ЖЭТФ, 2011, том 140, вып. 2 (8), стр. 236–240

RELATIVISTIC COULOMB GREEN's FUNCTION IN d DIMENSIONS

*R. N. Lee**, *A. I. Milstein***, *I. S. Terekhov****

Budker Institute of Nuclear Physics
630090, Novosibirsk, Russia

Novosibirsk State University
630090, Novosibirsk, Russia

Received January 28, 2011

Using the operator method, we derive the Green's functions of the Dirac and Klein–Gordon equations in the Coulomb potential $-Z\alpha/r$ for an arbitrary space dimensionality d. Nonrelativistic and semiclassical asymptotic forms of these Green's functions are considered in detail.

1. INTRODUCTION

At the calculation of the amplitudes and probabilities of QED processes in the field of heavy atoms applicable, the parameter $Z\alpha$ (where Z is the atomic charge number and α is the fine structure constant) is not small. The effect of higher orders in $Z\alpha$ can change the Born result by several times. Therefore, it is often required to calculate the probabilities of QED processes in such a strong field exactly in $Z\alpha$. The most convenient way to perform this calculation is to use the exact Green's functions of the Dirac equation (or the Klein–Gordon equation) for a charged particle in a field (the Furry representation). Deriving the Green's functions for specific field configurations is very important for applications. For the Coulomb potential, a convenient integral representation of the Green's function $G(\mathbf{r},\mathbf{r}'|\varepsilon)$ was derived in Ref. [1] using the O(2,1) algebra. The representation obtained is valid in the whole complex plane of the energy ε and does not contain contour integrals. Another integral representation for the Green's function in the Coulomb field was derived in Ref. [2] using an explicit form of the expansion of $G(\mathbf{r},\mathbf{r}'|\varepsilon)$ with respect to the eigenfunctions of the corresponding wave equation. The representation of the Green's function obtained in Ref. [2] contains a contour integral, which complicates its use in applications.

In the calculation of loop diagrams, it is often required to regularize the divergent integrals. One of the most convenient methods of the regularization is the dimensional regularization. To use the dimensional regularization within the approach based on the Furry representation, it is necessary to derive the exact Green's function in the Coulomb field in an arbitrary, not necessarily integer, space dimensionality d (the space–time dimensionality is $d+1$). In this paper, we solve this problem by generalizing the Green's function obtained in Ref. [1] for $d=3$ to arbitrary d. Our derivation closely follows the path of derivation in Ref. [1]. In contrast to the conventional approach, the operator method used in Ref. [1] and in this paper does not require the knowledge of the explicit form of the wave functions, which is difficult to define for noninteger d. To fix the explicit form of the Green's function for arbitrary d unambiguously, we use only the commutative and anticommutative relations for the operators and γ-matrices.

2. CALCULATION OF THE GREEN'S FUNCTION

Following Ref. [1], we represent the Green's function in the Coulomb potential

$$U(r) = -Z\alpha/r$$

(the system of units $\hbar = c = 1$ is used),

$$G(\mathbf{r},\mathbf{r}'|\varepsilon) = \frac{1}{\hat{\mathcal{P}} - m + i0}\delta(\mathbf{r}-\mathbf{r}'),\qquad \hat{\mathcal{P}} = \gamma^0(\varepsilon + Z\alpha/r) - \boldsymbol{\gamma}\mathbf{p}, \tag{1}$$

as

*E-mail: r.n.lee@inp.nsk.su
**E-mail: a.i.milstein@inp.nsk.su
***E-mail: i.s.terekhov@inp.nsk.su

$$G(\mathbf{r},\mathbf{r}'|\varepsilon) = (\hat{\mathcal{P}} + m) D(\mathbf{r},\mathbf{r}'|\varepsilon),$$
$$D(\mathbf{r},\mathbf{r}'|\varepsilon) = = -i\int_0^\infty ds \exp\left\{2iZ\alpha\varepsilon s - is\left[rp_r^2 + \kappa^2 r + \frac{K}{r}\right]\right\} \times \times \frac{\delta(r-r')}{r^{d-2}}\delta(\mathbf{n}-\mathbf{n}'), \quad (2)$$
$$\kappa = \sqrt{m^2-\varepsilon^2}, \quad p_r = -\frac{i}{r^{(d-1)/2}}\frac{\partial}{\partial r} r^{(d-1)/2},$$
$$\mathbf{n} = \mathbf{r}/r, \quad \mathbf{n}' = \mathbf{r}'/r',$$
$$K = \mathbf{l}^2 - iZ\alpha\boldsymbol{\alpha}\cdot\mathbf{n} - (Z\alpha)^2 + \frac{1}{4}(d-1)(d-3),$$
$$\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma},$$

where $-\mathbf{l}^2$ is the angular part of the Laplacian determined by

$$\Delta = \frac{1}{r^{d-1}}\partial_r r^{d-1}\partial_r - \frac{1}{r^2}\mathbf{l}^2 \quad (3)$$

and the γ-matrices obey the usual relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}.$$

