t)$ as

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{27}$$

We were motivated to choose such a time-dependent deceleration parameter by the fact that the universe is presently under going an accelerated expansion, as is observed in the recent observations of SNe Ia [17–21], and cosmic microwave background anisotropies [22–24], and experienced a decelerated expansion in the past. For a universe that was decelerating in the past and accelerating at the present time, the deceleration parameter must show signature flipping (see [25–27]). In general, therefore, the deceleration parameter is not a constant but a time-dependent variable.

Equation (27) can be rewritten as

$$\frac{\ddot{a}}{a} + q\frac{\dot{a}^2}{a^2} = 0. \tag{28}$$

To solve Eq. (28), we assume that $q = q(a(t))$. Integrating Eq. (28) gives [28]

$$a(t) = [\sinh(\alpha t)]^{1/n}. \tag{29}$$

From Eqs. (8), (26), and (29), we obtain that

$$B = k^{-1/(m+2)}[\sinh(\alpha t)]^{3/n(m+2)}, \tag{30}$$

$$A = k^{(1-m)/(m+2)}[\sinh(\alpha t)]^{3m/n(m+2)}. \quad (31)$$

Now using Eqs. (25), (30), and (31) in (22), we find

$$\rho = \frac{\phi}{8\pi}\left[\frac{6mk+1}{(m+2)^2} - \frac{3c\gamma mk - 2c\gamma}{m+2} - \frac{wc^2\gamma^2}{2}\right] \times \times \frac{\alpha^2}{n^2}\coth^2(\alpha t). \quad (32)$$

Using Eqs. (25), (30), and (31) in (20) yields

$$\lambda = \frac{\phi}{8\pi}\left\{\frac{3}{(m+2)^2} + \frac{3wc^2\gamma^2}{2} - \frac{2c\gamma}{m+2}\right\} \times \times \frac{\alpha^2}{n^2}\coth^2(\alpha t) - \frac{\phi}{8\pi} \times \times \left\{\frac{2}{m+2} + c\gamma\right\}\frac{\alpha^2}{n}\,\mathrm{cosech}^2(\alpha t). \quad (33)$$

Now, subtracting (33) from (32) we obtain

$$\rho_p = \rho - \lambda = \frac{\phi}{8\pi}\left\{\frac{6mk-2}{(m+2)^2} - - \frac{3c\gamma mk - 4c\gamma}{m+2} - 2wc^2\gamma^2\right\}\frac{\alpha^2}{n^2}\coth^2(\alpha t) + + \frac{\phi}{8\pi}\left\{\frac{2}{m+2} + c\gamma\right\}\frac{\alpha^2}{n}\,\mathrm{cosech}^2(\alpha t). \quad (34)$$

The deceleration parameter q, the volume scale factor V, the Hubble parameter H, the expansion scalar θ, the shear scalar σ, and the mean anisotropic parameter A_m are expressed as

$$q = -1 + n\,\mathrm{sech}^2(\alpha t), \quad (35)$$

$$V = k^{-(1+m)/(m+2)}[\sinh(\alpha t)]^{3/n}, \quad (36)$$

$$H = \frac{1}{3}\left(\frac{3mkk+1}{m+2}\right)\frac{\alpha}{n}\coth(\alpha t), \quad (37)$$

$$\theta = 3H = \left(\frac{3mkk+1}{m+2}\right)\frac{\alpha}{n}\coth(\alpha t), \quad (38)$$

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{3mkk-1}{m+2}\right)\frac{\alpha}{n}\coth(\alpha t), \quad (39)$$

$$A_m = \frac{1}{3}\left[\frac{4+2(3mk+1)^2}{(3mk+2)^2}\right]. \quad (40)$$

4. INTERPRETATION OF THE SOLUTIONS OBTAINED IN THE MODEL

The expressions for the proper energy density ρ, the string tension λ, and the particle density ρ_p for the model in Eq. (1) are given by Eqs. (32), (33), and (34).

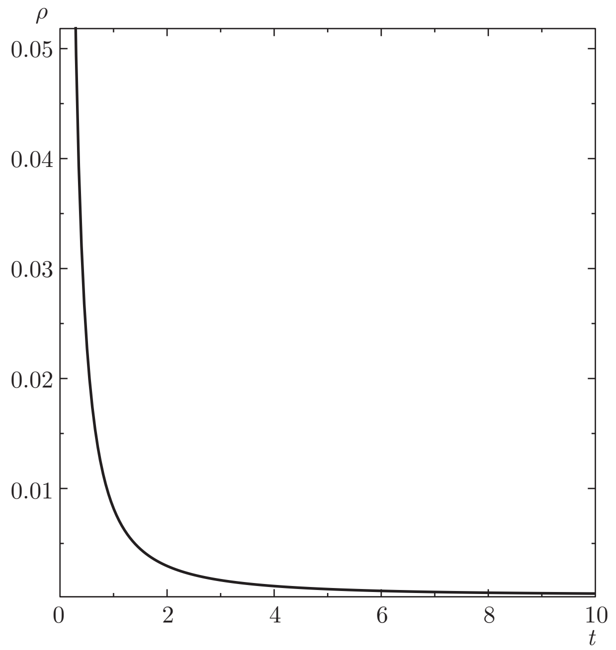

Fig. 1. The proper energy density ρ versus the cosmic time t for $\alpha = 0.0922$, $\gamma = 0.5$, $n = 1$, $m = 0.5001$, $c = 0.5$, $k = 1$

