ЖЭТФ, 2015, том 148, вып. 3 (9), стр. 470–484 © 2015

NUCLEON QCD SUM RULES IN THE INSTANTON MEDIUM

*M. G. Ryskin, E. G. Drukarev**, V. A. Sadovnikova*

National Research Center "Kurchatov Institute",
Konstantinov Petersburg Nuclear Physics Institute
188300, Gatchina, Leningrad Region, Russia

Received April 2, 2015

We try to find grounds for the standard nucleon QCD sum rules, based on a more detailed description of the QCD vacuum. We calculate the polarization operator of the nucleon current in the instanton medium. The medium (QCD vacuum) is assumed to be a composition of the small-size instantons and some long-wave gluon fluctuations. We solve the corresponding QCD sum rule equations and demonstrate that there is a solution with the value of the nucleon mass close to the physical one if the fraction of the small-size instantons contribution is $w_s \approx 2/3$.

DOI: 10.7868/S0044451015090059

1. INTRODUCTION

The idea of the QCD sum rule approach is to express the characteristics of the observed hadrons in terms of vacuum expectation values of the QCD operators often referred to as condensates. This idea was suggested in [1] for the calculation of the characteristics of mesons. Later, it was used for nucleons [2]. It succeeded in describing the nucleon mass, the anomalous magnetic moment, the axial coupling constant, etc. [3].

The QCD sum rule approach is based on the dispersion relation for the function describing propagation of the system that carries the quantum numbers of a hadron. This function is usually referred to as the "polarization operator" $\Pi(q)$, with q being the four-momentum of the system. The dispersion relation (in which we do not take care of subtractions)

$$\Pi(q^2) = \frac{1}{\pi}\int dk^2 \frac{\operatorname{Im}\Pi(k^2)}{k^2 - q^2} \qquad (1)$$

is analyzed at large and negative values of q^2. Due to the asymptotic freedom of QCD, the polarization operator can be calculated in this domain. Operator product expansion (OPE) [4] enables to represent the polarization operator for a power series in q^{-2} as $q^2 \to -\infty$. The coefficients of the expansion are the QCD condensates, such as the scalar quark condensate $\langle 0|\bar{q}(0)q(0)|0\rangle$, the gluon condensate $\langle 0|G^{a\mu\nu}G^a_{\mu\nu}|0\rangle$, etc. The nonperturbative physics is contained in these condensates. A typical value of a condensate with the dimension $d = n$ is $\langle 0|O_n|0\rangle \sim (\pm 250\ \text{MeV})^n$. Hence, we expect the series $\Pi(q) = \sum_n \langle 0|O_n|0\rangle/(q^2)^n$ to converge at $-q^2 \sim 1\ \text{GeV}^2$.

The left-hand side of Eq. (1) is calculated as an OPE series. The imaginary part in the right-hand side describes physical states with the baryon quantum number and charge equal to unity. These are the proton, described by the pole of $\operatorname{Im}\Pi(k^2)$, the cuts corresponding to systems containing a proton and pions, and so on. The right-hand side of Eq. (1) is usually approximated by the "pole + continuum" model [1, 2], in which the lowest-lying pole is written exactly, while the higher states are described by the continuum. The main aim is to obtain the value of the nucleon mass.

The polarization operator can be written as

$$\Pi(q^2) = i\int d^4x\, e^{i(q\cdot x)}\langle 0|T[j(x)\bar{j}(0)]|0\rangle, \qquad (2)$$

where $j(x)$ a local operator with the proton quantum numbers, often referred to as the "current". It is a composition of quark operators. Therefore, the integrand in Eq. (2) contains the nonlocal expectation values $\langle 0|\bar{q}(0)q(x)|0\rangle$. The nonlocal condensates have been considered previously (see [5] and the references therein), mainly in the studies of pion wave functions.

We note that the product $\bar{q}(0)q(x)$ is not gauge invariant. This expression makes sense if we define $q(x)$ as the expansion near the point $x = 0$, i. e.,

*E-mail: drukarev@thd.pnpi.spb.ru

$$q(x) = \left(1 + x^\mu D_\mu + \frac{x^\mu x^\nu}{2} D_\mu D_\nu + \ldots\right) q(0), \quad (3)$$

with D^μ standing for covariant derivatives. The Fock–Schwinger (fixed-point) gauge $x_\mu A^\mu(x) = 0$, where $A^\mu(x)$ is the gluon field, is usually employed. This allows replacement the covariant derivatives by ordinary ones. Thus, the condensate $\langle 0|\bar{q}(0)q(x)|0\rangle$ can be expressed as a Taylor series containing a set of new condensates, such as $\langle 0|\bar{q}(0)\partial^2 q(0)|0\rangle$.

In this approach, the QCD condensates are considered as phenomenological parameters. Extracting their values from experimental data, supported by certain theoretical ideas, does not always lead to unique conclusions.

The Borel transform is usually applied, which converts functions of q^2 into functions of the Borel mass M^2. We also note that the Borel transform removes divergent contributions caused by the behavior of the integrand on the right-hand side of Eq. (2) as $x \to 0$. An important assumption is that there is an interval of the values of M^2 where the two sides of the sum rules have a good overlap, also approximating the true functions. This interval is in the range of 1 GeV^2. Thus, one actually tries to expand the OPE from the high-momentum region to the region of $|q^2| \sim 1$ GeV^2.

To calculate the polarization operator defined by Eq. (2), we must clarify the form of the current $j(x)$. It is not unique. We can write

$$j(t, x) = j_1(x) + t j_2(x), \quad (4)$$

with

$$j_1(x) = (u_a^T(x) C d_b(x))\gamma_5 u_c(x)\varepsilon^{abc},$$

$$j_2(x) = (u_a^T(x) C\gamma_5 d_b(x)) u_c(x)\varepsilon^{abc},$$

while t is an arbitrary coefficient. Following [6], we use the current determined by Eq. (4) with $t = -1$, which can be written (up to a factor of 1/2) as

$$j(x) = (u_a^T(x) C\gamma_\mu u_b(x))\gamma_5\gamma^\mu d_c(x)\varepsilon^{abc}. \quad (5)$$

This current is often used in the QCD sum rules calculations. One of the strong points of the choice is that it makes the domination of the lowest pole over the higher states on the right-hand side of the sum rules more pronounced. We use only this current in the present paper.

Any model of the QCD vacuum should explain the origin and the values of QCD condensates. A currently popular standpoint (see, e. g., [7]) is that the QCD vacuum is filled with strong gluon fields (instantons). The values of the QCD condensates are determined by the space-time structure of the instantons. Hence, the instantons provide a more detailed description of the vacuum than the QCD condensates do.

We try to write the QCD sum rules in terms of the instanton vacuum parameters. Our aim is not to replace the OPE approach but to study a possible role of a more detailed structure of the QCD vacuum.

The instanton medium is characterized by a distribution of the instantons over their sizes ρ, to be denoted by $n(\rho)$, and by the distances R between instantons, which also have a certain distribution. The distribution $n(\rho)$ is known to peak at $\rho \approx 0.33$ fm [7]. A summary of a number of lattice calculations of the distribution is presented in [8]. A detailed analysis of the distribution over sizes is given in [9]. As regards the distance between the instantons, the conventional assumption is that the average separation is $R \approx 1$ fm [7]. We use the simplest model that reproduces the essential physics of the process. We assume that the QCD vacuum consists of "small-size" instantons with $\rho_s \approx 0.33$ fm (we vary this value in what follows) and some long-wave gluon field fluctuations characterized by a scale $\rho_\ell \gg 1$ GeV^{-1}. Thus, the quarks composing the polarization operator move in a superposition of the fields of small-size instantons and some long-wave fluctuations.

We treat the quarks in the field of small instantons following the approach developed in [10, 11]. In this approach, the light quarks move in the self-consistent field of interacting small-size instantons. They are described by the propagator (in the Eucledian metric)

$$S_{ab}(p) = \frac{\hat{p} + im(p)}{p^2 + m^2(p)}\delta_{ab}, \quad (6)$$

with the effective dynamical mass $m(p)$ found in [10, 11]. We note that the value $R = 1$ fm allows reproducing the value of the gluon condensate in such an instanton vacuum model. This instanton–instanton separation $R = 1$ fm is much larger than the inverse Borel mass $1/M \approx 0.2$ fm. Hence, the size of the system described by polarization operator (2) is much smaller than R and can accommodate only one ("nearest") instanton. We recall that this is a part of a self-consistent system of interacting instantons. This leads to several consequences. We can write the quark propagator in the "nearest-instanton approximation" (NIA) as

$$S_{ab}(p) = S_Z + S_{NZ}, \quad S_Z(p) = \frac{im(p)}{p^2}\delta_{ab}, \quad S_{NZ}(p) = \frac{\hat{p}}{p^2}\delta_{ab}, \quad (7)$$

where S_Z is the zero-mode contribution. The sum of all nonzero-mode contributions S_{NZ} is approximated by the free propagator of a massless quark. We note that in the NIA, we include only the terms that are proportional to the instanton density.

