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FREE-FIELD REPRESENTATIONS AND GEOMETRY OF SOME GEPNER MODELS

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The geometry of the k^K Gepner model, where $k+2=2K$, is investigated by a free-field representation known as the "$bc\beta\gamma$" system. Using this representation, we directly show that the internal sector of the model is given by Landau–Ginzburg $\mathbb{C}^K/\mathbb{Z}_{2K}$ orbifold. Then we consider the deformation of the orbifold by a marginal antichiral–chiral operator. Analyzing the chiral de Rham complex structure in the holomorphic sector, we show that it coincides with chiral de Rham complex of some toric manifold, where toric data are given by certain fermionic screening currents. This allows relating the Gepner model deformed by the marginal operator to a σ-model on the CY manifold realized as a double cover of $\mathbb{P}^{K-1}$ with ramification along a certain submanifold.

1. INTRODUCTION

Geometric aspects underlying purely algebraic, conformal field theory (CFT) construction of the superstring vacua by Gepner [1] are an important and interesting area of study. It has two decades history of research with a number of remarkable results. For example, the relationship between σ-models on Calabi–Yau (CY) manifolds and Gepner models has been clarified essentially (see [2] for the review and references to the original papers).

However, the question of how to directly relate the σ-model geometry to the algebraic data of Gepner's construction (and when this is possible) is still open.

In the important work of Borisov [3], the vertex operator algebra endowed with an $N=2$ Virasoro superalgebra action has been constructed for each pair of dual reflexive polytopes defining a toric CY manifold. Borisov thus directly constructed the holomorphic CFT sector from toric data of the CY manifold. His approach is based essentially on the important work by Malikov, Schechtman and Vaintrob [4], where a certain sheaf of vertex algebras called the chiral de Rham complex was introduced. Roughly speaking, the construction in [4] is a kind of free-field representation known as the "$bc\beta\gamma$" system, which is in the case of Gepner models is closely related to the Feigin and Semikhatov free-field representation [7] of $N=2$ supersymmetric minimal models. This circumstance is probably the key for understanding the string geometry of Gepner models and their relationship to σ-models on toric CY manifolds.

A significant step in this direction has been made in paper [5], where the vertex algebra of a certain Landau–Ginzburg (LG) orbifold was related to the chiral de Rham complex of a toric CY manifold by a spectral sequence. The CY manifold was realized as an algebraic surface of degree K in the projective space $\mathbb{P}^{K-1}$; one of the key points in [5] is that the free-field representation of the corresponding LG orbifold is given by K copies of the $N=2$ minimal model free-field representation in [7].

The Gepner model can be characterized by a K-dimensional vector

$$\boldsymbol{\mu}=(\mu_1,\ldots,\mu_K), \tag{1}$$

where

$$\mu_i=2,3,\ldots,\quad i=1,\ldots,K, \tag{2}$$

define central charges of the individual $N=2$ minimal models

$$c_i=3\left(1-\frac{2}{\mu_i}\right). \tag{3}$$

In what follows, the $\boldsymbol{\mu}$ is specified as

$$\boldsymbol{\mu}=(\mu,\mu,\ldots,\mu) \tag{4}$$

and hence the total central charge of the model is

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$$c = \sum_{i=1}^{K} c_i = 3K\left(1 - \frac{2}{\mu}\right). \qquad (5)$$

There are two cases where the central charge is integer and a multiple of 3:

$$\mu = K,\ 2K. \qquad (6)$$

The geometry underlying the first case was investigated in [5].

In the second case the geometry is more interesting. The total central charge is

$$c = 3(K-1), \qquad (7)$$

and hence the complex dimension of the compact manifold is $K-1$. We show in this paper that the internal geometry of the Gepner model corresponds in this case to the σ-model on the CY manifold that is a double cover of $\mathbb{P}^{K-1}$ with ramification along a certain submanifold. This means, in particular, that the center of mass of the string is allowed to move only along the base $\mathbb{P}^{K-1}$, but some twisted sectors are added along the fiber of the double cover.

