

SOME ASPECTS OF VIRTUAL BLACK HOLES

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We first consistently third-quantize modified gravity. We then analyze certain aspects of virtual black holes in this third-quantized modified gravity. We see how a statistical mechanical origin for the Bekenstein–Hawking entropy naturally arises in this model. Furthermore, the area and hence the entropy of a real macroscopic black hole is quantized in this model. Virtual black holes cause a loss of quantum coherence, which gives an intrinsic entropy to all physical systems that can be used to define a direction of time and hence provide a solution to the problem of time.

1. INTRODUCTION

It is expected that quantum fluctuations of spacetime can cause the topology of spacetime to change at the Planck scale, giving it a foam-like structure called the spacetime foam [1, 2]. Spacetime foam has largely been discussed via the formation of baby universes that render the spacetime multiply connected [3–5]. In this model, the spacetime manifold therefore has a large value of the first Betti number B_1 and the second Betti number vanishes, $B_2 = 0$. The problem with this model is that it predicts a wrong value of the QCD θ -parameter [6] and the cosmological constant [7, 8].

However, there is an alternative model of spacetime foam that seems to predict a correct value of the QCD θ -parameter [9]. In this model, the topology of spacetime changes by the formation of virtual black holes and thus the spacetime remains single connected [10, 11]. In this model, the spacetime manifold has a large value of the second Betti number B_2 and the first and third Betti numbers vanish, $B_1 = B_3 = 0$. In this picture, there is also an elegant way to describe black hole evaporation without the appearance of a naked singularity. Macroscopic real black holes evaporate to the Planck size by emitting Hawking radiation. At this stage, they are left with no energy or charge. They then disappear in a sea of virtual black holes. Because this picture of spacetime foam seems to be more realistic, we analyze certain aspects of it in this paper.

To study the physical effects of virtual black holes, we should analyze the collision of particles with ener-

gy less than the Planck energy in a small region containing a virtual black hole. For this, we would need to find a Euclidean solution for this process. But it is very difficult to find such a solution, and we therefore analyze virtual black holes via third quantization. The third quantization has been discussed implicitly in Refs. [12, 13] and explicitly in Refs. [14, 15]. The modification of the Wheeler–De Witt equation by the addition of nonlinear terms and the third quantization of the resultant theory was formally analyzed in Ref. [18]. Third quantization of Brans–Dicke theories [19] and Kaluza–Klein theories [20] has also been done. However, all this work has been done in the baby universe model of spacetime foam. We therefore apply third quantization to the virtual black hole model in this paper. It may be noted that the idea of canonical quantization of gravity has progressed into loop quantum gravity [21, 22]. Furthermore, the idea of third quantization now appears as a group field theory [23, 24] in loop quantum gravity. Hence, this present work should be translated into the language of group field theory. However, it is not clear how to deal with virtual black holes in group field theory. To understand that, it might be useful to first analyze virtual black holes in two dimensions via matrix models [25, 26]. This is because group field theory can be viewed as a higher-dimensional generalization of matrix models.

2. WHEELER–DE WITT EQUATION

It is hoped that a corrected gravitational potential could fit galaxy rotation curves without the need of

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dark matter [27, 28]. This is why $f(R)$ gravity theories have become very important [29, 30]. We therefore study the virtual black holes in $f(R)$ gravity theories. The Lagrangian density describing a generic $f(R)$ theory of gravity is given by

$$\mathcal{L} = \sqrt{-g} (f(R) - 2\Lambda_c), \quad f'' \neq 0, \quad (1)$$

where $f(R)$ is an arbitrary function of the scalar curvature and the primes denote differentiation with respect to the scalar curvature. The Hamiltonian constraint for the $f(R)$ gravity is given by [31]

$$\mathcal{H} = \frac{1}{2\kappa} \left[\frac{\mathcal{P}}{6} \left({}^{(3)}R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2 \right) + V(\mathcal{P}) - \frac{1}{3}g^{ij}\mathcal{P}|_{ij} - 2p^{ij}K_{ij} \right], \quad (2)$$