In this approach, one of the nonvanishing contributions comes from the configurations where all quarks are described by the propagators S_{NZ}. Another contribution comes from the configuration where the u quarks are described by S_{NZ}, i. e., do not feel the instantons, while the d quark is described by the propagator S_Z. The other configurations do not contribute to the polarization operator because two u quarks cannot be in a zero mode of the same instanton. The configuration in which u and d quarks are in zero modes correspond to the SU(2) version of the instanton-induced 't Hooft interaction [12]. For current (5), this configuration does not contribute because it contains the trace of an odd number (three) of γ matrices, which vanishes.

Comparing the structures of chirality-conserving and chirality-flipping components of the polarization operator in the condensate and small-size nearest instanton "languages", we see that they differ. In both "languages", the chirality-conserving structure contains the loop of three free quarks. But there is no such thing as a four-quark condensate in the "nearest instanton approximation"[1]. In the nucleon QCD sum rules, M^2 is of the order of 1 GeV^2, and hence $\tau = M^2\rho_s^2 \sim 1$. On the other hand, the quark condensate created by the small-size instantons can be represented by the general relation

$$\langle 0|\bar{q}(0)q(0)|0\rangle_s = i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,S(p) = = -4N_c\int \frac{d^4p}{(2\pi)^4}\,\frac{m(p)}{p^2} \qquad (8)$$

(the subscript s means the small-size instantons, and N_c is the number of colors), creating a bridge between the instanton and condensate languages. In the limit $\tau \gg 1$, the two languages provide the same result, and the contribution is proportional to the quark condensate $\langle 0|\bar{q}(0)q(0)|0\rangle$. In the instanton picture at $\tau \sim 1$, the contribution can be viewed as coming from the nonlocal scalar condensate $\langle 0|\bar{q}(0)q(x)|0\rangle_s$. The nonlocal condensate is not a new subject, it was used previously, for example, in the pion QCD sum rules in [5].

[1] Going beyond the terms that are linear in the instanton density, we would obtain a configuration with two u quarks in the instanton field. In the limit $M^2\rho_s^2 \to \infty$, the contribution corresponds to the factorized four-quark condensate in the OPE language (see below).

The radiative corrections to the chirality-conserving structure that contain the terms $\alpha_s \ln M^2$ (the leading logarithmic approximation, LLA) are the same as in the OPE case. The same refers to LLA corrections to the chirality-flipping structure, because they originate from the u quark loop and are determined by large momenta of the virtual gluons, which strongly exceed the momentum carried by the current.

We demonstrate that the QCD sum rules constructed in such a way do not have a physical solution. Therefore, we must assume that the small instantons create only part of the scalar condensate. At first glance, this contradicts the results in [10, 11], where the small instantons reproduced the conventional value of $\langle 0|\bar{q}(0)q(0)|0\rangle$. But in these papers, the instanton density, which is proportional to $1/R^4$, is tied to the gluon condensate, which is known with large uncertainties (up to a factor of 2) [13]. This leaves some room for other contributions to the quark scalar condensate. Here we assume that the small-size instantons provide a fraction w_s of the total scalar condensate. Our model assumption is that the rest part $(1-w_s)\langle 0|\bar{q}(0)q(0)|0\rangle$ is due to interactions at a large correlation length $\rho_l \gg 1$ GeV^{-1}. It can be approximated by a local condensate. Thus the chirality flipping component of the polarization operator is determined by terms that describe interactions of the d quark with the nearest small-size instanton and by a local condensate. We write the expectation value as

$$\begin{gathered}\langle 0|\bar{q}(0)q(x)|0\rangle = \langle 0|\bar{q}(0)q(x)|0\rangle_s + \langle 0|\bar{q}(0)q(0)|0\rangle_\ell, \\ \langle 0|\bar{q}(0)q(x)|0\rangle_{s,\ell} = \langle 0|\bar{q}(0)q(x)|0\rangle w_{s,\ell}, \\ w_s + w_\ell = 1,\end{gathered} \qquad (9)$$

where we do not account for nonlocalities of the second term. This realizes the old idea [14, 15] that the large-size instanton contributions are included in condensates, while the small-size instantons provide nonperturbative contributions written explicitly. As a special case, the condensate $\langle 0|\bar{q}(0)q(0)|0\rangle_\ell$ can be treated as due to the long-size instantons with $\rho_\ell M \gg 1$.

The polarization operator now obtains a contribution from a configuration in which one of the u quarks moves in the zero mode of a small-size instanton, while the second quark is described by a local scalar condensate. In the limit $\tau \gg 1$, the leading term of the expansion in powers of $1/\tau$ is equal to that given by the standard total condensate of the OPE approach.

For M^2 of the order of 1 GeV^2, we have $\rho_s^2 M^2 \sim 1$, and the convergence of the OPE series is obscure. For the chirality-flipping sum rule, the function of M^2 on the left-hand side can be viewed as coming from the nonlocality of the scalar quark condensate. The contribution of the four-quark condensate presented in the instanton picture then makes a much smaller contribution because one instanton can produce only one $\bar{q}q$ pair of a fixed flavor. On the other hand, some of the condensates that contribute to the OPE sum rules are not accounted for in our model, where all the nonzero mode contribution is approximated by (included in) the free quark propagator S_{NZ}.

We calculate the polarization operator $\Pi(q)$ in the instanton vacuum in the NIA and analyze the corresponding sum rules. We demonstrate that the sum rules have a solution with the value of the nucleon mass not far from the physical one for all $w_s < 0.6$–0.7. At $w_s \approx 2/3$, the value of the nucleon mass is $m_N \approx \approx 1$ GeV. Comparing with the sum rules in terms of condensates, we find that the consistency between the left- and right-hand sides of the sum rules is improved. At the conventional values of the quark condensate $\langle 0|\bar{q}(0)q(0)|0\rangle \approx (-250\ \text{MeV})^3$, the value of the threshold does not change much, while that of the nucleon residue becomes noticeably smaller. At larger values of w_s, the sum rules have only an unphysical solution with the contribution of the continuum much exceeding that of the nucleon pole. In Sec. 2, we recall the main features of the nucleon sum rules in terms of condensates. In Sec. 3, we calculate the polarization operator in the instanton vacuum. In Sec. 4, we solve the sum rule equations. We discuss the results in Sec. 5.

2. QCD SUM RULES IN TERMS OF CONDENSATES

In the case of a nucleon (we consider the proton), the polarization operator takes the form

$$\Pi(q) = \hat{q}\Pi^q(q^2) + I\Pi^I(q^2), \quad (10)$$

where q is the four-momentum of the system, $\hat{q} = q_\mu\gamma^\mu$, and I is the unit matrix. The first and the second terms on the right-hand side respectively correspond to the chirality-conserving and the chirality-flipping contributions. The dispersion relations are

$$\Pi^i(q^2) = \frac{1}{\pi}\int dk^2 \frac{\operatorname{Im}\Pi^i(k^2)}{k^2 - q^2}, \quad i = q, I. \quad (11)$$

As noted above, we do not take care of the subtractions.

We present the results of the calculation of the polarization operator defined by Eq. (2) with the current determined by (5). The left-hand side of Eq. (11) can be written as

$$\Pi^{q\ OPE}(q^2) = \sum_{n=0} A_n(q^2),$$
$$\Pi^{I\ OPE}(q^2) = \sum_{n=3} B_n(q^2), \quad (12)$$

where the lower index n is the dimension of the corresponding QCD condensate (A_0 stands for the three-quark loop). The most important terms for $n \le 8$ were obtained earlier [2, 3]. For the chirality-conserving structure, they are

$$A_0 = \frac{-Q^4 \ln Q^2}{64\pi^4}, \quad A_4 = \frac{-b \ln Q^2}{128\pi^4},$$
$$A_6 = \frac{1}{24\pi^4}\frac{a^2}{Q^2}, \quad A_8 = -\frac{1}{6\pi^4}\frac{m_0^2 a^2}{Q^4}, \quad (13)$$

where $Q^2 = -q^2 > 0$, while a and b are the scalar and gluon condensates multiplied by certain numerical factors

$$a = -(2\pi)^2\langle 0|\bar{q}q|0\rangle,$$
$$b = (2\pi)^2\langle 0|\frac{\alpha_s}{\pi}G^{a\mu\nu}G^a_{\mu\nu}|0\rangle, \quad (14)$$

and

$$m_0^2 \equiv \frac{\langle 0|\bar{q}\sigma_{\mu\nu}\mathcal{G}_{\mu\nu}q|0\rangle}{\langle 0|\bar{q}q|0\rangle}, \quad \mathcal{G}_{\mu\nu} = \frac{\alpha_s}{\pi}\sum_h \frac{G^h_{\mu\nu}\lambda^h}{2}. \quad (15)$$

We discuss the value of m_0^2 in Sec. 5. For the chirality-flipping structure, we find

$$B_3 = \frac{aQ^2 \ln Q^2}{16\pi^4}, \quad B_5 = 0. \quad (16)$$

The leading contribution to the chirality-conserving structure A_0 is the loop containing three free quarks. The leading contribution to the chirality-odd structure B_3 is proportional to the scalar quark condensate. Here, the free u quarks form a loop, while the d quarks are exchanged with the vacuum condensate, see Fig. 1.