We can generalize the second case and consider the models where

$$\mu = 3K, 4K, \ldots \qquad (8)$$

Although the total central charge is no longer integer and these models cannot be used as models of superstring compactification, the orbifold projection consistent with modular invariance still exits [6], which makes these $N=2$ supersymmetic CFT models interesting from the geometric standpoint. The geometry of these models has been partly investigated in [8].

In Sec. 2, we collect the known facts on the $N=2$ minimal models, fix the notation, and briefly recall Gepner's construction of the partition function in the internal sector of the Gepner model. In Sec. 3, the free-field representation in [7] is used to relate the model to the LG $\mathbb{C}^K/\mathbb{Z}_{2K}$-orbifold. In Sec. 4, a resolution of the orbifold singularity in the chiral sector is considered. It is given by adding some new fermionic screening charge coming from the twisted sector of the Gepner model. We show that this additional screening charge together with the old charges define the toric data of the total space of an $O(K)$ bundle over $\mathbb{P}^{K-1}$, as well as the potential on this space. The chiral sector space of states of the model has the structure of the chiral de Rham complex on the $O(K)$ bundle total space restricted to zeroes of the gradient of the potential. Then we consider the rest of the orbifold group action on the space of states and relate the model to a σ-model on the CY manifold that is a double cover of the projective space $\mathbb{P}^{K-1}$.

2. THE INTERNAL SECTOR PARTITION FUNCTION OF GEPNER MODELS

In this section, we recall the construction of the partition function of the Gepner model in the internal sector. To be more specific, the Ramond–Ramond (RR) partition function of the internal sector is important for investigating the geometry. But as a preliminary step, we collect some known facts about the $N=2$ minimal models and fix the notation.

2.1. Products of $N=2$ minimal models

The tensor product of K $N=2$ unitary minimal models can be characterized by a K-dimensional vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$, where $\mu_i \geq 2$ are integers defining the central charge of the model as

$$c_i = 3\left(1 - \frac{2}{\mu_i}\right).$$

For each individual minimal model, we let $M_{h,t}$ denote the irreducible unitary $N=2$ Virasoro superalgebra representation in the Neveu–Schwartz (NS) sector and $\chi_{h,t}(q,u)$ denote the character of the representation,

$$\chi_{h,t}(q,u) = \mathrm{Tr}_{h,t}(q^{L[0]-\frac{c}{24}} u^{J[0]}), \qquad (9)$$

where $h = 0, \ldots, \mu-2$ and $t = 0, \ldots, h$. There are the following important automorphisms of the irreducible modules and characters [7, 9]:

$$\begin{aligned} M_{h,t} &\equiv M_{\mu-h-2,t-h-1}, \\ \chi_{h,t}(q,u) &= \chi_{\mu-h-2,t-h-1}(q,u), \end{aligned} \qquad (10)$$

$$M_{h,t} \equiv M_{h,t+\mu}, \quad \chi_{h,t+\mu}(q,u) = \chi_{h,t}(q,u), \qquad (11)$$

where μ is odd, and

$$\begin{aligned} &M_{h,t} \equiv M_{h,t+\mu}, \quad \chi_{h,t+\mu}(q,u) = \chi_{h,t}(q,u), \quad h \neq \frac{\mu}{2}-1, \\ &M_{h,t} \equiv M_{h,t+\frac{\mu}{2}}, \quad \chi_{h,t+\frac{\mu}{2}}(q,u) = \chi_{h,t}(q,u), \quad h = \frac{\mu}{2}-1, \end{aligned} \qquad (12)$$

where μ is even. In what follows, we extend the set of admissible t:

$$t = 0, \ldots, \mu-1 \qquad (13)$$

using the automorphisms above.