where

$$V(\mathcal{P}) = \sqrt{h} [Rf'(R) - f(R)] \quad (3)$$

and

$$G_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}). \quad (4)$$

Here, K_{ij} is the second fundamental form, $K = h^{ij}K_{ij}$ is its trace, and ${}^{(3)}R$ is the three-dimensional scalar curvature. We also have

$$\mathcal{P} = -6\sqrt{h}f'(R), \quad (5)$$

and hence

$$\mathcal{H} = \frac{1}{2\kappa} \left[-\sqrt{h}f'(R) \left({}^{(3)}R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2 \right) + V(\mathcal{P}) + 2g^{ij} \left(\sqrt{h}f'(R) \right) |_{ij} - 2p^{ij}K_{ij} \right]. \quad (6)$$

This can be expressed as

$$\mathcal{H} = f'(R) \left[2\kappa G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{2\kappa} \left({}^{(3)}R - 2\Lambda_c \right) \right] + \frac{1}{2\kappa} \left[\sqrt{g}f'(R) (2K_{ij}K^{ij}) + V(\mathcal{P}) + 2g^{ij} \left(\sqrt{h}f'(R) \right) |_{ij} - 2p^{ij}K_{ij} \right]. \quad (7)$$

Using

$$p^{ij} = \sqrt{h}K^{ij} \quad (8)$$

and passing to canonical momenta, we write the Hamiltonian constraint for the $f(R)$ gravity as

$$\mathcal{H} = f'(R) \left[2\kappa G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{2\kappa} \left({}^{(3)}R - 2\Lambda_c \right) \right] + 4\kappa \left[G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] (f'(R) - 1) + \frac{1}{2\kappa} \left[V(\mathcal{P}) + 2g^{ij} \left(\sqrt{h}f'(R) \right) |_{ij} \right]. \quad (9)$$

The Wheeler–De Witt equation is the quantum mechanical version of this Hamiltonian constraint

$$H\phi(h) = 0, \quad (10)$$

where we use

$$\pi^{ij} = -i \frac{\delta}{\delta h_{ij}}. \quad (11)$$

We note that when

$$f(R) = R,$$

then

$$V(\mathcal{P}) = 0$$

and the Wheeler–De Witt equation for the $f(R)$ gravity reduces to the usual Wheeler–De Witt equation. In most interpretations of quantum gravity (e. g., naive [32], conditional probability [33], and WKB approximation [34]), the Wheeler–De Witt equation is analogous to the Schrödinger wave equation, in the sense that it represents the quantum state of a single universe. But in the third-quantized formalism, it is seen as a classical field equation that has to be third quantized [14, 15].

The third-quantized formalism describes the quantum state of an ensemble of geometries. It is therefore the natural formalism for analyzing any model of spacetime foam. Much work on third quantization is done in analogy with quantum field theory in flat spacetime [12, 13, 18]. However, there is no timelike Killing vector for the Wheeler–De Witt equation [16, 17], and hence consistent third quantization should be done in analogy with quantum field theory in curved spacetime, and this is done in the next section.

3. THIRD QUANTIZATION

In this section, we third-quantize the Wheeler–De Witt equation for the $f(R)$ gravity. We first interpret Eq. (10) as a classical field equation

and then quantize it. For this, we assume that $\{\phi(P, h)\}$ and $\{\phi^*(P, h)\}$ form a complete set of solutions of this Wheeler–De Witt equation and satisfy a Klein–Gordon-type symplectic product with the properties

$$\int Dh \mathcal{J}(\phi(P, h), \phi(Q, h)) = \mathcal{M}(P, Q), \quad (12)$$

$$\int Dh \mathcal{J}(\phi(P, h), \phi^*(Q, h)) = 0, \quad (13)$$

$$\int Dh \mathcal{J}(\phi^*(P, h), \phi^*(Q, h)) = -\mathcal{M}(P, Q). \quad (14)$$

In quantum field theory, the condition given in Eq. (13) does not hold in general, and it is therefore a requirement on the complete set of solutions of the Wheeler–De Witt equation [35]. We also chose $\mathcal{M}(P, Q)$ to have positive eigenvalues only. This again is not always true and hence this is again a requirement on the complete set of solutions of the Wheeler–De Witt equation.