We then represent the angular part of the δ-function as

$$\delta(\mathbf{n}-\mathbf{n}') = \sum_\lambda P_\lambda(\mathbf{n},\mathbf{n}'), \quad (4)$$

where the projection operators $P_\lambda(\mathbf{n},\mathbf{n}')$ satisfy the relations

$$KP_\lambda(\mathbf{n},\mathbf{n}') = \lambda(\lambda+1)P_\lambda(\mathbf{n},\mathbf{n}'),$$
$$\int d\mathbf{n}' P_\lambda(\mathbf{n},\mathbf{n}')P_{\lambda'}(\mathbf{n}',\mathbf{n}'') = \delta_{\lambda\lambda'}P_\lambda(\mathbf{n},\mathbf{n}''). \quad (5)$$

Because the operator K contains only one matrix operator $\boldsymbol{\alpha}\cdot\mathbf{n}$, the matrix structure of the projection operator $P_\lambda(\mathbf{n},\mathbf{n}')$ is given by the linear combination of the unit matrix I and matrices $\boldsymbol{\alpha}\cdot\mathbf{n}$, $\boldsymbol{\alpha}\cdot\mathbf{n}'$, and $(\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')$. All other matrices, such as $(\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')(\boldsymbol{\alpha}\cdot\mathbf{n})$, can be reduced to the four above matrices using the anticommutation relations. Taking this property into account, we seek the projection operators $P_\lambda(\mathbf{n},\mathbf{n}')$ in the form

$$P_\lambda(\mathbf{n},\mathbf{n}') = a_1\Lambda_+(\mathbf{n})\Lambda_+(\mathbf{n}') + a_2\Lambda_+(\mathbf{n})\Lambda_-(\mathbf{n}') + + a_3\Lambda_-(\mathbf{n})\Lambda_+(\mathbf{n}') + a_4\Lambda_-(\mathbf{n})\Lambda_-(\mathbf{n}'),$$
$$\Lambda_\pm(\mathbf{n}) = \frac{1}{2}(1\pm\boldsymbol{\alpha}\cdot\mathbf{n}), \quad (6)$$

where a_i are some functions of $x = \mathbf{n}\cdot\mathbf{n}'$. From Eqs. (5), we obtain

$$a_1 = \beta(\lambda + iZ\alpha)B_n(x),$$
$$a_2 = a_3 = \beta\,(n+\nu+1/2)\,A_n(x),$$
$$a_4 = \beta(\lambda - iZ\alpha)B_n(x),$$
$$\lambda = \pm\gamma, \quad \gamma = \sqrt{(n+\nu+1/2)^2 - (Z\alpha)^2},$$
$$\beta = \frac{\Gamma(\nu+1)}{2\lambda\pi^{\nu+1}}, \quad (7)$$
$$A_n(x) = \frac{1}{2\nu}\frac{\partial}{\partial x}[C_{n+1}^\nu(x) + C_n^\nu(x)],$$
$$B_n(x) = \frac{1}{2\nu}\frac{\partial}{\partial x}[C_{n+1}^\nu(x) - C_n^\nu(x)],$$
$$\nu = \frac{d}{2} - 1,$$

where $C_n^\nu(x)$ is the Gegenbauer polynomial, and $n = 0, 1, 2, \ldots$ is an integer. This integer appears from the requirement that the functions a_i have no singularities at $x = 1$. The result for a_i in (7) was obtained using the identity

$$\int (1 + \mathbf{n}\cdot\mathbf{n}' + \mathbf{n}\cdot\mathbf{n}'' + \mathbf{n}'\cdot\mathbf{n}'') \times \times B_n(\mathbf{n}\cdot\mathbf{n}')B_n(\mathbf{n}'\cdot\mathbf{n}'')\,d\mathbf{n}' = = \Omega_d(1+\mathbf{n}\cdot\mathbf{n}'')B_n(\mathbf{n}\cdot\mathbf{n}''), \quad (8)$$
$$\Omega_d = \int d\mathbf{n} = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{2\pi^{\nu+1}}{\Gamma(\nu+1)}.$$

We finally obtain for projection operator

$$P_\lambda(\mathbf{n},\mathbf{n}') = = \frac{\beta}{2}\left\{\Big[\lambda[1+(\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')] + iZ\alpha(\boldsymbol{\alpha}\cdot\mathbf{n}+\boldsymbol{\alpha}\cdot\mathbf{n}')\Big]B_n(x) + \right. \left. + (n+\nu+1/2)\,[1-(\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')]A_n(x)\right\}. \quad (9)$$

For $d = 3$, this projection operator coincides with that found in Ref. [3].