The parameter $t \in \mathbb{Z}$ labels the spectral flow automorphisms [10] of the $N = 2$ Virasoro superalgebra in the NS sector,

$$\begin{gathered} G^{\pm}[r] \to G^{\pm}_t[r] \equiv U^t G^{\pm}[r] U^{-t} \equiv G^{\pm}[r \pm t], \\ L[n] \to L_t[n] \equiv U^t L[n] U^{-t} \equiv \\ \equiv L[n] + tJ[n] + t^2 \frac{c}{6}\delta_{n,0}, \\ J[n] \to J_t[n] \equiv U^t J[n] U^{-t} \equiv J[n] + t\frac{c}{3}\delta_{n,0}, \end{gathered} \tag{14}$$

where U^t denotes the spectral flow operator generating twisted sectors. Here, r is half-integer for the modes of the spin-3/2 fermionic currents $G^{\pm}(z)$, and n is integer for the modes of the stress-energy tensor $T(z)$ and the $U(1)$ current $J(z)$ of the $N = 2$ Virasoro superalgebra. Thus, allowing t to be half-integer, we recover the irreducible representations and characters in the R sector.

The $N = 2$ Virasoro superalgebra generators in the product of minimal models are given by the sums of generators of each minimal model,

$$\begin{gathered} G^{\pm}[r] = \sum_i G^{\pm}_i[r], \\ J[n] = \sum_i J_i[n], \quad T[n] = \sum_i T_i[n], \\ c = \sum_i 3\left(1 - \frac{2}{\mu_i}\right). \end{gathered} \tag{15}$$

This algebra obviously acts in the tensor product

$$M_{\mathbf{h},\mathbf{t}} = \otimes_{i=1}^K M_{h_i,t_i}$$

of the irreducible $N = 2$ Virasoro superalgebra representations of each individual model. We use a similar notation for the corresponding product of characters:

$$\chi_{\mathbf{h},\mathbf{t}}(q,u) = \prod_{i=1}^K \chi_{h_i,t_i}(q,u). \tag{16}$$

2.2. Partition function of the internal sector

In what follows, the characters with a fermionic number operator insertion play an important role:

$$\tilde{\chi}_{h_i,t_i}(q,u) = \mathrm{Tr}_{h_i,t_i}((-1)^F q^{L_i[0] - \frac{c_i}{24}} u^{J_i[0]}). \tag{17}$$

The internal sector partition function of the Gepner model in the RR sector is given by

$$\begin{gathered} Z(q,\bar{q},u,\bar{u}) = \frac{1}{2K2^K} \sum_{n,m}^{2K-1} \prod_{i=1}^K \times \\ \times \sum_{h_i,t_i} \tilde{\chi}_{h_i,t_i+n+\frac{1}{2}}(\tau, \upsilon + m) \tilde{\chi}^*_{h_i,+t_i+n+\frac{1}{2}}(\tau,\upsilon) \end{gathered} \tag{18}$$

where

$$q = \exp[i2\pi\tau], \quad u = \exp[i2\pi\upsilon],$$

and $*$ denotes complex conjugation. The summation over n is due to the spectral-flow-twisted sectors generated by the product of spectral flow operators $\prod_{i=1}^K U_i^n$. The summation over m corresponds to the projection on $\mathbb{Z}_{2K}$-invariant states with respect to the operator $\exp[i2\pi J[0]]$. Therefore, (18) is the Z_{2K}-orbifold partition function in the RR sector with a periodic spin structure along both cycles of the torus.

3. FREE-FIELD REPRESENTATIONS AND THE LG ORBIFOLD GEOMETRY OF GEPNER MODELS

In this section, we relate the Gepner models to the LG orbifolds $\mathbb{C}^K/\mathbb{Z}_{2K}$ essentially using the free-field construction of irreducible representations of the $N = 2$ minimal models found in [7].