We note the last equality $B_5 = 0$, however. There are indeed two contributions of dimension $d = 5$, and we can therefore write $B_5 = B_5^a + B_5^b$. The term B_5^a comes from the Taylor expansion of the product $\bar{d}(0)d(x)$ and is proportional to the condensate $\langle 0|\bar{d}(0)D^2 d(0)|0\rangle$. In this case, the u quarks are described by free propagators that are diagonal in color variables. But the product of the operators $d^a_\alpha \bar{d}^b_\beta G^h_{\mu\nu}$ make a contribution to the d quark propagator, proportional to the product

$$\langle 0|\bar{q}\mathcal{G}_{\mu\nu}\sigma_{\mu\nu}q|0\rangle\sigma_{\alpha\beta}\lambda^h_{ab}/2.$$

The contribution to the polarization operator B_5^b is thus proportional to the condensate $\langle 0|\bar{q}\mathcal{G}_{\mu\nu}\sigma_{\mu\nu}q|0\rangle$, and the

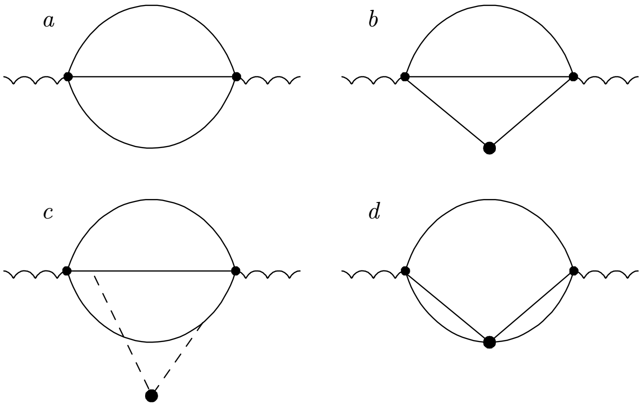

Fig. 1. The set of diagrams for the lowest OPE terms of the nucleon sum rules. Wavy lines are for the nucleon current, solid lines stand for the quarks, and dashed lines denote the gluons. The circles stand for the quark and gluon condensates

propagator of one of the u quarks of the polarization operator should include interaction with this gluon field (and cannot be treated as a free one). Due to the equation of motion

$$\left(D^2 - \frac{1}{2}\sigma_{\mu\nu}\mathcal{G}_{\mu\nu}\right) q = m_q^2 q,$$

where m_q is the current mass of the quark, we find that

$$\langle 0|\bar{d}(0)D^2 d(0)|0\rangle = \frac{1}{2}\langle 0|\bar{q}\sigma_{\mu\nu}\mathcal{G}_{\mu\nu}q|0\rangle \tag{17}$$

for the massless quark. Thus, the contributions B_5^a and B_5^b can be expressed in terms of the same condensate. Direct calculation [16] demonstrates that $B_5^a + B_5^b = 0$. We note that this cancelation occurs only for current (5). If we use current (4) with $t \neq -1$, then $B_5 \neq 0$.

Usually, sum rules for the operators

$$\mathcal{P}^i(M^2) = 32\pi^4 \mathcal{B}\Pi^{i\,OPE}(q^2),$$

where $\mathcal{B}$ is the Borel transform operator, are actually considered. The factor $32\pi^4$ is introduced in order to deal with the values of the order of unity (in GeV units). After the Borel transform, we write (12) as

$$\begin{aligned} \mathcal{P}^q(M^2) &= \sum_{n=0} A'_n(M^2), \\ \mathcal{P}^i(M^2) &= \sum_{n=3} B'_n(M^2), \\ A'_n(M^2) &= 32\pi^4 \mathcal{B}A_n(q^2), \\ B'_n(M^2) &= 32\pi^4 \mathcal{B}B_n(q^2). \end{aligned} \tag{18}$$

We here present the most important terms:

$$\begin{aligned} A'_0(M^2) = M^6, \quad A'_4(M^2) = \frac{bM^2}{4}, \\ A'_6 = \frac{4}{3}a^2, \quad B'_3(M^2) = 2aM^4. \end{aligned} \tag{19}$$

The Borel-transformed sum rules (11) can now be written as

$$\mathcal{P}^i(M^2) = \mathcal{F}_p^i(M^2) + \mathcal{F}_c^i(M^2), \tag{20}$$

where the two terms on the right-hand side are the contributions to the right-hand side of the Borel-transformed Eq. (11) made by the nucleon pole with the mass m_N and by the continuum:

$$\begin{aligned} \mathcal{F}_p^i(M^2) &= \xi_i \lambda^2 \exp\left(-\frac{m_N^2}{M^2}\right), \\ \mathcal{F}_c^i(M^2) &= \int_{W^2}^{\infty} dk^2 \exp\left(-\frac{k^2}{M^2}\right) \Delta[\mathcal{B}\Pi_1(k^2)]. \end{aligned} \tag{21}$$

Here, λ^2 is the residue at the nucleon pole (multiplied by $32\pi^4$), W^2 is the continuum threshold, and $\xi_q = 1$, $\xi_I = m_N$.

The conventional form of the sum rules is

$$\mathcal{L}^q(M^2, W^2) = R^q(M^2), \tag{22}$$

and

$$\mathcal{L}^I(M^2, W^2) = R^I(M^2), \tag{23}$$

where $\mathcal{L}^i$ and R^i are the respective Borel transforms of the left- and right-hand sides of Eqs. (11):

$$\begin{aligned} R^q(M^2) &= \lambda^2 \exp\left(-\frac{m_N^2}{M^2}\right), \\ R^I(M^2) &= m_N \lambda^2 \exp\left(-\frac{m_N^2}{M^2}\right), \end{aligned} \tag{24}$$

with $\lambda^2 = 32\pi^4\lambda_N^2$. The contribution of the continuum is moved to the left-hand sides of Eqs. (22) and (23), which can be written as

$$\begin{aligned} \mathcal{L}^q &= \sum_{n=0} \tilde{A}_n(M^2, W^2), \\ \mathcal{L}^I &= \sum_{n=3} \tilde{B}_n(M^2, W^2), \end{aligned} \tag{25}$$

(see Eq. (18)). Here,

$$\begin{aligned} \tilde{A}_0 = \frac{M^6 E_2(\gamma)}{L(M^2)}, \quad \tilde{A}_4 = \frac{bM^2 E_0(\gamma)}{4L(M^2)}, \\ \tilde{A}_6 = \frac{4}{3}a^2 L, \quad \tilde{B}_3 = 2aM^4 E_1(\gamma), \quad \gamma = \frac{W^2}{M^2}, \end{aligned} \tag{26}$$

with

$$E_0(\gamma) = 1 - e^{-\gamma}, \quad E_1(\gamma) = 1 - (1+\gamma)e^{-\gamma}, \quad E_2(\gamma) = 1 - (1 + \gamma + \gamma^2/2)e^{-\gamma}. \tag{27}$$

The factor

$$L(M^2) = \left(\frac{\ln M^2/\Lambda^2}{\ln \mu^2/\Lambda^2}\right)^{4/9} \tag{28}$$

includes the most important radiative corrections of the order $\alpha_s \ln Q^2$ (LLA). These contributions were summed to all orders of $(\alpha_s \ln Q^2)^n$. In Eq. (28), $\Lambda = \Lambda_{QCD}$ is the QCD scale, while μ is the normalization point, its standard choice being $\mu = 0.5$ GeV.

The position of the nucleon pole m_N, its residue λ^2, and the continuum threshold W^2 are the unknowns of sum rule equations (22) and (23). The nucleon sum rule equations (22) and (23) are usually solved at $M^2 \sim 1$ GeV^2, namely,

$$0.8 \text{ GeV}^2 \le M^2 \le 1.4 \text{ GeV}^2. \tag{29}$$

The range of M^2 where the sum rules hold is usually referred to as the "duality interval".

After the inclusion of several condensates of higher dimensions and of the lowest-order radiative corrections beyond the leading logarithmic approximation [17], the sum rules yield the solution (for $\Lambda_{QCD} = 230$ MeV) $m_N = 928$ MeV, $\lambda^2 = 2.36$ GeV^6, $W^2 = 2.13$ GeV^2.

3. QCD SUM RULES IN THE INSTANTON VACUUM

3.1. Instanton representation and the OPE approximation

We recall that a typical value of the condensate of dimension $d = n$ is $\langle 0|O_n|0\rangle \sim c^n$ with $c = 250$ MeV. Because we have $c^2M^2 \ll 1$ at $M \approx 1$ GeV, we could expect the convergence of the OPE. In the instanton language, we have $\rho_s^2 M^2 \sim 1$ and cannot expect the convergence of the series in powers of $1/M^2$.