We note that the functions $A_n(x)$ and $B_n(x)$ have a nonsingular limit as $\nu\to 0$ (or $d\to 2$),

$$\lim_{\nu\to 0} A_n(x) = \frac{\sin((n+1)\phi) + \sin(n\phi)}{\sin\phi},$$

$$\lim_{\nu\to 0} B_n(x) = \frac{\sin((n+1)\phi) - \sin(n\phi)}{\sin\phi},$$

where $\phi = \arccos x$.

To complete the calculation of $D(\mathbf{r},\mathbf{r}'|\varepsilon)$ in Eq. (2), it is necessary to find the result of the action of the operator

$$\exp\left\{-is\left[rp_r^2 + \kappa^2 r + \lambda(\lambda+1)/r\right]\right\}$$

on the function

$$\delta(r-r')/r^{2\nu}.$$

This can be done exactly in the same way as in Ref. [1]. The method in Ref. [1] is based on the commutation relations of the operators

$$T_1 = \frac{1}{2}[rp_r^2 + \lambda(\lambda+1)/r], \quad T_2 = rp_r, \quad T_3 = r,$$

which coincide with those of the O(2,1) algebra generators (some other examples of applying the O(2,1) algebra in a Coulomb field can be found in Refs. [4, 5]). The only difference between the case of arbitrary d and $d=3$ is the value of the parameter δ in the equation

$$T_1 r^\delta = 0.$$

For arbitrary d, we have

$$\delta = \begin{cases} \lambda + \dfrac{3-d}{2}, & \text{for } \lambda > 0, \\ |\lambda| + \dfrac{1-d}{2}, & \text{for } \lambda < 0. \end{cases} \tag{10}$$

The final result for the function $D(\mathbf{r},\mathbf{r}'|\varepsilon)$ in Eq. (2) is

$$\begin{aligned} D(\mathbf{r},\mathbf{r}'|\varepsilon) = -\frac{i\Gamma(\nu+1)}{2\pi^{\nu+1}(rr')^{\nu+1/2}} \times \\ \times \sum_{n=0}^{\infty}\int_0^{\infty} ds \exp[2iZ\alpha\varepsilon s + i\kappa(r+r')\operatorname{ctg}(\kappa s) - i\pi\gamma] \times \\ \times \left\{ \frac{y}{2} J'_{2\gamma}(y)[1 + (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')]B_n(x) + \right. \\ + iZ\alpha J_{2\gamma}(y)(\boldsymbol{\alpha}\cdot\mathbf{n} + \boldsymbol{\alpha}\cdot\mathbf{n}')B_n(x) + \\ + (n+\nu+1/2)\, J_{2\gamma}(y) \times \\ \left. \times [1 - (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')]A_n(x) \right\}, \quad y = \frac{2\kappa\sqrt{rr'}}{\sin(\kappa s)}, \end{aligned} \tag{11}$$

where $J_{2\gamma}(y)$ is the Bessel function and $A_n(x)$, $B_n(x)$, ν, and γ are defined in Eq. (7). The corresponding result for the Coulomb Green's function of the Dirac equation in d spatial dimension is

