

THEORY OF A RANDOM FIBER LASER

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We develop the theory explaining the role of nonlinearity in generation radiation in a fiber laser that is pumped by external light. The pumping energy is converted into the generating signal due to the Raman scattering supplying an effective gain for the signal. The signal is generated with frequencies near the one corresponding to the maximum value of the gain. Generation conditions and spectral properties of the generated signal are examined. We focus mainly on the case of a random laser where reflection of the signal occurs on impurities of the fiber. From the theoretical standpoint, kinetics of a wave system close to an integrable one are investigated. We demonstrate that in this case, the perturbation expansion in the kinetic equation has to use the closeness to the integrable case.

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1. INTRODUCTION

We consider the theory of random fiber lasers. The concept of random lasers exploiting multiple scattering of photons in an amplifying disordered medium in order to generate coherent light without a traditional laser resonator has attracted much attention in recent years. This research area lies at the interface of the fundamental physics of disordered systems and laser science. The idea of a random laser was originally proposed in the context of astrophysics in the 1960s by V. S. Letokhov, who studied scattering with "negative absorption" of the interstellar molecular clouds. Research on random lasers has developed into a mature experimental and theoretical field. A simple design of such lasers would be promising for potential applications.

In traditional random lasers, the properties of the output radiation are typically characterized by complex features in the spatial, spectral, and temporal domains, making them less attractive than standard laser systems in terms of practical applications. Recently, an interesting and novel type of random lasers that operate in a conventional telecommunication fibers without any predesigned resonator mirrors was demonstrated. The feedback required for laser generation in the random fiber laser is provided by Rayleigh scattering from the inhomogeneities of the refractive index that are naturally present in silica glass. In the proposed laser concept, the randomly backscattered light is amplified through the Raman effect, providing distributed gain over distances up to 100 km. Although an effective reflection due to the Rayleigh scattering is extremely small, the lasing threshold may be exceeded when a sufficiently large distributed Raman gain is supplied.

The random distributed feedback fiber laser has a number of interesting and attractive features. The fiber waveguide geometry provides transverse confinement, and the effectively one-dimensional random distributed feedback leads to the generation of a stationary beam with a narrow spectrum. The random distributed feedback fiber laser has efficiency and performance that are comparable to and even exceed those of similar conventional fiber lasers. The key features of the generated radiation of random distributed feedback fiber lasers include a stationary narrow-band continuous modeless spectrum that is free of mode competition, nonlinear power broadening, and an output beam with a Gaussian profile in the fundamental transverse mode (generated both in single-mode and multi-mode fibers). Details of the random laser performance can be found in recent review [1].

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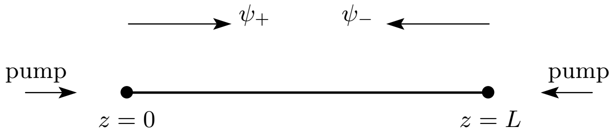

Scheme illustrating arrangement of a fiber laser

2. BASIC DYNAMIC EQUATIONS

The random fiber laser is a piece of optical fiber of length L that is optically pumped from the fiber ends. As a result, randomly backscattered light in the fiber is amplified through the Raman effect, and the system starts to lase at some level of the amplification (see Ref. [1]). Two electromagnetic waves propagating to the right and to the left are generated in the fiber. A schematic distribution of the generated waves along the fiber is presented in the figure. Due to pumping, their amplitudes increase during the propagation and achieve maxima near the ends of the fiber, before passing outside the fiber. We stress that the nonlinear interaction of the generated waves propagating to the right and to the left is weak because their maxima are achieved at the opposite ends of the fiber. Therefore, they can be considered independently.

We begin with the dynamic equation describing the evolution of the envelope of the generation electromagnetic field, ψ, over the evolution coordinate z within the fiber, at $0 < z < L$, where L is the fiber length. The equation for the generation wave propagating in the fiber to the right is

$$i\left(\partial_z - \hat{g}\right)\psi = \beta\partial_t^2\psi + \frac{\gamma}{2}\psi|\psi|^2, \tag{1}$$

where t is the time, γ is the Kerr nonlinear coefficient, and β is the quadratic dispersion coefficient. We consider the generation processes high above the generation threshold and therefore neglect noise terms in Eq. (1). An equation analogous to Eq. (1) can be formulated for the signal propagating to the left, the only difference being in the sign of the derivative ∂_z.