Therefore, the structure of the left-hand sides of the sum rules differs from that in the condensate representation. The leading contribution A_0 to the chirality-conserving operator Π^q remains unchanged. However, as long as we consider only the nearest small-size instanton, there is no contribution of two zero-mode u quarks (this contribution plays the role of a four-quark condensate in the "condensate language"), because only one u quark can be placed in the zero mode of the field of an instanton.

In the chirality-flipping structure Π^I, we describe the d quark by the propagator S_Z given by Eq. (7). The Borel-transformed contribution $\mathcal{B}\Pi^I(M^2)$ depends on the parameter $\rho_s^2 M^2 \sim 1$ and cannot be represented as a $1/M^2$ series. In the condensate language, this means that it includes a nonlocal scalar quark condensate.

We note that our form for the propagator S_{NZ} means that we did not include some of contributions that were present in the condensate picture. In the terms A_4 and B_5^b, the propagator of one of the u quarks should include the influence of the gluon field. Therefore, its propagator is not diagonal in color indices, while both S_Z and S_{NZ} are.

We now assume that in the NIA, the small-size instantons create only a part $w_s < 1$ of the scalar condensate. The contribution of the remaining part of the condensate $(1 - w_s)a$ to the chirality-flipping structure is expressed by the term B_3' given by Eq. (19) with a replaced by a_ℓ. In the chirality-flipping structure, one of the u quarks is in the zero mode of the nearest small-size instanton, while the other is described by a local condensate. The latter provides the factor a_ℓ in the contribution to the polarization operator, and the former provides a factor containing a nonlocal scalar condensate. We note that we do not use the factorization hypothesis here.

Considering only the small-size instantons, we do not have an analogue of the OPE four-quark condensates because two u quarks cannot be in the zero-mode of the same instanton. We find such an analogue in going beyond the NIA.

3.2. Calculation of the polarization operator

Instead of the condensate $\langle 0|\bar{q}(0)q(0)|0\rangle$, we use the parameter a defined by Eq. (14). We also introduce $a_s = aw_s$ and $a_\ell = aw_\ell = a(1 - w_s)$. With these variables, Eq. (8) with $N_c = 3$ takes the form

$$a_s = 6\int_0^\infty dp\, p\, m(p), \quad a_s = -(2\pi)^2 \langle 0|\bar{q}q|0\rangle_s. \tag{30}$$

As we have noted, the leading contribution A_0 to the $\hat{Q}$ structure remains unchanged. The contribution to the chirality-flipping structure is now

$$\Pi_1^I(q^2) = 2a(1 - w_s)Q^2 \ln Q^2 + Y_s, \tag{31}$$

where the two terms are the respective contributions of large-size and small-size instantons. The last one can be written as $Y_s = 32\pi^4 X_s$, with

$$X_s = 12 \int \frac{d^4p}{(2\pi)^4} \gamma_\mu \frac{m(p)}{p^2} \gamma_\nu T_{\mu\nu}(Q - p), \tag{32}$$

where

$$T_{\mu\nu}(Q-p) = \int d^4x e^{-i(Q-p,x)} \,\mathrm{Tr}[t_{\mu\nu}(x)], \qquad (33)$$

with

$$t_{\mu\nu}(x) = \gamma_\mu G_0(x) \gamma_\nu G_0(x). \qquad (34)$$

Here,

$$G_0(x) = -\frac{1}{2\pi^2}\frac{\hat{x}}{x^4} \qquad (35)$$

is the Fourier transform of the propagator S_{NZ} determined by Eq. (30). We note that the quark in the zero mode now carries a nonzero momentum p. In the condensate language, it carries the momentum $p=0$. Neglecting the momentum p in the factor $T_{\mu\nu}(Q-p)$ on the right-hand side of Eq. (32), we would obtain

$$X_s = \frac{3Q^2 \ln Q^2}{8\pi^4} \int\limits_0^\infty dp\, p\, m(p) = B_3(Q^2), \qquad (36)$$

with $B_3(Q^2)$ defined by Eq. (16) and with a replaced by a_s. Thus, in the limit $Q^2 \to \infty$, we obtain the lowest OPE term. We can view the calculation of the contribution given by Eq. (32) as the inclusion of nonlocality in the scalar quark condensate.

The four-quark contribution can emerge only if one of the $\bar{u}u$ pairs comes from small-size instantons, while the other originates from large-size fluctuation. Following the previous discussion, we can write the contribution to the polarization operator as

$$A_6 = \frac{4aw_s(1-w_s)}{\pi^2} \int \frac{d^4p}{(2\pi)^4} \frac{m(p)}{p^2} \frac{\hat{Q}-\hat{p}}{(Q-p)^2}. \qquad (37)$$

The lower index 6 here shows that neglecting p in the last factor on the right-hand side, we would obtain the factorized OPE term $\tilde{A}_6$ determined by Eq. (19) times $2w_s(1-w_s)$. The set of diagrams included in the sum rules is shown in Fig. 2.

It is instructive to trace how the contributions to the spin-conserving part of the polarization operator change if we go beyond the NIA. In the loop corresponding to A_0 the quark propagators then have their masses squared in denominators, and the contribution of A_0 diminishes. Two u quarks can now be described by the chirality-flipping parts of their propagators. The corresponding contribution of the small-size instantons is

$$A_6' = 12\cdot 8 \int \frac{d^4p}{(2\pi)^4} \frac{m(p)}{p^2} \times \int \frac{d^4p'}{(2\pi)^4} \frac{m(p')}{p'^2} \frac{\hat{Q}-\hat{p}-\hat{p}'}{(Q-p-p')^2}. \qquad (38)$$

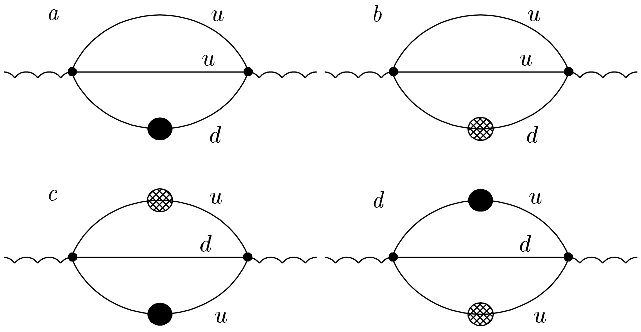

Fig. 2. The set of diagrams for the quarks in the field of instantons. Dark and dashed blobs on the quark lines stand for small-size and large-size instantons

In the limit $Q^2 \to \infty$, we neglect p and p' in the last factor in the integrand, coming to the factorized form of the OPE contribution.

To obtain results in analytic form, we parameterize the dynamical quark mass caused by the small-size instantons as

$$m(p) = \frac{\mathcal{A}}{(p^2+\eta^2)^3}, \qquad (39)$$

with $\mathcal{A}$ and η being the fitting parameters. The power in the denominator insures the proper behavior $m(p) \sim p^{-6}$ as $p \to \infty$ [10]. Now Eq. (30) can be written as

$$a_s = \frac{3\mathcal{A}}{2\eta^4}. \qquad (40)$$

Calculating the tensor $T_{\mu\nu}$, we write

$$X_s = \frac{3}{\pi^2} \int \frac{d^4p}{(2\pi)^4} \times \frac{\mathcal{A}}{p^2(p^2+\eta^2)^3} (Q-p)^2 \ln(Q-p)^2. \qquad (41)$$

Further details of the calculation are presented in the Appendix. We find the Borel transformed contribution

$$B'(M^2) = 2a_\ell M^4 + 2a_s M^4 F(\beta),$$
$$F(\beta) = \frac{1}{3}\left(\frac{2(1-e^{-\beta})}{\beta} + e^{-\beta}(1-\beta) + \beta^2 \mathcal{E}(\beta)\right), \qquad (42)$$
$$\beta = \eta^2/M^2,$$

where

$$\mathcal{E}(\beta) = \int\limits_\beta^\infty dt \frac{e^{-t}}{t}. \qquad (43)$$

In the literature, our function $\mathcal{E}$ is usually denoted by E_1. We avoid this notation because in the QCD sum