3.1. Free-field realization of $N = 2$ minimal models

Let $X(z)$ and $X^*(z)$ be free bosonic fields, and $\psi(z)$ and $\psi^*(z)$ be free fermionic fields (in the left-moving sector) with the OPEs given by

$$\begin{gathered} X^*(z_1)X(z_2) = \ln(z_{12}) + \text{reg.}, \\ \psi^*(z_1)\psi(z_2) = z_{12}^{-1} + \text{reg.}, \end{gathered} \tag{19}$$

where $z_{12} = z_1 - z_2$. For an arbitrary number μ, the currents of the $N = 2$ super-Virasoro algebra are given by

$$\begin{aligned} &G^+(z) = \psi^*(z)\partial X(z) - \frac{1}{\mu}\partial\psi^*(z), \\ &G^-(z) = \psi(z)\partial X^*(z) - \partial\psi(z), \\ &J(z) = \psi^*(z)\psi(z) + \frac{1}{\mu}\partial X^*(z) - \partial X(z), \\ &T(z) = \partial X(z)\partial X^*(z) + \frac{1}{2}(\partial\psi^*(z)\psi(z) - \\ &\quad - \psi^*(z)\partial\psi(z)) - \frac{1}{2}(\partial^2 X(z) + \frac{1}{\mu}\partial^2 X^*(z)), \end{aligned} \tag{20}$$

and the central charge is

$$c = 3\left(1 - \frac{2}{\mu}\right). \tag{21}$$

As usual, the fermions are expanded into half-integer modes in the NS sector and into integer modes in the R sector,

$$\begin{aligned}
\psi(z) &= \sum_r \psi[r] z^{-\frac{1}{2}-r}, \\
\psi^*(z) &= \sum_r \psi^*[r] z^{-\frac{1}{2}-r}, \\
G^{\pm}(z) &= \sum_r G^{\pm}[r] z^{-\frac{3}{2}-r}.
\end{aligned} \tag{22}$$

The bosons are expanded into the integer modes in both sectors:

$$\begin{aligned}
\partial X(z) &= \sum_{n\in Z} X[n] z^{-1-n}, \\
\partial X^*(z) &= \sum_{n\in Z} X^*[n] z^{-1-n}, \\
J(z) &= \sum_{n\in Z} J[n] z^{-1-n}, \\
T(z) &= \sum_{n\in Z} L[n] z^{-2-n}.
\end{aligned} \tag{23}$$

In the NS sector, the $N=2$ Virasoro superalgebra acts naturally in the Fock module F_{p,p^*} generated by the fermionic operators $\psi^*[r]$, $\psi[r]$, $r<1/2$, and by the bosonic operators $X^*[n]$, $X[n]$, $n<0$ from the vacuum state $|p,p^*\rangle$, such that

$$\begin{gathered}
\psi[r]|p,p^*\rangle = \psi^*[r]|p,p^* >= 0, r \geq \frac{1}{2}, \\
X[n]|p,p^*\rangle = X^*[n]|p,p^*\rangle = 0, n \geq 1, \\
X[0]|p,p^*\rangle = p|p,p^*\rangle, \quad X^*[0]|p,p^*\rangle = p^*|p,p^*\rangle.
\end{gathered} \tag{24}$$

The $|p,p^*\rangle$ state is a primary state with respect to the $N=2$ Virasoro algebra,

$$\begin{aligned}
G^{\pm}[r]|p,p^*\rangle &= 0, \quad r>0, \\
J[n]|p,p^*\rangle = L[n]|p,p^*\rangle &= 0, \quad n>0, \\
J[0]|p,p^*\rangle &= \frac{j}{\mu}|p,p^*\rangle = 0, \\
L[0]|p,p^*\rangle &= \frac{h(h+2)-j^2}{4\mu}|p,p^*\rangle = 0,
\end{aligned} \tag{25}$$

where $j = p^* - \mu p$ and $h = p^* + \mu p$.

If $\mu - 2$ is integer and nonnegative, the Fock module is a highly reducible representation of the $N=2$ Virasoro algebra.