$$\begin{aligned} G(\mathbf{r},\mathbf{r}'|\varepsilon) = -\frac{i\Gamma(\nu+1)}{2\pi^{\nu+1}(rr')^{\nu+1/2}} \times \\ \times \sum_{n=0}^{\infty}\int_0^{\infty} ds \exp[2iZ\alpha\varepsilon s + \\ + i\kappa(r+r')\operatorname{ctg}(\kappa s) - i\pi\gamma]\,\mathcal{T}, \\ \mathcal{T} = [1 + (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')] \times \\ \times \Big[\frac{y}{2} J'_{2\gamma}(y)(\gamma^0\varepsilon + m) - \\ - iZ\alpha J_{2\gamma}(y)\gamma^0\kappa\operatorname{ctg}(\kappa s)\Big] B_n(x) + \\ + \Big[[1 - (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')](\gamma^0\varepsilon + m) - \\ - \kappa\operatorname{ctg}(\kappa s)(\boldsymbol{\gamma}\cdot\mathbf{n} - \boldsymbol{\gamma}\cdot\mathbf{n}')\Big] \times \\ \times J_{2\gamma}(y)\,(n+\nu+1/2)\,A_n(x) + \\ + \left[\frac{i\kappa^2(r-r')}{2\sin^2(\kappa s)} + imZ\alpha\gamma^0\right] \times \\ \times (\boldsymbol{\gamma}\cdot\mathbf{n} + \boldsymbol{\gamma}\cdot\mathbf{n}')J_{2\gamma}(y)B_n(x). \end{aligned} \tag{12}$$

For $d=3$, this result coincides with the corresponding result in Ref. [1]. The function $G(\mathbf{r},\mathbf{r}'|\,\varepsilon)$ has cuts in the complex plane ε along the real axis from $-\infty$ to $-m$ and from m to ∞, which correspond to the continuous spectrum, and also has simple poles in the interval $(0,m)$ for an attractive field and in the interval $(-m,0)$ for a repulsive field. Integral representation (12) is valid for any ε that belongs to the domain $\operatorname{Re}\varepsilon<0$, $\operatorname{Im}\varepsilon<0$ or $\operatorname{Re}\varepsilon>0$, $\operatorname{Im}\varepsilon>0$. If $\operatorname{Re}\varepsilon<0$, $\operatorname{Im}\varepsilon>0$ or $\operatorname{Re}\varepsilon>0$, $\operatorname{Im}\varepsilon<0$, then the integration over s must be performed in Eq. (12) from zero to $-\infty$.

For real ε in the interval $-m<\varepsilon<m$, we obtain (cf. Ref. [1])

$$\begin{aligned} G(\mathbf{r},\mathbf{r}'|\varepsilon) = \\ = \frac{\Gamma(\nu+1)}{4\kappa\sin[\pi(Z\alpha\varepsilon/\kappa - \gamma)]\pi^{\nu+1}(rr')^{\nu+1/2}} \times \\ \times \sum_{n=0}^{\infty}\int_{-\pi/2}^{\pi/2} ds \exp[-2iZ\alpha\varepsilon s/\kappa + \\ + i\kappa(r+r')\operatorname{tg} s]\,\mathcal{T}, \\ \mathcal{T} = [1 + (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')] \times \\ \times \Big[\frac{v}{2} J'_{2\gamma}(v)(\gamma^0\varepsilon + m) - \\ - iZ\alpha J_{2\gamma}(v)\gamma^0\kappa\operatorname{tg} s\Big] B_n(x) + \\ + \Big[[1 - (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')](\gamma^0\varepsilon + m) - \\ - \kappa\operatorname{tg} s(\boldsymbol{\gamma}\cdot\mathbf{n} - \boldsymbol{\gamma}\cdot\mathbf{n}')\Big] \times \\ \times J_{2\gamma}(v)\,(n+\nu+1/2)\,A_n(x) + \\ + \left[\frac{i\kappa^2(r-r')}{2\cos^2 s} + imZ\alpha\gamma^0\right] \times \\ \times (\boldsymbol{\gamma}\cdot\mathbf{n} + \boldsymbol{\gamma}\cdot\mathbf{n}')J_{2\gamma}(v)B_n(x), \quad v = \frac{2\kappa\sqrt{rr'}}{\cos s}. \end{aligned} \tag{13}$$

The denominator in Eq. (13) vanishes at the points

$$Z\alpha\varepsilon/\kappa - \gamma = k$$

for any integer k. But the integral over s also vanishes at these points with negative k (see Ref. [1]), and hence expression (13) has poles only for $k = 0, 1, 2 \ldots$.

Taking into account that γ is positive, we find that the simple poles corresponding to the discrete spectrum are at the points

$$\varepsilon = \frac{m\,\mathrm{sign}Z}{\sqrt{1+\left(\dfrac{Z\alpha}{k+\gamma}\right)^2}}. \qquad (14)$$

The maximal value of Z when all the results obtained are applicable is defined by the relation

$$(Z\alpha)_{max} = \frac{d-1}{2}$$

(see the definition of γ in Eq. (7)).