The gain operator $\hat{g}$ is determined by an interplay of the pumping and the attenuation of the signal. In the frequency domain, it is a frequency-dependent factor

$$g = g_R P(z) - \alpha_l,$$

where g_R the Raman gain coefficient, $P(z)$ is the power of the pumping wave, and α_l is the linear attenuation coefficient in the fiber. The distribution of the pumping over the evolution coordinate z is defined by the factor $P(z)$, which is assumed to be known. The lasing is realized for frequencies near the frequency where the gain g achieves a maximum. We take the frequency as the carrying frequency for the envelop ψ. Then we obtain

$$g(\omega) = g_0 - \varpi\omega^2, \tag{2}$$

which is an expansion of the gain coefficient near its maximum. Here, ω is the frequency shift from the carrying frequency. We note that above the generation threshold, the condition $g_0 > 0$ has to be fulfilled.

We stress that in reality, the power P of the pumping wave is dependent on the generation wave ψ: they are related via the balance equation [1, 2]. Therefore, the problem should be solved in two steps. First, we have to solve the balance equations to find $P(z)$. Then, $P(z)$ can be involved in calculating ψ. Here, we concentrate on the second step.

In a random fiber, almost all generated radiation is coupled out from the fiber end. Only a small part of the energy is reflected back via Rayleigh backscattering processes. Because the amplitudes of generated waves increased during evolution, the scattering process is effective only at the end of the fiber. This implies an effective initial condition for the generation wave ψ_+, propagating to the right, in terms the amplitude of the generation wave ψ_-, propagating to the left. Formally, the initial conditions for the waves have the form

$$\begin{aligned} \psi_+(0,t) &= \hat{R}_l\psi_-(0,t), \\ \psi_-(L,t) &= \hat{R}_r(\omega)\psi_+(L,t), \end{aligned} \tag{3}$$

where R_l and R_r are reflection coefficients on the left end and on the right end of the fiber, defined in the frequency domain. They have different ω-dependences in different situations. In the case of the random-fiber laser, $|R| \ll 1$. The reflection smallness leads to the conclusion that the signal is weakly disturbed by the reflection, thus justifying conditions (3).

The spectrum of the generated wave in the random-fiber laser is relatively broad (compared to traditional lasers) and consists of a high number of spectral components near the carrying frequency (see [1]). The main challenge here is to describe the influence of nonlinearity on the generation spectrum. For this, we use the standard kinetic approach dealing with averaged quantities. We assume that the dispersion length $(\beta\Delta^2)^{-1}$ (where Δ is the spectral width) is small in comparison with the fiber length L. Then the harmonics with different frequencies possess essentially different phases and therefore, under averaging over a length larger than the dispersion length, the harmonics can be treated as approximately independent.

The main object in the kinetic theory is the pair correlation function

$$\langle\psi(z,t_1+t)\psi^*(z,t_1)\rangle = \int \frac{d\omega}{2\pi} e^{-i\omega t} F(z,\omega), \quad (4)$$

where angular brackets mean averaging over a distance larger than the dispersion length and "$*$" denotes complex conjugation. Due to the assumed time homogeneity, the average in (4) depends solely on the time difference t and is independent of t_1. However, in examining real fibers, it is useful to average over time (integrate over t_1) to eliminate effects related to different fluctuations (noises) neglected in our formalism. We stress that due to the z-dependence of the generation wave, the system is not homogeneous in space, in contrast to the time behavior. The function F is no other than the spectrum of the generated signal. We note that the signal intensity I can be expressed via the spectrum as the integral

$$I \equiv \langle|\psi|^2\rangle = \int \frac{d\omega}{2\pi} F(\omega). \quad (5)$$