In the third-quantized formalism, $\phi(h)$ is promoted to a Hermitian operator and is expressed as [18]

$$\hat{\phi}(h) = \int DP [a(P)\phi(P, h) + a^\dagger(P)\phi^*(P, h)], \quad (15)$$

where $a(P)$ and $a^\dagger(P)$ satisfy the relations

$$\begin{aligned} [a(P), a^\dagger(Q)] &= \delta(P, Q), \\ [a^\dagger(P), a^\dagger(Q)] &= 0, \\ [a(P), a(Q)] &= 0, \end{aligned} \quad (16)$$

where $\delta(P, Q)$ is defined by

$$\int DP \delta(P, Q)\phi(P, h) = \phi(Q, h). \quad (17)$$

For this choice of the complete set of solutions of the Wheeler–De Witt equation, we define the vacuum state $|0\rangle$ as the state that is annihilated by $a(P)$:

$$a(P)|0\rangle = 0. \quad (18)$$

Now $a^\dagger(P)$ and $a(P)$ can be respectively called the creation and annihilation operators in analogy with those for the simple quantum harmonic oscillator. They create and annihilate geometries in the third-quantized formalism.

We note that the division between $\{\phi(P, h)\}$ and $\{\phi^*(P, h)\}$ is not unique even after imposing the conditions given by Eqs. (12)–(14) [35]. Due to this

nonuniqueness, the vacuum state is also defined not uniquely. This can be seen by considering $\{\phi'(P, h)\}$ and $\{\phi'^*(P, h)\}$ as another complete set of solutions of Eq. (10), satisfying conditions given by Eqs. (12)–(14). We then have

$$\hat{\phi}(h) = \int DP [a'(P)\phi'(P, h) + a'^\dagger(P)\phi'^*(P, h)], \quad (19)$$

where the vacuum state $|0'\rangle$ is the state annihilated by $a'(P)$,

$$a'(P)|0'\rangle = 0. \quad (20)$$

Many geometry states can be built by repeated action of $a'^\dagger(P)$ on $|0'\rangle$. Because $\phi(P, h)$ and $\phi^*(P, h)$ form a complete set of solutions of the field equation, Eq. (10), we can express $\phi'(P, h)$ as a linear combination of $\phi(P, h)$ and $\phi^*(P, h)$:

$$\begin{aligned} \phi'(P, h) &= \\ &= \int DQ [\alpha(P, Q)\phi(Q, h) + \beta(P, Q)\phi^*(Q, h)]. \end{aligned} \quad (21)$$

Substituting Eq. (21) in Eq. (19) and comparing the resulting expression with Eq. (15), we find

$$a(P) = \int DQ [\alpha(P, Q)a'(Q) + \beta^*(P, Q)a'^\dagger(Q)], \quad (22)$$

$$a^\dagger(P) = \int DQ [\alpha^*(P, Q)a'^\dagger(Q) + \beta(P, Q)a'(Q)]. \quad (23)$$

The two Fock spaces based on these choices of a complete set of solutions of the field equation, Eq. (10), are different as long as $\beta(P, Q) \neq 0$. In particular, $a(P)|0'\rangle$ does not vanish because

$$\begin{aligned} a(P)|0'\rangle &= \int DQ [\alpha(P, Q)a'(Q) + \\ &+ \beta^*(P, Q)a'^\dagger(Q)]|0'\rangle = \\ &= \int DQ \beta^*(P, Q)a'^\dagger(Q)|0'\rangle \neq 0, \end{aligned} \quad (24)$$

but

$$a(P)|0\rangle = 0. \quad (25)$$

Hence, $a(P)|0'\rangle$ is a one-geometry state. In fact, we have

$$\begin{aligned} \langle 0'|a(P)^\dagger a(P)|0'\rangle &= \\ &= \int DU DQ \beta(P, U)\beta^*(P, Q)\mathcal{M}(U, Q). \end{aligned} \quad (26)$$