We observe that the proper energy density conditions $\rho \geq 0$, $\rho_p \geq 0$ are satisfied under conditions

$$\left[\frac{6mk+1}{(m+2)^2}\right] \geq \left[\frac{3c\gamma mk - 2c\gamma}{m+2} + \frac{wc^2\gamma^2}{2}\right]. \quad (41)$$

Figure 1 depicts the proper energy density ρ versus time t. We see that $\rho(t)$ is a positive decreasing function of time, which approaches a small positive value at the present epoch.

We also observe that the string tension density $\lambda \geq \geq 0$ under the conditions

$$\left[\frac{3}{(m+2)^2} + \frac{3wc^2\gamma^2}{2} - \frac{2c\gamma}{m+2}\right]\frac{\cosh^2(\alpha t)}{n} \geq \geq \left[\frac{2}{m+2} + c\gamma\right]. \quad (42)$$

It follows from Eq. (33) that λ is negative. We see from Fig. 2 that λ is an increasing function of time, but is negative. But λ can be only positive under condition (42). It is pointed out in [29, 30] that λ may be positive or negative. When $\lambda < 0$, the string phase of the universe disappears, i. e., we have an anisotropic fluid of particles.

It can be seen from Eq. (34) that the particle density ρ_p is a decreasing function of time and $\rho_p > 0$ under the condition

$$\left[\frac{6mk-2}{(m+2)^2}\right] \geq \left[\frac{3c\gamma mk - 4c\gamma}{m+2} + \frac{2wc^2\gamma^2}{2}\right]. \quad (43)$$

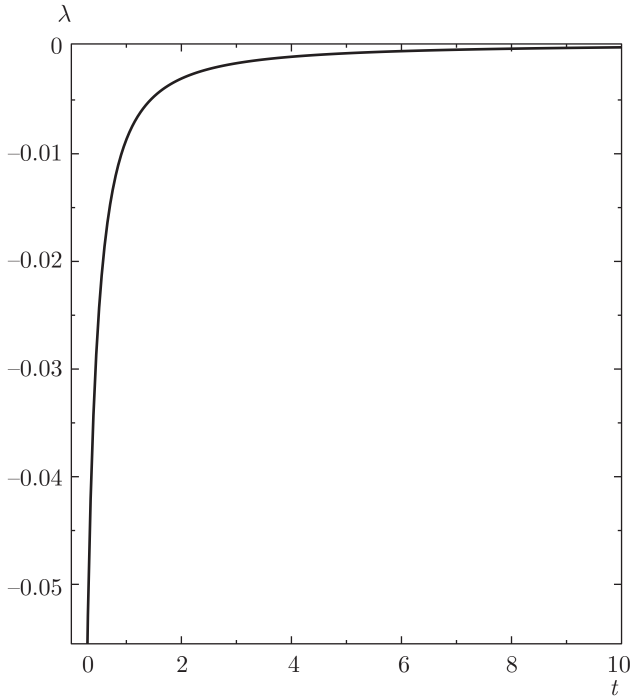

Fig. 2. The string tension density λ versus the cosmic time t for $\alpha = 0.0922$, $\gamma = 0.5$, $n = 1$, $m = 0.5001$, $c = 0.5$, $k = 1$

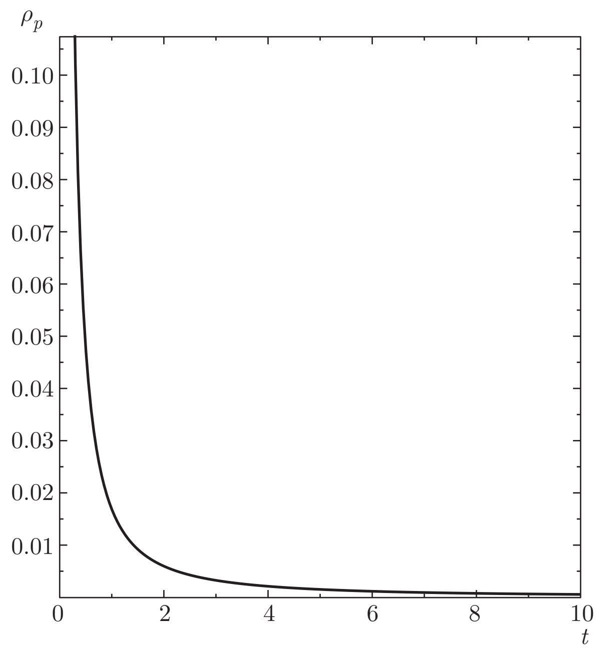

Fig. 3. The particle energy density ρ_p versus the cosmic time t for $\alpha = 0.0922$, $\gamma = 0.5$, $n = 1$, $m = 0.5001$, $c = 0.5$, $k = 1$

This nature of ρ_p is clearly shown in Fig. 3.