Boundary conditions (3) lead to the following relations for the averages:

$$\begin{aligned} F_+(0,\omega) &= |R_l(\omega)|^2 F_-(0,\omega), \\ F_-(L,\omega) &= |R_r(\omega)|^2 F_+(L,\omega), \end{aligned} \quad (6)$$

where F_+ and F_- correspond to the respective generation waves propagating to the right and to the left. In what follows, we consider the symmetric stationary situation where $R_l = R_r$ and $F_+(z) = F_-(L-z)$. Then we obtain the condition

$$F(0,\omega) = |R(\omega)|^2 F(L,\omega), \quad (7)$$

for the signal propagating to the right. The condition relates values of the correlation function F taken at different ends of the fiber.

3. KINETICS

We assume weak nonlinearity of the system. Then a perturbation theory has to be developed to examine nonlinear effects in the random laser. The starting point for the theory is the basic equation (1) for the envelope $\psi(z,t)$. We treat the nonlinear term in Eq. (1) as a perturbation and expand the solution of the equation with respect to the nonlinearity. Then we use the expansion for calculating average (4).

Below, our aim is to derive a dynamic equation for the spectrum F. The equation enables analyzing the form of the spectrum and its dependence on the system parameters. Our derivation is performed in the spirit of the derivation of the standard kinetic equation [3, 4] for classic waves. However, our system is close to an integrable one because at $g = 0$, the basic equation (1) is the nonlinear Schrödinger equation, which is completely integrable and has an infinite number of integrals of motion. There are no kinetics in the system of waves described by the nonlinear Schrödinger equation [5]. Therefore, the kinetics are related to the presence of the gain term g, which makes the situation absolutely different from the standard kinetic equation and requires a consistent derivation of the generalized kinetic equation.

A formal solution of Eq. (1) can be written as

$$\psi(z,t) = \int dt'\, G(z,z_\star,t-t')\psi(z_\star,t') - \frac{i\gamma}{2} \times \\ \times \int dt' \int_{z_\star}^{z} dz'\, G(z,z',t-t')\psi(z',t')|\psi(z',t')|^2, \quad (8)$$

where $z_\star$ is an arbitrary point. Here,

$$G(z,z',t) = \theta(z-z') \int \frac{d\omega}{2\pi} \times \\ \times \exp\left[-i\omega t + \int_{z'}^{z} dz''\,(g + i\beta\omega^2)\right] \quad (9)$$

is the Green's function determining a linear response of the system to an external influence. Analogously, it is possible to consider the evolution "backward" in z. For example, in the linear approximation,

$$\psi(z',t') \approx \int dt\, G(z,z',t-t')\psi(z,t). \quad (10)$$

We now pass to obtaining an equation for the function F in (4). It follows directly from Eq. (1) that

$$\partial_z F(z,\omega) = 2gF - \frac{i\gamma}{2} \int dt\, e^{i\omega t} \times \\ \times \left\langle \psi(z,t)\psi^*(z,0)\left[|\psi(z,t)|^2 - |\psi(z,0)|^2\right]\right\rangle. \quad (11)$$

Here, as above, the angular brackets denote averaging over a distance larger than the dispersion length. Equation (11) implies that both the gain g and the correlation function F are slowly varying functions at the averaging length. We assume that $\beta\Delta^2 \gg g_0$, and then we can choose the averaging length l much smaller than g_0^{-1}. In addition, the inequality $\varpi\Delta^2 \ll g_0$ has to be satisfied, which is a manifestation of the spectrum narrowness in comparison with the characteristic frequency range of Raman scattering. Therefore, we arrive at the chain of the inequalities $\varpi\Delta^2 \ll g_0 \ll \beta\Delta^2$ to be satisfied for the validity of our theoretical scheme.