The irreducible module $M_{h,j}$ is given by the cohomology of some complex built up from Fock modules. This complex was constructed in [7]. We first consider the free-field construction for the chiral module $M_{h,0}$. In this case, the complex (which is known due to Feigin and Semikhatov as the butterfly resolution) can be represented by the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \\
 & & \uparrow & & \uparrow & & & & \\
\ldots & \leftarrow & F_{1,h+\mu} & \leftarrow & F_{0,h+\mu} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
\ldots & \leftarrow & F_{1,h} & \leftarrow & F_{0,h} & & & & \\
 & & & & & \nwarrow & & & \\
 & & & & & & F_{-1,h-\mu} \leftarrow & F_{-2,h-\mu} & \leftarrow \ldots \\
 & & & & & & \uparrow & \uparrow & \\
 & & & & & & F_{-1,h-2\mu} \leftarrow & F_{-2,h-2\mu} & \leftarrow \ldots \\
 & & & & & & \uparrow & \uparrow & \\
 & & & & & & \vdots & \vdots &
\end{array} \tag{26}$$

The horizontal arrows in this diagram are given by the action of

$$Q^+ = \oint dz\, S^+(z), \quad S^+(z) = \psi^* \exp(X^*)(z). \tag{27}$$

The vertical arrows are given by the action of

$$Q^- = \oint dz\, S^-(z), \quad S^-(z) = \psi \exp(\mu X)(z). \tag{28}$$

The diagonal arrow at the middle of the butterfly resolution is given by the action of Q^+Q^-. It is a complex because

$$(Q^+)^2 = (Q^-)^2 = \{Q^+, Q^-\} = 0. \tag{29}$$

The main statement in [7] is that complex (26) is exact except at the $F_{0,h}$ module, where the cohomology is given by the chiral module $M_{h,0}$.

To obtain the resolution for the irreducible module $M_{h,t}$, we can use the observation in [7] that all irreducible modules can be obtained from the chiral module $M_{h,0}$, $h=0,\ldots,\mu-2$, by the spectral flow action $U^{-t}, t=1,\ldots,\mu-1$. The spectral flow action on the free fields can be easily described if we bosonize the fermions as

$$\psi(z)=\exp(-\phi(z)),\quad \psi^*(z)=\exp(\phi(z)) \tag{30}$$

and introduce the spectral flow vertex operator

$$U^t(z)=\exp\left(-t\left(\phi+\frac{1}{\mu}X^*-X\right)(z)\right). \tag{31}$$

Using resolution (26) and the spectral flow, we obtain the following expression for the character [9]:

$$\begin{aligned}\chi_{h,-t}(u,q) &= q^{\frac{h}{2\mu}+\frac{c}{6}t^2+\frac{th}{\mu}-\frac{c}{24}}\times\\ &\times q^{\frac{1-\mu}{8}}u^{\frac{h}{\mu}+\frac{ct}{3}}\left(\frac{\eta(q^\mu)}{\eta(q)}\right)^3\times\\ &\times\prod_{n=0}\frac{1+uq^{\frac{1}{2}+t+n}}{1+u^{-1}q^{-\frac{1}{2}-t+n\mu}}\times\\ &\times\frac{1+u^{-1}q^{\frac{1}{2}-t+n}}{1+uq^{\frac{1}{2}+t+(n+1)\mu}}\,\frac{1-q^{n+1}}{1-q^{(n+1)\mu}}\times\\ &\times\prod_{n=0}\frac{1-q^{-1-h+n\mu}}{1+uq^{-\frac{1}{2}-h+t+n\mu}}\times\\ &\times\frac{1-q^{1+h+(n+1)\mu}}{1+u^{-1}q^{\frac{1}{2}+h-t+(n+1)\mu}},\end{aligned} \tag{32}$$

where

$$\eta(q)=q^{\frac{1}{24}}\prod_{n=1}(1-q^n). \tag{33}$$

The resolutions and irreducible modules in the R sector are generated from those in the NS sector by the spectral flow operator $U^{1/2}$.