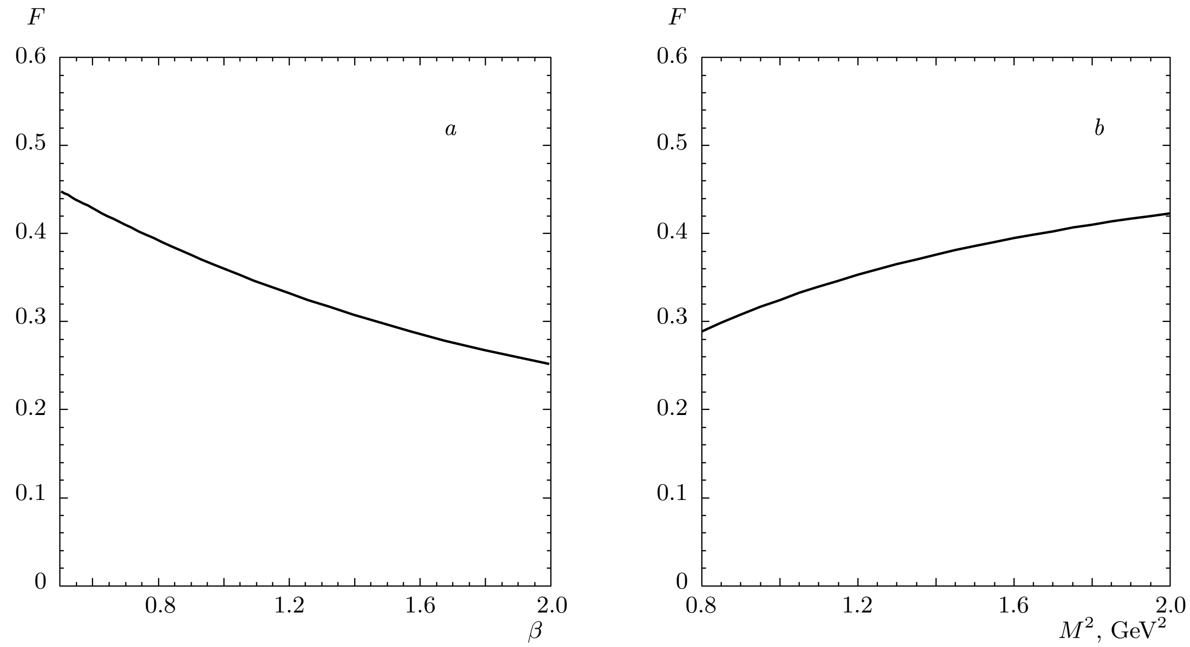

Fig. 3. *a*) The function $F(\beta)$ defined by Eq. (42). *b*) Dependence of the functions $F(\eta^2/M^2)$ for $\eta^2 = 1.26\ \mathrm{GeV}^2$, corresponding to the size $\rho_s = 0.33$ fm

rule publications, the notation E_1 has another meaning (see Eq. (27)).

Combining Eq. (40) with the relation $m(0) = \mathcal{A}/\eta^6$ coming from Eq. (39), we find that $\eta^2 = 2a_s/3m(0)$. It was demonstrated in [10, 11] that $a_s \sim R^{-2}\rho^{-1}$, while $m(0) \sim R^{-2}\rho$. Thus, η^2 depends only on ρ, and $\eta^2 = 1.26\ \mathrm{GeV}^2$ at $\rho = 0.33$ fm. In duality interval (29), $0.9 \le \beta \le 1.6$. The function $F(\beta)$ is plotted in Fig. 3*a*. The dependence of F on M^2 for $\eta^2 = 1.26\ \mathrm{GeV}^2$ is shown in Fig. 3*b*. As expected, we find $B = B_3'$ in the limit $M^2 \to \infty$ (see Eq. (19)).

A similar calculation yields

$$A_6' = \frac{8}{3}a^2 w_s(1-w_s)\frac{1-e^{-\beta}}{\beta}. \qquad (44)$$

3.3. Parameterization of the nonlocal scalar condensate

It is reasonable to try to establish a connection with the OPE approach. We write Eq. (42) as

$$B_3'(M^2) = 2M^4 a(M^2), \qquad (45)$$

where

$$a(M^2) = a\left(1 - w_s + w_s F\left(\frac{\eta^2}{M^2}\right)\right), \qquad (46)$$

with F defined by Eq. (42). We have $a(M^2) \to a$ for $M^2 \to \infty$. We now define

$$K(M^2) = \frac{a(M^2)}{a}, \qquad (47)$$

and try to parameterize the function $K(M^2)$ in the duality interval by a power series in $1/M^2$:

$$K(M^2) = 1 + \sum_{n=1}^{N}\frac{C_n}{M^{2n}}. \qquad (48)$$

If the second term on the right-hand side can be approximated by one or two terms, such a representation can be related to the parameterization of the expectation value $\langle 0|\bar{q}(0)q(x)|0\rangle$ by a polynomial in x^2. We can write the polarization operator Π_I as

$$\Pi_I(q^2) = \frac{2}{\pi^4}\int\frac{d^4x}{x^6}f(x^2)e^{iqx}, \qquad (49)$$

with $f(x^2) = \langle 0|\bar{q}(0)q(x)|0\rangle$. Assuming that $f(x)$ can be parameterized as

$$f(x) = f(0)(1 + c_1x^2 + c_2x^4) \qquad (50)$$

(we recall that we are in a Euclidean metric), we find

$$B_3'(M^2) = 2M^4 f(0)\left(1 - \frac{8c_1}{M^2} + \frac{32c_2}{M^4}\right), \qquad (51)$$

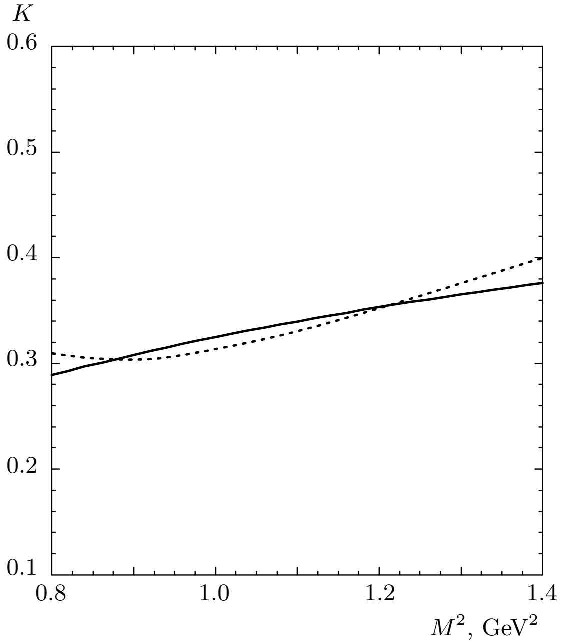

Fig. 4. Approximation of the function $K(M^2)$ defined by Eq. (47) (solid line) by the series on the right-hand side of Eq. (48) with parameters determined by Eq. (52) (dotted line)

and hence

$$C_1 = 8c_1, \quad C_2 = 32c_2 \tag{52}$$

in (48). We note that the right-hand side of Eq. (50) cannot be treated as the lowest terms of the Taylor expansion. The terms x^{2n} with $n \geq 3$ yield integrals that are divergent on the upper limit and cannot be eliminated by the Borel transformation.

For the medium consisting solely of small-size instantons, i. e., for $w_s = 1$, keeping the first three terms in (48), we find that $C_1 = -1.23$ GeV^2 and $C_2 = 0.54$ GeV^4, and hence $c_1 = -0.15$ GeV^2 and $c_2 = 0.017$ GeV^4 in the duality interval $0.8 \text{ GeV}^2 \leq M^2 \leq 1.4 \text{ GeV}^2$ determined by Eq. (29). The accuracy of the parameterization is illustrated by Fig. 4. In the interval $0.8 \text{ GeV}^2 \leq M^2 \leq 2.0 \text{ GeV}^2$, we find a slightly different set of values: $C_1 = -1.38$ GeV^2 and $C_2 = 0.68$ GeV^4, corresponding to $c_1 = -0.17$ GeV^2 and $c_2 = 0.021$ GeV^4. Thus we can assume that parameterization (50) with $c_1 \approx -0.2$ GeV^2 and $c_2 \approx 0.02$ GeV^4 can be used for the Borel masses in the GeV region. This point was discussed in more detail in [18].

4. SOLUTION OF THE SUM RULE EQUATIONS

We return to the Minkowski metric and analyze Eqs. (22) and (23) with

$$\mathcal{L}^q = \tilde{A}_0(M^2, W^2) + \tilde{A}_6(M^2), \quad \mathcal{L}^I = \tilde{B}(M^2, W^2), \tag{53}$$

where $\tilde{A}_0(M^2, W^2)$ is given by Eq. (26), $\tilde{A}_6 = A'_6$ is presented in Eq. (44), and

$$\tilde{B}(M^2, W^2) = 2a_\ell M^4 E_2(\gamma) + 2a_s M^4 \Phi(M^2, W^2), \tag{54}$$

$$\Phi(M^2, W^2) = \frac{1}{3}\left(\frac{2}{\beta}(1 - e^{-\beta}) + e^{-\beta}(1-\beta) - e^{-\gamma}(1 - \beta + \gamma) + \beta^2(\mathcal{E}(\beta) - \mathcal{E}(\gamma))\right).$$

The functions $E_i (i = 0, 1, 2)$ are determined by Eq. (27).