The phase randomization caused by dispersion leads to approximately Gaussian statistics of the field ψ since it appears to be a sum of a large number of independent terms. Therefore, in calculating averages like (11), we can use the Wick theorem (that is, the presentation of the average of some product of ψ fields via its pair correlation functions). But applying the Wick theorem to the combination in the right-hand side of Eq. (11) gives zero. Therefore, we have to take into account a weak correlation between different harmonics caused by nonlinearity. Technically, we must use the nonlinear contribution to ψ from Eq. (8),

$$\psi_{non}(z,t) = -\frac{i\gamma}{2}\int dt' \int\limits_0^z dz'\, G(z,z',t-t') \times \\ \times \int dt_1\ G(z,z',t_1-t')\psi(z,t_1) \times \\ \times \left|\int dt_2\ G(z,z',t_2-t')\psi(z,t_2)\right|^2, \quad (12)$$

where we substituted expression (10). The goal of the substitution is to express the variation $\delta\psi$ in terms of $\psi(z,t)$. Then averages in the right-hand side of Eq. (12) are expressed in terms of the function $F(z,\omega)$ in (4).

Using the Wick theorem, substituting expression (9) and taking integrals over time, we obtain from Eq. (12) that

$$(\partial_z - 2g)\,F = \gamma^2 \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \\ \times \left[\frac{g_a\, F F_2 F_3}{g_a^2+\Omega^2} + \frac{g_c\, F_1 F_2 F_3}{g_c^2+\Omega^2} - \frac{2g_b\, F F_1 F_3}{g_b^2+\Omega^2}\right], \quad (13)$$

where $F = F(z,\omega)$, $F_1 = F(z,\omega_1)$, and so on, and the following notations are introduced:

$$\Omega = \beta(\omega^2+\omega_1^2-\omega_2^2-\omega_3^2),$$

$$g_a = g(\omega)+g(\omega_2)+g(\omega_3)-g(\omega_1),$$

$$g_b = g(\omega)+g(\omega_1)+g(\omega_3)-g(\omega_2),$$

$$g_c = g(\omega_1)+g(\omega_2)+g(\omega_3)-g(\omega).$$

Equation (13) is a generalized kinetic equation derived for interacting waves in an unstable medium (due to pumping). We see that in Eq. (13), the usual δ-functions (that ensure the wave vector conservation) in the collision integral (right-hand side) are substituted by Lorentzians, where the gain g is present. This is a manifestation of the system inhomogeneity in z caused by the gain. Other properties of the generalized kinetic equation are close to those of the usual wave kinetic equation. For example, the integral over ω of the collision integral is equal to zero. This is a consequence of the wave action (number of waves) conservation law which is valid without gain.

It is possible to substitute $g \to g_0$ in the collision integral because we assume $\varpi\Delta^2 \ll g_0$ and the nonlinear stage (where the collision integral is relevant) is relatively short. However, generally, we should keep the term $\varpi\omega^2$ in the left-hand side of Eq. (13) since it is relevant at the linear stage of wave evolution. As a result, we arrive at the equation

$$\left(\partial_z - 2g_0 + 2\varpi\omega^2\right) F = \\ = \gamma^2 \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \\ \times \frac{2g_0}{4g_0^2+\Omega^2}\left[FF_2F_3+F_1F_2F_3-FF_1F_2-F_\omega F_1F_3\right], \quad (14)$$

which is a starting point for the subsequent analysis.

In this paper, we examine the case of relatively strong dispersion (wide spectrum), when $\beta\Delta^2 \gg g_0$ (where Δ is the spectrum width). (The case of a narrow spectrum was considered in [2].) The inequality $\beta\Delta^2 \gg g_0$ means that we can pass to the limit of small g in Eq. (13) or (14). However, we should be careful because of the noted cancelations. In the limit as $g_0 \to 0$, the Lorentzian in the collision integral (the right-hand side of Eq. (14)) turns into a δ-function of Ω, thus acquiring the form of the usual collision integral [4]. But the collision integral vanishes in this limit. This is a consequence of the complete integrability of the one-dimensional nonlinear Schrödinger equation. The existence of an infinite number of integrals of motion leads in this case to the absence of kinetics in all orders in nonlinearity [5].