3.2. Free-field realization of the product of minimal models

It is clear how to generalize the free-field representation to the tensor product of K $N=2$ minimal models. In the left-moving sector, we introduce free bosonic fields $X_i(z), X_i^*(z)$ and free fermionic fields $\psi_i(z), \psi_i^*(z)$, $i=1,\ldots,K$, such that their singular OPEs are given by (19). The $N=2$ Virasoro currents for each of the models are given by (20). To describe the products of irreducible representations $M_{\mathbf{h},\mathbf{t}}$, we introduce the fermionic screening currents and their charges

$$\begin{gathered}S_i^+(z)=\psi_i^*\exp(X_i^*)(z),\\ S_i^-(z)=\psi_i\exp(\mu_i X_i)(z),\\ Q_i^\pm=\oint dz\, S_i^\pm(z).\end{gathered} \tag{34}$$

Then the module $M_{\mathbf{h},0}$ is given by the cohomology of the product of butterfly resolutions (26). The resolution of $M_{\mathbf{h},\mathbf{t}}$ is generated by the spectral flow operator

$$U^{\mathbf{t}}=\prod_i U_i^{t_i},\quad t_i=1,\ldots,\mu_i-1,$$

where $U_i^{t_i}$ is the spectral flow operator from the ith minimal model (31). Allowing t_i to be half-integer, we generate the corresponding objects in the R sector. In what follows, we consider the case $\mu_1=\ldots=\mu_K=2K$.

3.3. The LG orbifold geometry of Gepner models

The holomorphic factor of the space of states of model (18) in the R sector is also given by cohomology of a complex. It is an orbifold of the complex that is the sum of butterfly resolutions for the modules $M_{\mathbf{h},\mathbf{t}}$. The cohomology of this complex can be calculated in two steps.

At the first step, we take the cohomology with respect to the operator

$$Q^+=\sum_{i=1}^K Q_i^+. \tag{35}$$

The cohomology is generated by the "$bc\beta\gamma$" system of fields

$$\begin{gathered}a_i(z)=\exp[X_i](z),\quad \alpha_i(z)=\psi_i\exp[X_i](z),\\ a_i^*(z)=(\partial X_i^*-\psi_i\psi_i^*)\exp[-X_i](z),\\ \alpha_i^*(z)=\psi_i^*\exp[-X_i](z).\end{gathered} \tag{36}$$

The singular operator product expansions of these fields are

$$\begin{gathered}a_i^*(z_1)a_j(z_2)=z_{12}^{-1}\delta_{ij}+\ldots,\\ \alpha_i^*(z_1)\alpha_j(z_2)=z_{12}^{-1}\delta_{ij}+\ldots\end{gathered} \tag{37}$$

In terms of the fields in (36), the $N=2$ Virasoro currents (15) are given by

$$
\begin{aligned}
G^- &= \sum_i \alpha_i a_i^*, \\
G^+ &= \sum_i \left(1 - \frac{1}{2K}\right) \alpha_i^* \partial a_i - \frac{1}{2K} a_i \partial \alpha_i^*, \\
J &= \sum_i \left(1 - \frac{1}{2K}\right) \alpha_i^* \alpha_i + \frac{1}{2K} a_i a_i^*, \\
T &= \sum_i \frac{1}{2} \left(\left(1 + \frac{1}{2K}\right) \partial \alpha_i^* \alpha_i - \right. \\
&\left. - \left(1 - \frac{1}{2K}\right) \alpha_i^* \partial \alpha_i \right) + \left(1 - \frac{1}{4K}\right) \partial a_i a_i^* - \\
&- \frac{1}{4K} a_i \partial a_i^*.
\end{aligned} \tag{38}
$$

We note that the zero mode $G^-[0]$ acts in the space of states generated by the "$bc\beta\gamma$" system of fields similarly to how the de Rham differential acts in the de Rham complex of $\mathbb{C}^K$. Due to this observation and taking (37) into account, we can make the following geometric interpretation of the fields in (36). The fields $a_i(z)$ correspond to the coordinates a_i on the complex space $\mathbb{C}^K$, and the fields $a_i^*(z)$ correspond to the operators $\frac{\partial}{\partial a_i}$. The fields $\alpha_i(z)$ correspond to the differentials da_i, and $\alpha_i^*(z)$ correspond to the variables conjugate to da_i.