For completeness, we also present the final result for the Coulomb Green's function of the Klein–Gordon equation,

$$G_0(\mathbf{r},\mathbf{r}'|\varepsilon) = -\frac{\Gamma(\nu+1)}{2\pi^{\nu+1}(rr')^\nu}\sum_{n=0}^{\infty}\frac{n+\nu}{\nu}C_n^\nu(x)\times \\ \times\int_0^\infty \frac{\kappa\,ds}{\sin(\kappa s)}\exp[2iZ\alpha\varepsilon s + \\ + i\kappa(r+r')\,\mathrm{ctg}(\kappa s) - i\pi\mu]J_{2\mu}(y), \\ \mu = \sqrt{(n+\nu)^2-(Z\alpha)^2}. \qquad (15)$$

We note that there is no singularity in this formula at $d=2$ because

$$\lim_{\nu\to 0}\frac{n+\nu}{\nu}C_n^\nu(x) = \cos(n\phi).$$

3. ASYMPTOTIC FORMS

We derive the Coulomb Green's function $G_{nr}(\mathbf{r},\mathbf{r}'|E)$ of the Schrödinger equation in d spatial dimensions. For this, we calculate the nonrelativistic asymptotic form of the Coulomb Green's function of the Klein–Gordon equation, applicable at $|E|\ll m$ and $Z\alpha\ll 1$, where $E=\varepsilon-m$. Neglecting $(Z\alpha)^2$ in μ, using the summation formula (cf. [2]),

$$S_0 = \sum_{n=0}^{\infty}(-1)^n\frac{\nu+n}{\nu}C_n^\nu(x)J_{2(n+\nu)}(y) = \\ = \frac{\sqrt{\pi}\,y^{2\nu}J_{\nu-1/2}(w)}{2^{3\nu+1/2}\Gamma(\nu+1)w^{\nu-1/2}}, \quad w = y\sqrt{\frac{1+x}{2}}, \qquad (16)$$

and multiplying by $2m$, we obtain

$$G_{nr}(\mathbf{r},\mathbf{r}'|E) = -\frac{m}{(2\pi)^{\nu+1/2}}\int_0^\infty ds\left(\frac{\kappa}{\sin(\kappa s)}\right)^{2\nu+1}\times \\ \times\exp[2iZ\alpha ms+i\kappa(r+r')\,\mathrm{ctg}(\kappa s)-i\pi\nu]\frac{J_{\nu-1/2}(w)}{w^{\nu-1/2}}, \\ \kappa=\sqrt{-2mE}. \qquad (17)$$

This formula is in agreement with the corresponding result in Ref. [6].

At high energies and small scattering angles of the particles, the characteristic angular momenta are large and the semiclassical approximation is applicable. The semiclassical Green's function of the Dirac equation in a Coulomb potential for $d=3$ was first derived in Refs. [7, 8]. Another representation of this function was obtained in Refs. [9, 10]. The semiclassical Green's function for an arbitrary spherically symmetric localized potential was found in Refs. [11, 12]. In Ref. [13], the semiclassical Green's function for an arbitrary localized potential was found with the next-to-leading semiclassical correction taken into account. In Ref. [13], the spherical symmetry of the potential was not required.

We consider the semiclassical Green's function of the Dirac equation in a Coulomb potential for an arbitrary spatial dimension d. In this case, $\varepsilon\gg m$ and $1+x\ll 1$, and hence the leading contribution to the sum over n in Eq. (12) is given by $n\gg 1$. We can therefore neglect the term $(Z\alpha)^2$ in γ, Eq. (7), and sum over n analytically. We need to calculate two sums,

$$S_A = \sum_{n=0}^{\infty}(-1)^n(\nu+n+1/2)\times \\ \times A_n(x)J_{2(n+\nu+1/2)}(y), \qquad (18) \\ S_B = \sum_{n=0}^{\infty}(-1)^n B_n(x)J_{2(n+\nu+1/2)}(y),$$

where the functions $A_n(x)$ and $B_n(x)$ are defined in Eq. (7). Using the recurrent relations for the Bessel functions and the Gegenbauer polynomials, it is easy to show that

$$S_A = (1+x)\frac{\partial}{\partial x}S_B + (\nu+1/2)S_B, \\ S_B = -\frac{2}{y}\frac{\partial}{\partial x}S_0, \qquad (19)$$

so that

$$S_A = \frac{\sqrt{\pi}\,y^{2\nu+1}J_{\nu-1/2}(w)}{2^{3\nu+5/2}\Gamma(\nu+1)w^{\nu-1/2}}, \\ S_B = \frac{\sqrt{\pi}\,y^{2\nu+1}J_{\nu+1/2}(w)}{2^{3\nu+3/2}\Gamma(\nu+1)w^{\nu+1/2}}. \qquad (20)$$