All the physical quantities ρ, ρ_p, and λ tend to infinity as $t \to 0$ and tends to 0 as $t \to \infty$. Model (1) therefore starts with a Big Bang at $t = 0$ and it continues expanding until it comes to a rest at $t = \infty$. There is a point-type singularity [31] in the model at $t = 0$.

The deceleration parameter is given as

$$q = -1 + n(1 - \tanh^2(\alpha t)).$$

From the above equation, we observe that $q > 0$ for $t < \alpha^{-1}\tanh^{-1}(1 - 1/n)^{1/2}$ and $q < 0$ for $t > \alpha^{-1}\tanh^{-1}(1 - 1/n)^{1/2}$ [28]).

Also, recent observations of type Ia supernovae [21–25] reveal that the present-day universe is in an accelerating phase and the deceleration parameter lies somewhere in the range $-1 < q \le 0$ (Fig. 4). It follows that our model of the universe is consistent with the recent observations. The expression for the proper volume V is given in (36). It can be seen that the spatial volume is zero at $t = 0$ and increases with an increase in the time t, which shows consistency with the concept of an expanding universe.

The expression for the mean Hubble parameter H, the expansion scalar θ, the shear scalar σ^2, and the mean anisotropy parameter A_m for the model are respectively given by Eqs. (37), (38), (39), and (40).

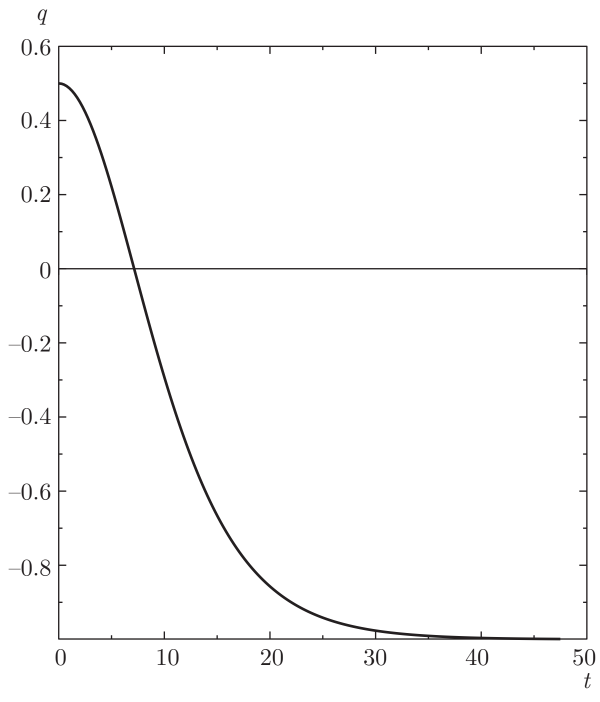

Fig. 4. The deceleration parameter q versus the cosmic time t for $\alpha = 0.0922$, $n = 1.5$

The shear scalar diverses at $t = 0$. The expansion scalar and shear scalar all tend to zero as $t \to \infty$. The mean anisotropy parameter is uniform throughout whole expansion of the universe. This shows that the universe is expanding with the cosmic time, but the rate of expansion and the shear scalar decrease to zero and tend to be isotropic. At the initial stage of expansion, when ρ is large, the Hubble parameter is also large and H and θ decrease with the expansion of the universe as does ρ because $\sigma^2/\theta^2 = \text{const}$.

5. CONCLUDING REMARKS

We have found that the proper volume and the shear scalar approach zero at the initial stage and approach an infinitely large value in the limit as $t \to \infty$. The proper energy density ρ and the particle energy density ρ_p are infinitely large at the initial stage and tend to zero as $t \to \infty$, which is consistent with the Big Bang theory. We have also found that the string tension density (λ) is negative at the initial stage and tends to zero as $t \to \infty$, which allows concluding that our model starts from an anisotropic one. The model describes an expanding cosmological model with a variable deceleration parameter. The mean anisotropy parameter is uniform throughout whole expansion of the universe. This shows that the universe is expanding with the cosmic time, but the rate of expansion and the shear scalar decrease to zero and tend to isotropic ones. At the time of the beginning of this model, (universe), the scalar field is found to be a constant quantity; it then increases gradually with time. Very close to the Big Bang singularity, matter is in a highly dense exotic form, which may include viscosity, heat flow, and null radiation flow as well as cosmic strings [32]. Therefore, it is very important that we have a space–time metric with a time-dependent scalar field capable of describing almost all these attributes for suitable values for certain parameters. The energy conditions are satisfied. Thus, we have found a new solution for inflation that deserves attention.

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