The Wightman two-point function is then given by

$$G(h, h') = \langle 0|\hat{\phi}(h)\hat{\phi}(h')|0\rangle. \quad (27)$$

This can be written as

$$\begin{aligned}
 G(h, h') &= \int DPDQ \langle 0 | (a(P)\phi(P, h) + \\
 &\quad + a(P)^\dagger \phi^*(P, h)) \times \\
 &\quad \times (a(Q)\phi(Q, h') + a^\dagger(Q)\phi(Q, h')) | 0 \rangle = \\
 &= \int DPDQ \phi(P, h) \phi^*(Q, h') \mathcal{C}(P, Q), \quad (28)
 \end{aligned}$$

where

$$\mathcal{C}(P, Q) = \langle 0 | [a(P), a^\dagger(Q)] | 0 \rangle \quad (29)$$

is the commutator. Next, because $\hat{\phi}$ is Hermitian, we have [35]

$$[(\phi(P), \hat{\phi}), (\hat{\phi}, \phi(Q))] = \mathcal{M}(P, Q), \quad (30)$$

and therefore

$$(\phi(P), \hat{\phi}) = \int DU a(U) \mathcal{M}(P, U), \quad (31)$$

$$\begin{aligned}
 (\hat{\phi}, \phi(Q)) &= [(\phi(Q), \hat{\phi})]^\dagger = \\
 &= \int DW a^\dagger(W) \mathcal{M}(W, Q). \quad (32)
 \end{aligned}$$

It then follows from Eqs. (30)–(32) that

$$\begin{aligned}
 \int DUDW \mathcal{M}(P, U) [a(U), a^\dagger(W)] \mathcal{M}(W, Q) &= \\
 &= \mathcal{M}(P, Q). \quad (33)
 \end{aligned}$$

Using Eqs. (29) and (33), we obtain

$$\begin{aligned}
 \int DUDW \mathcal{M}(P, U) \mathcal{C}(U, W) \mathcal{M}(W, Q) &= \\
 &= \mathcal{M}(P, Q). \quad (34)
 \end{aligned}$$

This equation in matrix notation is written as

$$MCM = M. \quad (35)$$

Assuming that $\mathcal{M}(P, Q)$ has only positive eigenvalues, i.e., that it is invertible, we obtain

$$\mathcal{C} = M^{-1}. \quad (36)$$

Therefore, the two-point function is given by

$$\begin{aligned}
 G(h, h') &= \\
 &= \int DPDQ \phi(P, h) \phi^*(Q, h') M^{-1}(P, Q). \quad (37)
 \end{aligned}$$

In this section, we developed a third quantization of the Wheeler–De Witt equation for the $f(R)$ gravity. In the next section, we use it to analyze the formation of virtual black holes.

4. VIRTUAL BLACK HOLES

The Wheeler–De Witt equation in the third-quantized formalism represents the quantum state of an ensemble of noninteracting geometries. However, this is still not enough to account for topology change. To obtain a theory consistent with topology change, we need to include interaction terms. We therefore modify the original Wheeler–De Witt equation by adding interaction terms,

$$\left[H\phi - \frac{\delta \mathcal{V}[\phi]}{\delta \phi(h)} \right] = 0, \quad (38)$$

where $\mathcal{V}[\phi]$ is a potential summarizing all the interactions. We can now apply this third-quantized formalism of quantum gravity to virtual black holes. It has been argued that virtual black holes can form in loops like other virtual particles form in the conventional quantum field theory [9]. But this discussion on virtual black hole loops so far has not been made precise. This can be very easily done in the third-quantized formalism. We can thus write the amplitude for the formation of a virtual black hole loop in the third-quantized formalism as

$$\mathcal{A} = \int DhDh' G(h, h') G(h', h). \quad (39)$$

From Eqs. (37) and (39), we then obtain

$$\begin{aligned}
 \mathcal{A} &= \int DhDh' DPDQDUDW \times \\
 &\quad \times \phi(P, h) \phi^*(Q, h') \mathcal{M}^{-1}(P, Q) \times \\
 &\quad \times \phi(U, h') \phi^*(W, h) \mathcal{M}^{-1}(U, W). \quad (40)
 \end{aligned}$$

It follows that these virtual black holes form as off-shell contributions in the third-quantized formalism of quantum gravity.