Substituting these results in Eq. (12), we arrive at the final expression for the semiclassical Green's function

$$\begin{gathered}G_{qc}(\mathbf{r},\mathbf{r}'|\varepsilon) = -\frac{1}{2^{\nu+3/2}\pi^{\nu+1/2}} \times \\ \times \int\limits_0^\infty \frac{ds}{u^{\nu-1/2}}\left(\frac{p}{\operatorname{sh}(ps)}\right)^{2\nu+1} \times \\ \times \exp[2iZ\alpha\varepsilon s + ip(r+r')\operatorname{cth}(ps) - i\pi\nu]\,\mathcal{M}, \\ \mathcal{M} = J_{\nu-1/2}(u) \times \\ \times\left[\gamma^0\varepsilon + m - \frac{p}{2}\operatorname{cth}(ps)(\boldsymbol{\gamma}\cdot\mathbf{n} - \boldsymbol{\gamma}\cdot\mathbf{n}')\right] + \\ + i\frac{J_{\nu+1/2}(u)}{u}\left\{\left[\frac{p^2(r-r')}{2\operatorname{sh}^2(ps)} + mZ\alpha\gamma^0\right] \times \right. \\ \times(\boldsymbol{\gamma}\cdot\mathbf{n} + \boldsymbol{\gamma}\cdot\mathbf{n}') - Z\alpha\gamma^0 p\operatorname{cth}(ps) \times \\ \left. \times[1 + (\boldsymbol{\alpha}\cdot\mathbf{n})(\boldsymbol{\alpha}\cdot\mathbf{n}')]\right\}, \quad u = \frac{p\sqrt{2rr'(1+x)}}{\operatorname{sh}(ps)},\end{gathered} \tag{21}$$

where

$$p = \sqrt{\varepsilon^2 - m^2} = i\kappa.$$

For $d = 3$, the result in (21) agrees with that obtained in Refs. [7, 8]. The term $(Z\alpha)^2$ in γ can also be neglected in the nonrelativistic approximation when $Z\alpha \ll 1$, $p \ll m$, and $Z\alpha m/p$ is fixed. In this case, we immediately obtain from Eq. (21) that the nonrelativistic approximation for the Green's function of the Dirac equation is given by

$$G(\mathbf{r},\mathbf{r}'|m+E) = \frac{\gamma^0+1}{2}G_{nr}(\mathbf{r},\mathbf{r}'|E), \tag{22}$$

where $G_{nr}(\mathbf{r},\mathbf{r}'|\varepsilon)$ is defined in Eq. (17).

To summarize, in d spatial dimensions, we have calculated the Green's functions of the Dirac and Klein–Gordon equations in the Coulomb field (Eqs. (12) and (15)). Nonrelativistic and semiclassical limit cases of these Green's functions are considered in detail. The results obtained can be applied for calculation of various QED amplitudes in the strong Coulomb field with the use of dimensional regularization.

This work was supported in part by the RFBR (Grant № 09-02-00024) and the Grant 14.740.11.0082 of the Federal Program "Personnel of Innovational Russia". I. S. T. was also supported by the "Dynasty" Foundation.

REFERENCES

1. A. I. Milstein and V. M. Strakhovenko, Phys. Lett. A **90**, 447 (1982).
2. L. Hostler, J. Math. Phys. **5**, 591 (1964).
3. P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958).
4. Y. Nambu, in: *Proc. 1967 Intern. Conf. Particles and Fields*, New York (1967).
5. V. F. Dmitriev and Yu. B. Rumer, Theor. Math. Phys. **5**, 276 (1970) (in Russian).
6. L. C. Hostler, J. Math. Phys. **11**, 2966 (1970).
7. A. I. Milstein and V. M. Strakhovenko, Phys. Lett. A **95**, 135 (1983).
8. A. I. Milstein and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **85**, 14 (1983) [JETP **58**, 8 (1983)].
9. R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **112**, 1921 (1997) [JETP **85**, 1049 (1997)].
10. R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, Phys. Rev. A **57**, 2325 (1998).
11. R. N. Lee and A. I. Milstein, Phys. Lett. A **198**, 217 (1995).
12. R. N. Lee and A. I. Milstein, Zh. Eksp. Teor. Fiz. **107**, 1393 (1995) [JETP **80**, 777 (1995)].
13. R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **117**, 75 (2000) [JETP **90**, 66 (2000)].