The next important property is the behavior of the "$bc\beta\gamma$" system under a local change of coordinates on $\mathbb{C}^K$ [4]. For each new set of coordinates

$$
b_i = g_i(a_1, \ldots, a_K), \quad a_i = f_i(b_1, \ldots, b_K), \tag{39}
$$

an isomorphic "$bc\beta\gamma$" system of fields is given by

$$
\begin{aligned}
b_i(z) &= g_i(a_1(z), \ldots, a_K(z)), \\
\beta_i(z) &= \frac{\partial g_i}{\partial a_j}(a_1(z), \ldots, a_K(z)) \alpha_j(z), \\
\beta_i^*(z) &= \frac{\partial f_j}{\partial b_i}(a_1(z), \ldots, a_K(z)) \alpha_j^*(z), \\
b_i^*(z) &= \frac{\partial f_j}{\partial b_i}(a_1(z), \ldots, a_K(z)) a_j^*(z) + \\
&+ \frac{\partial^2 f_k}{\partial b_i \partial b_j} \frac{\partial g_j}{\partial a_n}(a_1(z), \ldots, a_K(z)) \alpha_k^*(z) \alpha_n(z),
\end{aligned} \tag{40}
$$

where normal ordering is implied. This endows the "$bc\beta\gamma$" system (36) with the structure of a sheaf, known as the chiral de Rham complex [4].

All these properties give a geometric interpretation to the algebraic construction of Gepner models. Indeed, it was shown in the general toric setup in [3] that the screening charges Q_i^+ determine the toric data of some toric manifold and the cohomology of differential (35) gives sections of chiral de Rham complex on this manifold. In our case, this manifold is $\mathbb{C}^K$ and the chiral de Rham complex on this space is generated by "$bc\beta\gamma$" system (36).

The charges of fields (36) are given by

$$
\begin{aligned}
J(z_1) a_i(z_2) &= z_{12}^{-1} \frac{1}{2K} a_i(z_2) + \text{reg.}, \\
J(z_1) a_i^*(z_2) &= -z_{12}^{-1} \frac{1}{2K} a_i^*(z_2) + \text{reg.}, \\
J(z_1) \alpha_i(z_2) &= -z_{12}^{-1} \left(1 - \frac{1}{2K}\right) \alpha_i(z_2) + \text{reg.}, \\
J(z_1) \alpha_i^*(z_2) &= z_{12}^{-1} \left(1 - \frac{1}{2K}\right) \alpha_i^*(z_2) + \text{reg.}
\end{aligned} \tag{41}
$$

Hence, projecting on $\mathbb{Z}_{2K}$-invariant states and adding twisted sectors generated by $\prod_{i=1}^K (U_i)^n$, we obtain a toric construction of the chiral de Rham complex of the $\mathbb{C}^K/\mathbb{Z}_{2K}$ orbifold. The chiral de Rham complex on the orbifold was recently introduced in [11].

The second step in the cohomology calculation is to take the cohomology with respect to the differential $Q^- = \sum_{i=1}^K Q_i^-$. This operator survives the orbifold projection and is expressed in terms of fields (36) as

$$
Q^- = \oint dz \sum_{i=1}^K \alpha_i (a_i)^{2K-1}. \tag{42}
$$

Therefore, the second step of the cohomology calculation gives the restriction of the chiral de Rham complex to the points $dW = 0$ of the potential

$$
W = \sum_{i=1}^K (a_i)^{2K}. \tag{43}
$$

The total space of states in the holomorphic sector can be recovered by spectral flow operators $\prod_{i=1}^K (U_i)^{t_i}$, where the vector $(t_1, \ldots, t_K)$ is orthogonal to the vector $(1, \ldots, 1)$ generating the twisted sectors of the orbifold. Hence, the total space of states is the space of states of the LG orbifold $\mathbb{C}^K/\mathbb{Z}_{2K}$, whose partition function in the RR sector is given by (18).