4.1. The absence of a solution at $w_s = 1$

We can immediately guess that there is no solution for $w_s = 1$. Indeed, if the values m_N, λ^2, and W^2 compose a solution, we should obtain

$$\kappa(M^2) \equiv \frac{\mathcal{L}^I(M^2, W^2)}{\mathcal{L}^q(M^2, W^2)} \approx \text{const} = m_N. \tag{55}$$

Because the contribution of the continuum should not be too large, we should expect

$$\frac{\mathcal{L}^I(M^2)}{\mathcal{L}^q(M^2)} \approx \text{const} \approx m_N, \tag{56}$$

where we set $\mathcal{L}^i(M^2) = \mathcal{L}^i(M^2, W^2 \to \infty)$. For $w_s = 1$, Eq. (56) takes the form

$$\kappa(M^2) = \frac{2aF(\eta^2/M^2)}{M^2}. \tag{57}$$

Using the dependence of the function F on M^2 for $\eta^2 = 1.26$ GeV^2 presented in Fig. 3*b*, we can see that the values of κ range between $2a \cdot 0.36/\text{GeV}^2$ and $2a \cdot 0.27/\text{GeV}^2$ in the interval (29) of variation of M^2. For the distance $R = 1$ fm between small-size instantons, $a = 0.59$ GeV^3 [10, 11]. Hence, $m_N \approx 0.35$ GeV.

The unrealistic value of the nucleon mass obtained in such a way is not the main problem, however. We try to find the value of λ^2 using Eq. (21). We obtain $M^6 \exp(m_N^2/M^2) = \lambda^2$. But the left-hand side of

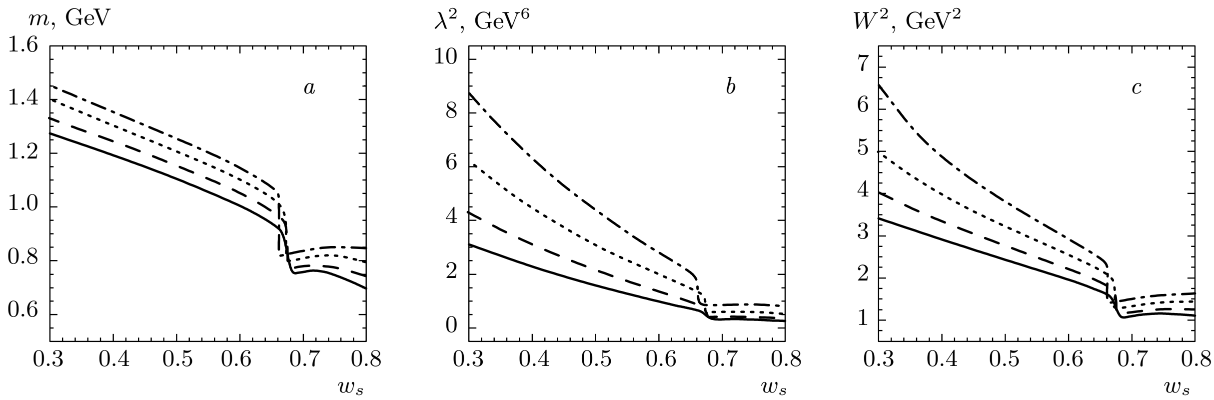

Fig. 5. Dependence of the solution of the sum rule equations on the value of w_s at $\rho_s = 0.33$ fm for the nucleon mass m (a), λ^2 (b), and W^2 (c). The solid, dashed, dotted, and dashed-dotted lines are for the respective scalar condensate values $a = 0.58$, 0.67, 0.80, and 0.96 GeV^3

this equality changes by a factor of 6 in duality interval (29). Therefore, the equality can be satisfied only if the contribution of the continuum, which was moved to the left-hand side, as discussed around Eqs. (22)–(25), changes the left-hand side strongly. Hence, we come to an unphysical solution of the sum rules [19]. As we see below, a more detailed analysis confirms this conclusion.

We note that at $w_s = 1$, the right-hand sides of the sum rules for both chirality-flipping and chirality-conserving structures suffered large changes comparing to the standard OPE sum rules. The most important change in the former case is the inclusion of the nonlocality of the scalar condensate. In the latter case, there is no four-quark condensate, which played an important role in the OPE case.

4.2. Dependence of solutions on the fraction of small-size instantons

The functions $\mathcal{L}^q$ and $\mathcal{L}^I$ depend explicitly on the scalar condensate a, on its fraction caused by the instantons of the small size $a_s = aw_s$, and on the parameter η^2. On the other hand, the medium of small instantons is determined by their average size ρ_s and the distance between the instantons $R(w_s)$. It was found in [10, 11] that

$$\langle 0|\bar{q}(0)q(0)|0\rangle_s = \frac{C}{R^2(w_s)\rho_s}, \qquad (58)$$

where $C = 25.0$ and R is the distance between small-size instantons. Hence, we can study the dependence of the solution of the sum rule equations on the fraction of small-size instantons w_s for several values of the scalar condensate

$$a = -(2\pi)^2\langle 0|\bar{q}(0)q(0)|0\rangle$$

and for different sizes of small instantons ρ_s.

We note that at $\rho_s = 0.33$ fm and $R(1) = 1$ fm, the scalar condensate $a = 0.58$ GeV^3 (at the conventional normalization point $\mu = 0.5$ GeV) [10, 11]. This enables us to find the dependence on w_s at any values of a and ρ_s.

The results for $\rho_s = 0.33$ fm are presented in Table 1 and in Fig. 5. We can see that at several reasonable values of the quark condensate, the sum rules have a physical solution for w_s not exceeding a certain value w_0. At $w_s = w_0 \approx 0.67$, the solutions jump to unphysical ones with a smaller value of the nucleon mass and the dominant contribution of the continuum [19]. At w_s about 0.6, the nucleon mass is close to the physical value.

In Table 1 and in Fig. 5, we present the results for four values of the scalar condensate a corresponding to $\rho_s = 0.33$ fm and the distances between the small instantons $R = 1.3$, 1.2, 1.1, and 1.0 fm at $w_s = 0.6$. The distances $R = 1.3$ fm and $R = 1.2$ fm correspond to the values $a = 0.58$ GeV^3 and $a = 0.67$ GeV^3, i.e., to the values of the scalar condensate $\langle 0|\bar{q}(0)q(0)|0\rangle$ equal to $(-244\ \mathrm{MeV})^3$ and $(-257\ \mathrm{MeV})^3$, close to the conventional values. The distances $R = 1.1$ fm and $R = 1.0$ fm correspond to $a = 0.80$ GeV^3 and $a = 0.96$ GeV^3, i.e., to somewhat larger values of $\langle 0|\bar{q}(0)q(0)|0\rangle$ equal to $(-273\ \mathrm{MeV})^3$ and a less realistic $(-290\ \mathrm{MeV})^3$. The consistency of the left- and right-hand sides of the sum rules is illustrated in Fig. 6.

Table 1. Solutions of the sum rule equations for $\rho = 0.33$ fm

a, GeV^3	w_s	m_N, GeV	λ^2, GeV^6	W^2, GeV^2	χ^2_N
0.96	0.30	1.45	8.7	6.6	3.7(−2)
	0.60	1.15	2.8	2.9	4.0(−2)
	0.66	1.05	1.9	2.3	3.9(−2)
	0.67	0.82	0.86	1.4	2.1(−2)
0.80	0.30	1.40	6.2	4.9	1.7(−2)
	0.60	1.10	2.0	2.6	2.3(−2)
	0.67	0.99	1.2	2.0	2.2(−2)
	0.68	0.80	0.60	1.3	1.2(−2)
0.67	0.30	1.33	4.3	4.0	8.3(−3)
	0.60	1.05	1.4	2.2	1.4(−2)
	0.67	0.95	0.83	1.7	1.3(−2)
	0.68	0.77	0.41	1.1	5.9(−3)
0.57	0.30	1.27	3.0	3.4	4.2(−2)
	0.60	1.00	0.95	1.9	8.9(−3)
	0.67	0.90	0.57	1.5	8.3(−3)
	0.68	0.75	0.30	1.0	3.0(−3)

As noted above, the pole-to-continuum ratio

$$r_i(M^2) = \mathcal{F}_i^p(M^2)/\mathcal{F}_i^c(M^2), \quad i = q, I, \qquad (59)$$

of the two contributions to the right-hand side of Eq. (20) characterizes the validity of the "pole + continuum" model for the spectrum of the polarization operator in Eqs. (20) and (21). For larger values of $r_i(M^2)$, the model is justified better. The values of the ratio are presented in Table 2 for $\rho_s = 0.33$ fm and $w_s = 0.60$. We take two cases for illustration. For $a = 0.58$ GeV^3, the solution is

$$m_N = 1.01\ \text{GeV}, \quad \lambda^2 = 1.2\ \text{GeV}^6, \quad W^2 = 2.0\ \text{GeV}^2. \qquad (60)$$

The pole-to-continuum ratio decreases with the value of M^2 (see Table 2). Although the sum rule equations can be solved with good accuracy in the broad interval of values of the Borel mass (see Table 3), the pole-to-continuum ratio becomes unacceptably small for $M^2 > 1.4$ GeV^2. In this case, we therefore stay in the traditional duality interval determined by Eq. (29).