Virtual black holes exist in dynamic equilibrium with flat spacetime at the Planck scale. However, if the probability of formation of virtual black holes, i.e., \mathcal{A}^2 , increases due to any process, then a phase transition can occur at a certain critical value of \mathcal{A}^2 . This can change the large-scale structure of spacetime and create a real black hole. Thus the formation of a real black hole can be seen as a phase transition induced by virtual black holes in analogy to how a ferromagnet is magnetised. This implies that spacetime foam can generate the microsates of a real black hole.

We therefore assume that the macroscopic state of a black hole is made up of microstates of spacetime foam. The area of a real macroscopic black hole A is then covered by microstates of foam like the structure

of spacetime at the Planck scale. If N is the number of microstates that completely cover the area of this macroscopic black hole at the Planck scale, then the total statistical mechanical entropy associated with the real black hole because of these micro-states is proportional to N [37]. This suggest that the entropy of a real black hole can have a statistical mechanical origin and the foam-like structure of spacetime may give rise to the microstates for the Bekenstein–Hawking entropy.

In fact, because A is completely covered by these microstates of spacetime foam, we have

$$A = N\tilde{A} = 4N\pi, \tag{41}$$

where $\tilde{A} = 4\pi$ is the unit Planckian area in the Planck units. Because the macroscopic black hole area is represented by N microscopic areas of the Planckian size, it is naturally quantized. Then the Bekenstein–Hawking entropy along with the quantum corrections to it for a macroscopic black hole is given by [36]

$$S = \frac{A}{4} + \alpha_0 \log A + \frac{\alpha_1}{A} + \frac{\alpha_2}{A^2} + \dots = N\pi + \alpha_0 \log(4N\pi) + \frac{\alpha_1}{4N\pi} + \frac{\alpha_2}{16N^2\pi^2} + \dots, \tag{42}$$

where $\alpha_0, \alpha_1, \alpha_2, \dots$ are constants. Because N is a very large number, we can neglect the higher-order corrections to this Bekenstein–Hawking entropy. In the leading order, this entropy is therefore given by

$$S \sim N\pi. \tag{43}$$

In the leading order, the Bekenstein–Hawking entropy is therefore proportional to N . This means that the entropy of a real black hole is quantized by the structure of spacetime foam.

5. QUANTUM COHERENCE AND THE PROBLEM OF TIME

There is always some probability of the formation of virtual black holes in flat spacetime, and hence the particles found in nature naturally interact with virtual black holes even in flat spacetime. If \mathcal{H}_{phy} is the Hilbert space of the particles found in nature and \mathcal{H}_{vb} is the Hilbert space of these virtual black holes, then the total Hilbert space \mathcal{H} for this physical theory is

$$\mathcal{H} = \mathcal{H}_{phy} \otimes \mathcal{H}_{vb}. \tag{44}$$

The density operator ρ can be expressed as $|\Psi\rangle\langle\Psi|$, where $|\Psi\rangle$ is a vector or a total wave function in \mathcal{H} . It is a pure state.

If ρ_- and ρ_+ are the respective density matrices for the total Hilbert space at past and future infinities and $\$$ is the superscattering operator, then we can write [38]

$$\rho_+ = \$\rho_-. \tag{45}$$

Because the set of creation operators for the total Hilbert space \mathcal{H} is a complete set of bases at both past and future infinities, we can write $\$$ as

$$\$ = \mathcal{S}\mathcal{S}^\dagger, \tag{46}$$

where \mathcal{S}^\dagger is the adjoint of the \mathcal{S} -matrix. To see how the density matrix evolves, we have to take the trace of the future density matrix, which is given by

$$\text{Tr}(\rho_+^2) = \text{Tr}((\$ \rho_-)(\$ \rho_-)). \tag{47}$$

Because the supperscattering matrix factorizes, we can write

$$\text{Tr}((\$ \rho_-)(\$ \rho_-)) = \text{Tr}(\mathcal{S}\rho_- \mathcal{S}^\dagger \mathcal{S}\rho_- \mathcal{S}^\dagger), \tag{48}$$

whence

$$\text{Tr}(\mathcal{S}\rho_- \mathcal{S}^\dagger \mathcal{S}\rho_- \mathcal{S}^\dagger) = \text{Tr}(\mathcal{S}\rho_-^2 \mathcal{S}^\dagger) = \text{Tr}(\rho_-^2). \tag{49}$$

This is the trace of the past density matrix.