Therefore, we should go beyond the zeroth order in g_0 (that gives the δ-function) and keep the first order in g_0. Hence, we can neglect g_0 in comparison with Ω in the denominator in Eq. (14) and keep g_0 in the numerator to obtain

$$(\partial_z - 2g)\,F(z) = \frac{2g_0\gamma^2}{\beta^2} \times \\ \times \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \\ \times (\omega^2+\omega_1^2-\omega_2^2-\omega_3^2)^{-2} \times \\ \times \left(FF_2F_3+F_1F_2F_3-FF_1F_2-FF_1F_3\right). \quad (15)$$

We note the presence of a singular denominator in Eq. (15). This does not lead to any divergence just due to the integrability (any divergence would mean that the coefficient at the δ-function is nonzero). This equation is a starting point of subsequent calculations.

As follows from Eq. (15), in the linear approximation,

$$F_{lin}(z,\omega) \propto \exp\left[2\int dz\ (g_0 - \varpi\omega^2)\right]. \tag{16}$$

This expression describes the exponential growth of the signal amplitude. Besides, relation (16) shows that in the linear regime, the laser spectrum becomes narrower following the gain spectral shape $g(\omega)$. If $\mathcal{A} > \Delta_0^{-2}$, where $\mathcal{A} = \int dz\ \varpi$ and Δ_0 is the initial spectrum width at $z = 0$, then the spectrum width Δ at the end of the linear stage can be estimated as $\Delta \sim \mathcal{A}^{-1/2}$. We note that the spectral width in this case does not depend on the initial spectral width at $z = 0$.

4. SOLUTION

The right-hand side of Eq. (15) can be estimated as $g_0 F(\gamma I/\beta\Delta^2)^2$. We first analyze the case where $\gamma I \ll \beta\Delta^2$ at the end of the fiber. That means that the inequality is satisfied everywhere because I increases monotonically as z increases. The inequality $\gamma I \ll \beta\Delta^2$ means that the linear term $2g$ in the left-hand side of Eq. (15) is larger than the collision integral (the right-hand side of the same equation). Then the leading contribution to the F-evolution by the collision integral is produced at the nearest-to-the-fiber-end interval of the length of the order of g_0^{-1}.

To calculate the nonlinear (collision) contribution to $F(L)$, we can use the linear law (16) (where the term with ϖ can be neglected) to obtain

$$F(z) = \exp[2g_0(z-L)]F(L).$$

Then in accordance with Eq. (15), the nonlinear correction to F can be written as

$$F_{non} = \frac{\gamma^2}{3\beta^2}\int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \times (\omega^2+\omega_1^2-\omega_2^2-\omega_3^2)^{-2} \times \times (FF_2F_3 + F_1F_2F_3 - FF_1F_2 - FF_1F_3), \tag{17}$$

where all functions, $F, F_1, \ldots$, are taken at $z = L$.

To achieve a statistically steady state, we have to satisfy relation (7). We assume here that the signal scattering is produced by impurities. Then the reflection coefficient R depends weakly on the frequency ω, because the impurity size is much smaller than the wavelength. In this case, the only relevant parameter is $\kappa = |R_\omega|^2 \ll 1$. Then it follows from Eq. (7)

$$F(0,\omega) = \kappa F(L,\omega). \tag{18}$$

To satisfy Eq. (18), we have to assume that the ϖ-contribution to the law (16) is small. Therefore,

$$\kappa \exp\left(2\int_0^L dz\ g_0\right) = 1 + \eta, \tag{19}$$

where $\eta \ll 1$.