For the condensate $a = 0.96$ GeV^3, corresponding to $R(0.6) = 1$ fm, the solution is

$$m_N = 1.15\ \text{GeV}, \quad \lambda^2 = 2.8\ \text{GeV}^6, \quad W^2 = 2.9\ \text{GeV}^2. \qquad (61)$$

Here, the sum rule equations can also be solved with good accuracy in a large interval of values of the Borel mass (see Table 3). We can see that both r_q and r_I decrease as M^2 increases. In this case, the pole-to-continuum ratio is much larger than it was for smaller values of the condensate. Hence, the interval of the values of M^2 where the sum rule equations can be solved becomes larger.

We also fix the value $R = 1.3$ fm and trace the dependence of the solutions on ρ_s. In Table 4, we present the results for $\rho = 0.25$ fm ($a = 0.76$ GeV^3 and $\langle 0|\bar{q}(0)q(0)|0\rangle = (-268\ \text{MeV})^3$) and $\rho_s = 0.40$ fm ($a = 0.48$ GeV^3 and $\langle 0|\bar{q}(0)q(0)|0\rangle = (-230\ \text{MeV})^3$). They are shown in Fig. 7. The situation is similar to the preceding case when we changed R. However, at $\rho_s = 0.40$ fm, the jump to the unphysical solution occurs at a larger value $w_s \approx 0.75$.

For $w_s = 0.65$, the function $K(M^2)$ determined by Eq. (47) is approximated by the series on the right-hand side of Eq. (48) with the parameters

$$C_1 = -0.80\ \text{GeV}^2, \quad C_2 = 0.35\ \text{GeV}^4, \qquad (62)$$

whence $c_1 = -0.10$ GeV^2 and $c_2 = 0.011$ GeV^4.

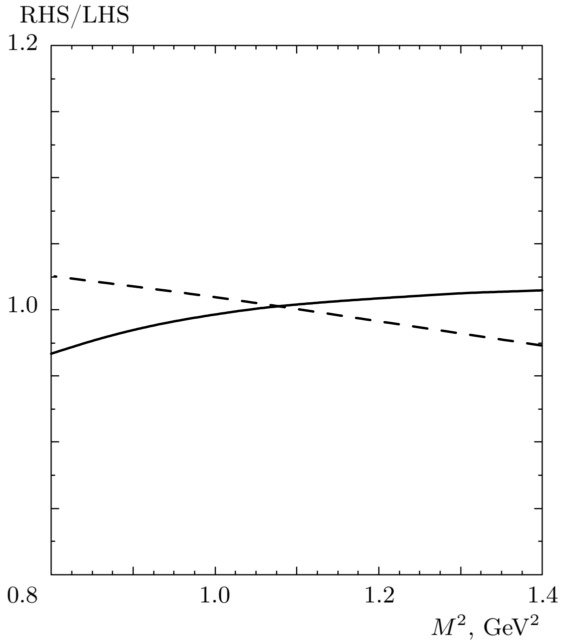

Fig. 6. Consistency of the left- and right-hand sides (LHS, RHS) of the sum rules for $a = 0.58$ GeV^3, $w_s = 0.60$. The solid and dashed lines respectively show the ratios of the right- to the left-hand sides for the sum rules for chirality-conserving and chirality-flipping equations

Table 2. Pole-to-continuum ratio $r(M^2)$ for solutions of the sum rules at $\rho = 0.33$ fm for $a = 0.58$ GeV^3 and $a = 0.96$ GeV^3; $w_s = 0.60$

a, GeV^3	M^2, GeV^2	$r_q(M^2)$	$r_I(M^2)$
0.58	0.8	1.25	1.84
	1.0	0.69	1.08
	1.2	0.43	0.72
	1.4	0.29	0.52
0.96	0.8	4.69	5.85
	1.0	2.30	2.99
	1.2	1.34	1.82
	1.4	0.86	1.23

5. SUMMARY

We calculated the polarization operator of the nucleon current in the instanton medium that we assumed to be a composition of the small-size instantons and some large-size gluon field fluctuations with the correlation length $\rho_\ell \gg 1$ GeV^{-1}. The instantons of large size $\rho \gg (1 \text{ GeV})^{-1}$ manifest themselves in terms of the local scalar quark condensate. The quark propagator in the field of small-size instantons contains the zero mode chirality-flipping part proportional to the effective quark mass $m(p)$ and a nonzero-mode part approximated by the propagator of a free massless quark [10, 11]. The zero-mode part can be expressed in terms of the nonlocal scalar condensate.

We solved the sum rule equations and traced the dependence of the solution on the fraction of small-size instantons w_s. We demonstrated that at $w_s \le 0.6$–0.7, the sum rules have a solution with a reasonable value of the nucleon mass. At $w_s \approx 2/3$, the value of the nucleon mass is very close to the physical one. The numerical values vary slightly with variation of the actual values of the size of small instantons and of the distance between them. Finally at the values of the scalar condensate close to the conventional value $(-250 \text{ MeV})^3$,

$$m_N \approx 1 \text{ GeV}, \quad \lambda^2 \approx 1 \text{ GeV}^6, \quad W^2 \approx 2 \text{ GeV}^2. \quad (63)$$

At larger values of w_s, the sum rules have only an unphysical solution with a strong domination of the continuum contribution over that of the nucleon pole and with a small value of the nucleon mass.

Solution (63) was found for $\rho_s = 0.33$ fm, with $R = 1.2$–1.3 fm. It is also valid for $R \approx 1.3$ fm with $\rho_s \approx 0.25$–0.40 fm. We note that in [10, 11], the value of R is tied to that of the gluon condensate, which is known with a large uncertainty [13], and $R = 1.2$ fm is not unrealistic. Also (see [9]), we can tie the gluon condensate to the total instanton density. For the conventional value

$$\langle 0|\frac{\alpha_s}{\pi} G^{a\mu\nu} G^a_{\mu\nu}|0\rangle \frac{1}{32\pi^2} = (200 \text{ MeV})^4$$

and the distance between small-size instantons $R = 1.2$ fm, the densities of small-size and large-size instantons are approximately the same.

At larger values of the quark condensate, the values of the nucleon residue and of the continuum threshold increase, reaching the values $\lambda^2 \approx 3$ GeV^6 and $W^2 \approx 3$ GeV^2 at $\langle 0|\bar{q}(0)q(0)|0\rangle = (-290 \text{ MeV})^3$.

Compared to the sum rules in the condensate representation, we included the nonlocality of the scalar condensate. Also, the instanton representation strongly diminished the role of the contribution corresponding to the four-quark condensate in the condensate language.

The consistency between the left- and right-hand sides of the sum rules appeared to be much better than in the sum rules in terms of local condensates, where the value of "χ^2 per point" was of the order 10^{-1} [17] assuming 10 % error bars. The mean relative difference

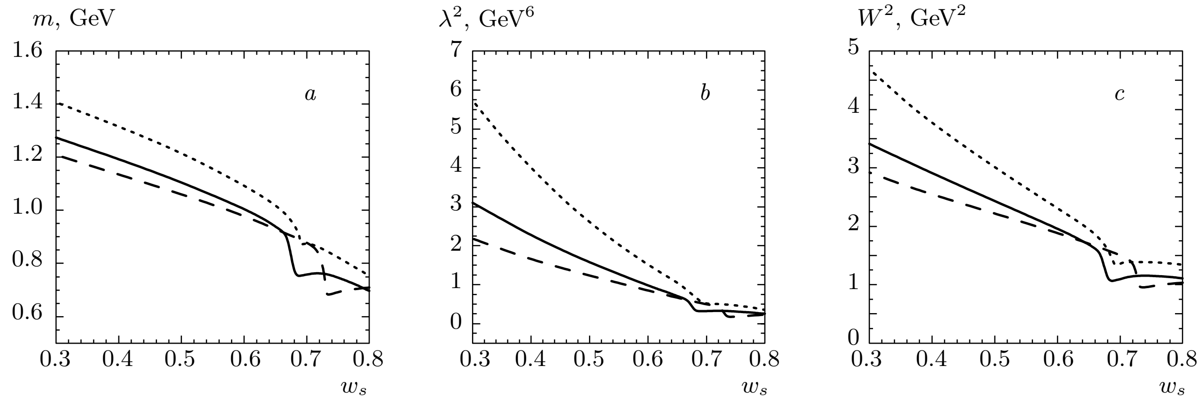

Fig. 7. Dependence of the solution of the sum rule equations on the value of w_s at $R \approx 1.3$ fm for the nucleon mass m (a), λ^2 (b), and W^2 (c). The solid, dashed, and dotted curves are for the respective values of the scalar condensate $a = 0.58$, 0.48, and 0.77 GeV^3

Table 3. Solutions of the sum rule equations in various intervals of the values of the Borel mass. The parameter values are the same as in Table 2

a, GeV^3	M^2, GeV^2	m_N, GeV	λ^2, GeV^6	W^2, GeV^2	χ^2_N
0.58	0.8–1.4	1.01	0.98	1.96	9.3(−3)
	0.8–1.6	1.02	1.01	1.99	1.2(−2)
	0.8–1.8	1.03	1.04	2.01	1.5(−2)
0.96	0.8–1.4	1.15	2.83	2.93	4.0(−2)
	0.8–1.6	1.17	3.03	3.02	5.1(−2)
	0.8–1.8	1.19	3.20	3.08	6.0(−2)

between the left- and right-hand sides is about 3 %. At larger values of the scalar condensate, the domination of the contribution of the pole over that of the continuum becomes more pronounced. Also, the duality interval becomes larger than that defined by Eq. (29) due to the shift of the upper limit.