However, the states of virtual black holes are not measurable, and in reality we therefore have to take the partial trace over \mathcal{H}_{vb} . If I_{vb} is the identity operator on \mathcal{H}_{vb} and A_{phy} is an observable in \mathcal{H}_{phy} , then physically meaningful measurements are given by

$$\text{Tr}(\rho I_{vb} \otimes A_{phy}). \tag{50}$$

Because the states in \mathcal{H}_{phy} do not form a complete set by themselves and hence the superscattering operator does not factorize into the \mathcal{S} -matrix and its adjoint, the evolution for \mathcal{H}_{phy} is nonunitary. This causes a loss of quantum coherence.

The total wave function $|\Psi\rangle$ can be written as a superposition, with the coefficients c_n such that

$$\sum_n |c_n|^2 = 1, \tag{51}$$

of single tensor products

$$|\Psi\rangle = \sum_n c_n |vb_n\rangle \otimes |phy_n\rangle, \tag{52}$$

where the $|vb_n\rangle$ and $|phy_n\rangle$ are orthonormal sets of basis vectors in \mathcal{H}_{vb} and \mathcal{H}_{phy} respectively. Hence,

$$\rho_{vb} = \sum_n |c_n|^2 |vb_n\rangle\langle vb_n|, \tag{53}$$

$$\rho_{phy} = \sum_n |c_n|^2 |phy_n\rangle\langle phy_n|.$$

The von Neumann entropy for the physical system is given by

$$S_{phy} = -\text{Tr}(\rho_{phy} \log \rho_{phy}). \quad (54)$$

We can then write

$$S_{phy} = -\sum_n |c_n|^2 \log |c_n|^2. \quad (55)$$

This is the entropy associated with the physical systems.

This entropy can be used to define a direction of time. To define time-like anything along the traditional lines, one needs a notion of a flow of time, represented by a one-parameter family of unitary operators, which we call a flow $t \mapsto U(t)$ on \mathcal{H} with t ranging over the nonnegative real numbers, mapping any initial density matrix

$$\rho_0 = |\Psi_0\rangle\langle\Psi_0|$$

at some initial instant $t = 0$ into the density matrix

$$\rho_t = |\Psi_t\rangle\langle\Psi_t|$$

at a later instant t according to the transformation

$$\rho_t = U(t)\rho_0U(t)^{-1}. \quad (56)$$

A single microstate at one instant therefore evolves to a single microstate at a later instant. For any initial state, the entanglement between \mathcal{H}_{vb} and \mathcal{H}_{phy} is less than at a later stage. Because the states keep becoming more and more entangled with the passage of time, this entanglement can also be used to identify the direction of time. The value of entropy then also increases uniformly as a state evolves to the future. We can equate the direction of the increase in this von Neumann entropy $S(t)_{phy}$ with time. This can give a solution to the problem of time in quantum gravity.

6. CONCLUSION

In this paper, modified gravity is consistently third quantized in analogy with the quantization of scalar field theory in curved spacetime. Then the virtual black hole model of spacetime foam, which currently seems to be the correct model of spacetime foam, is analyzed in this third-quantized modified gravity. This model is used to give a statistical origin of the Bekenstein–Hawking entropy. It is also shown that the area and hence the entropy of a real black hole is quantized in this model. Furthermore, the loss of quantum coherence occurs because virtual black hole states are not measurable. This in turn causes all physical systems to

acquire an intrinsic entropy. This entropy is used to give a definition of time in quantum gravity.

It would be interesting to analyze many other results that have been discussed for baby universes in the third-quantized formalism in terms of spacetime foam formed by virtual black holes. It might be possible to obtain a different value of the cosmological constant than the one obtained in the model of spacetime foam containing baby universes. In higher dimensions, spacetime is known to possess more exotic topologies like the black rings. It would also be interesting to analyze a model of spacetime containing virtual black rings. The results of this paper can easily be generalized to virtual black rings.

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