Using relations (16), (17), and (19), we find from the condition (18) that

$$(\eta - 2\mathcal{A}\omega^2)F + \frac{\gamma^2}{2\beta^2}\int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2}\delta(\omega+\omega_1-\omega_2-\omega_3) \times \times \frac{1}{(\omega^2+\omega_1^2-\omega_2^2-\omega_3^2)^2} \times \times [FF_2F_3 + F_1F_2F_3 - FF_1F_2 - FF_1F_3] = 0, \tag{20}$$

where all functions are taken at $z = L$ and $\mathcal{A} = \int_0^L dz\ \varpi$. As follows from Eq. (20), the spectrum width is determined by the balance of the terms in the left-hand side, that is,

$$\Delta = \sqrt{\frac{\eta}{2\mathcal{A}}}. \tag{21}$$

We note the smallness of Δ in η. Comparing different terms in Eq. (20) we find

$$I \sim \frac{\beta\eta^{3/2}}{\gamma\mathcal{A}}$$

and

$$I \sim \frac{\beta}{\gamma}\sqrt{\mathcal{A}}\,\Delta^3. \tag{22}$$

Hence, $\Delta \propto I^{1/3}$ in the regime.

Equation (20) admits a self-similar substitution

$$F(L,\omega) = \frac{\beta\eta}{\gamma\sqrt{\mathcal{A}}}\,\phi\left(\frac{\omega}{\Delta}\right), \tag{23}$$

where Δ is determined by Eq. (21). Then Eq. (20) leads to the universal form of the equation for the self-similar function

$$(x^2-1)\phi(x) = \int \frac{dx_2 dx_3}{(4\pi)^2} \times \times \frac{\phi\phi_2\phi_3 + \phi_1\phi_2\phi_3 - \phi\phi_1\phi_2 - \phi\phi_1\phi_3}{(x-x_2)^2(x-x_3)^2}, \tag{24}$$

where $x_1 = x_2 + x_3 - x$. Numerical solution of the equation gives the normalization factor

$$\int dx\ \phi(x) \approx 23.8.$$

We see from Eq. (22) that the spectral width Δ increases as the intensity I increases. At some level of pumping, γI becomes of the order of $\beta\Delta^2$. For higher pumping levels, the lasing regime completely changes. The regime requires a separate consideration. Our preliminary analysis shows that in this regime, the relation $\gamma I \sim \beta\Delta^2$ is satisfied during the nonlinear stage of the generation wave propagation (near the fiber end). The result needs an additional justification.

5. CONCLUSION

We analyzed the signal spectrum of a fiber laser that is pumped by external light (due to Raman scattering). We use a generalized kinetic equation for the analysis. A peculiarity of the wave system under consideration is its closeness to the completely integrable case of the one-dimensional nonlinear Schrödinger equation. We find a relation between the spectrum width and the intensity of the signal, that is characterized by a power law. We also establish exponential tails of the spectrum (with power-law corrections). The exponential character of the tails is ultimately caused by the frequency conservation law, satisfied due to homogeneity of the system in time. From the other side, our system is spatially inhomogeneous. However, the inhomogeneity was assumed to be weak in comparison with the phase variations caused by dispersion. The last condition implies that the spectrum width has to be large enough. The opposite case was analyzed in [2].

It is instructive to compare our generalized equation with the usual kinetic equation for weak wave turbulence [4]. The last one has two types of solutions: equilibrium solutions and flux solutions, both with power spectra. In our case, the collision integral is nonzero, because it must be balanced by some additional term appearing due to the spatial inhomogeneity of the system. That leads to the existence of a z-dependent characteristic spectrum width. Formally, it is a consequence of the "locality" property of our collision integral (it is determined by frequencies of the order of the external frequency); the "locality" property is also characteristic of the collision integral in usual weak wave turbulence.

Another peculiarity of our system, distinguishing it from the traditional weak wave turbulence, is the nearly integrable character of the system. Indeed, in the leading approximation, the wave propagation through a fiber is described by the nonlinear Schrödinger equation that is completely integrable. The wave kinetics in the integrable case are absent [5]. Therefore, the wave kinetics in our case are related mainly to the spatial nonhomogeneity of the fiber caused by the gain (and the relaxation). Therefore, we have to use the double perturbation theory, using weakness of both the nonlinearity and the nonintegrability. That is why the resulting wave kinetics appear to be essentially different from those in the traditional weak wave turbulence: instead of a power-law spectrum, we arrive at exponential tails in the spectrum.

Our predictions are in good agreement with experimental observations. The comparison will be published elsewhere.

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