We demonstrated that the contribution of the nonlocality of the scalar condensate can be approximated by two additional terms of the $1/M^2$ series. This corresponds to approximating the dependence of the nonlocal quark condensate $f(x^2) = \langle 0|\bar{q}(x)q(0)|0\rangle$ on x^2 by a polynomial of the second order. At $x^2 = 1$ GeV^{-2} (with the Euclidean metric), we found $f(x^2) - f(0) = tf(0)$ with $t = -0.14$ for $w_s = 1$ and $t = -0.09$ for $w_s = 0.65$. More complicated calculations in the framework of the instanton liquid model [20] yielded $t \approx -0.1$ for $x^2 = 1$ GeV^{-2}. The parameter m_0^2 defined by Eq. (15) determines the lowest-order term of the Taylor series of the condensate $f(x^2)$. Its value was estimated in the nucleon QCD sum rule analysis as providing the best fit of the two sides of the sum rules. The result in [16] is $m_0^2 \approx 0.8$ GeV^2, leading to $t \approx 0.2$, while the value $m_0^2 \approx 0.2$ GeV^2 yielding $t \approx -0.05$ was obtained in [21].

We note that these are to large extent the preliminary results. Representing the continuous distribution of instanton sizes as a superposition of small-size instantons and of some large-size gluon field fluctuations, we neglected their possible interactions. Another point is the interpretation of the condensate $(1 - w_s)\langle 0|\bar{q}q|0\rangle$ caused by the interactions at the large scale. A more general analysis should be carried out. The last but not the least, we plan to include interactions between the quarks composing the polarization operator, i. e., to take the radiative corrections into account. They are the same as in the condensate representation for the structure Π^q. However, additional work is required to find these corrections for the chirality-flipping structure Π^I. Hence, a more general analysis is required; the corresponding results will be published elsewhere.

Table 4. Solutions of the sum rule equations for $R \approx 1.3$ fm

a, GeV^3	w_s	m_N, GeV	λ^2, GeV^6	W^2, GeV^2	χ^2_N
0.77	0.60	1.09	1.52	2.31	3.8(−3)
	0.70	0.88	0.55	1.38	1.5(−3)
0.48	0.50	1.06	1.20	2.21	7.8(−3)
	0.60	0.98	0.82	1.87	1.1(−2)

We thank A. E. Dorokhov, N. I. Kochelev, and especially V. Yu. Petrov for the stimulating discussions.

APPENDIX

To calculate the integral on the right-hand side of Eq. (41), we write

$$\ln (Q-p)^2 = -\int_0^\infty \frac{dy}{(Q-p)^2+y}. \qquad (A.1)$$

Here and below, we omit polynomials in Q^2 because they are eliminated by the Borel transformation. Now we can write

$$X_s = -\frac{3i}{\pi^2}\int \frac{d^4p}{(2\pi)^4}\frac{\mathcal{A}}{p^2(p^2+\eta^2)^3} \times \int_0^\infty \frac{dy\, y}{(Q-p)^2+y}. \qquad (A.2)$$

We can verify that

$$\frac{1}{p^2(p^2+\eta^2)^3} = 3\int_0^1 \frac{dx\, x^2}{(p^2+\eta^2 x)^4}, \qquad (A.3)$$

whence

$$X_s = -3\int_0^1 dx\, x^2 \Psi(\eta^2 x), \qquad (A.4)$$

where

$$\Psi(\mu^2) = \frac{3\mathcal{A}}{\pi^2}\int_0^\infty dy\, y\Phi(\mu^2, y),$$
$$\Phi(\mu^2, y) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+\mu^2)^4}\frac{1}{(Q-p)^2+y}. \qquad (A.5)$$

Integrating over the angular variables, we find

$$\Phi(\mu^2, y) = \frac{1}{48\pi^2}\int_0^1 \frac{dt(1-t)^3}{(ty+\mu^2(1-t)+t(1-t)Q^2)^3} = \int_0^1 \frac{dt(1-t)^3}{t^3(y+\kappa)^3}, \quad \kappa = Q^2(1-t) + \frac{\mu^2(1-t)}{t^3}. \qquad (A.6)$$

Integrating over y, we obtain

$$\Psi(\mu^2) = \frac{\mathcal{A}}{32\pi^4}\int_1^\infty du \left(1-\frac{1}{u}\right)^2 \frac{u}{Q^2+\mu^2 u}. \qquad (A.7)$$

The divergence at the upper limit is not important, because this contribution is eliminated by the Borel transformation. Returning to Eq. (A.4), we can write it as

$$X_s = \frac{3\mathcal{A}}{32\pi^4}\int_0^1 dx\, x^2 \int_1^\infty du \times \left(1-\frac{1}{u}\right)^2 \frac{u}{Q^2+\eta^2 ux}. \qquad (A.8)$$

We can now integrate easily, with the result

$$X_s = \frac{3\mathcal{A}}{32\pi^4}\left[\frac{Q^4}{\eta^6}\ln\frac{Q^2+\eta^2}{Q^2} + \left(\frac{3Q^2}{\eta^4}+\frac{3}{\eta^2}+\frac{1}{Q^2}\right)\ln\frac{Q^2+\eta^2}{\eta^2}\right]. \qquad (A.9)$$

After the Borel transformation, we arrive at Eq. (42). We note that the Borel transform of the right-hand side of Eq. (A.8) is given by compact expression

$$\mathcal{B}X_s = \frac{3\mathcal{A}}{32\pi^4}\int_0^1 dx\, x^2 \int_1^\infty du\, u\left(1-\frac{1}{u}\right)^2 \times \exp\left(-\frac{\eta^2 xu}{M^2}\right). \qquad (A.10)$$

REFERENCES

1. M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385, 448, 519 (1979).

2. B. L. Ioffe, Nucl. Phys. B **188**, 317 (1981); B **191**, 591(E) (1981).

3. B. L. Ioffe, L. N. Lipatov, and V. S. Fadin, *Quantum Chromodynamics*, Campridge Univ. Press, Campridge (2010).

4. K. G. Wilson, Phys. Rev. **179**, 1499 (1969).

5. A. V. Radyushkin, CEBAF-TH-94-13, hep-ph/9406237; A. P. Bakulev and S. V. Mikhailov, Phys. Rev. D **65**, 114511 (2002).

6. B. L. Ioffe, Z. Phys. C **18**, 67 (1983).

7. E. V. Shuryak, *The QCD Vacuum, Hadrons and the Superdense Matter*, World Sci. Publ., Singapore (1988).

8. A. Ringwald and F. Schremmp, Phys. Lett. B **459**, 249 (1999).

9. H. Forkel, Phys. Rev. D **71**, 054008 (2005).

10. D. I. Dyakonov and V. Yu. Petrov, Zh. Eksp. Teor. Fiz. **89**, 361 (1985) [JETP **62**, 204 (1985)].

11. D. I. Dyakonov and V. Yu. Petrov, Nucl. Phys. B **272**, 457 (1986).

12. G.'t Hooft, Phys. Rev. Lett **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).

13. B. L. Ioffe, Progr. Part. Nucl. Phys. **56**, 232 (2006).

14. V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **174**, 378 (1980).

15. H. Forkel and M. K. Banerjee, Phys. Rev. Lett. **71**, 484 (1993).

16. B. L. Ioffe and A. V. Smilga, Nucl. Phys. B **232**, 109 (1984).

17. V. A. Sadovnikova, E. G. Drukarev, and M. G. Ryskin, Phys. Rev. D **72**, 114015 (2005).

18. M. G. Ryskin, E. G. Drukarev, and V. A. Sadovnikova, Phys. Atom. Nucl. (2015).

19. E. G. Drukarev, M. G. Ryskin, and V. A. Sadovnikova, Phys. Rev. D **80**, 014008 (2009).

20. E. V. Shuryak, Nucl. Phys. B **328**, 85 (1989).

21. Y. Chung, H. G. Dosch, M. Kremer, and D. Schall, Z. Phys. C **25**, 151 (1984).