4. THE LG σ-MODEL CORRESPONDENCE CONJECTURE

In this section, we relate the LG orbifold $\mathbb{C}^K/\mathbb{Z}_{2K}$ to a σ-model on the CY manifold that is a double cover of $\mathbb{P}^{K-1}$. The relation appears when we deform the LG orbifold by a marginal operator making the orbifold singularity resolution. According to the construction in [3, 5], the orbifold singularity resolution in the holomorphic sector is given by supplementary screening charges.

4.1. The $K = 2$ example

We first consider the simplest example where $K = 2$. In this case, we add the screening charge

$$D_{orb} = \oint dz \frac{1}{2}(\psi_1^* + \psi_2^*) \exp\left(\frac{1}{2}(X_1^* + X_2^*)\right)(z) \quad (44)$$

to the charges $Q^+_{1,2}$. It is easy to verify that this operator commutes with the total $N = 2$ Virasoro currents (15) and also commutes with the operators Q_i^-. The corresponding fermionic screening current is the holomorphic (chiral) factor of an antichiral–chiral marginal field [2, 13], coming from the twisted sector. The fermionic operators

$$D_n^+ = \oint dz \left(\frac{2-n}{4}\psi_1^* + \frac{2+n}{4}\psi_2^*\right) \times \\ \times \exp\left(\frac{2-n}{4}X_1^* + \frac{2+n}{4}X_2^*\right)(z), \quad n = -1, 1, \quad (45)$$

also commute with the $N = 2$ Virasoro algebra and with Q_i^-, but do not appear as marginal operators of the model because they should come from twisted sectors that are nonexistent in the model (see (18)).

To the set of screening charges Q_1^+, Q_2^+, D_{orb}, following the construction of Borisov, we associate the fan [12] consisting of two 2-dimensional cones σ_1 and σ_2 respectively generated in the lattice $(\frac{1}{2}\mathbb{Z})^2$ by the vectors $\left(e_1, \frac{1}{2}(e_1 + e_2)\right)$ and $\left(e_2, \frac{1}{2}(e_1 + e_2)\right)$. To each of the cones σ_i, a "$bc\beta\gamma$" system of fields is related by the cohomology of the differential $Q_i^+ + D_{orb}$, $i = 1, 2$. This is the first step of the cohomology calculation.

It can be shown that these two systems generate the space of sections of the chiral de Rham complex on open sets of the standard covering of the total space of an $O(2)$ bundle over $\mathbb{P}^1$.

The first step of the cohomology calculation can be split into two substeps. At the first substep, we take the $Q_1^+ + D_{orb}$ cohomology. It is given by the following "$bc\beta\gamma$" fields:

$$\begin{gathered} b_0(z) = \exp[2X_2](z), \\ \beta_0(z) = 2\psi_2 \exp[2X_2](z), \\ b_0^*(z) = \left(\frac{1}{2}(\partial X_1^* + \partial X_2^*) - \psi_2(\psi_1^* + \psi_2^*)\right) \times \\ \times \exp[-2X_2](z), \\ \beta_0^*(z) = \frac{1}{2}(\psi_1^* + \psi_2^*) \exp[-2X_2](z), \\ b_1(z) = \exp[X_1 - X_2](z), \\ \beta_1(z) = (\psi_1 - \psi_2) \exp[X_1 - X_2](z), \\ b_1^*(z) = (\partial X_1^* - (\psi_1 - \psi_2)\psi_1^*) \exp[X_2 - X_1](z), \\ \beta_1^*(z) = \psi_1^* \exp[X_2 - X_1](z). \end{gathered} \quad (46)$$

At the second substep, we calculate the Q_2^+ cohomology.

Equivalently, we can take the $Q_2^+ + D_{orb}$ cohomology at the first substep and apply Q_1^+ at the second substep. This way, we obtain different "$bc\beta\gamma$" fields:

$$\begin{gathered} \tilde{b}_0(z) = \exp[2X_1](z), \\ \tilde{\beta}_0(z) = 2\psi_1 \exp[2X_1](z), \\ \tilde{b}_0^*(z) = \left(\frac{1}{2}(\partial X_1^* + \partial X_2^*) - \psi_1(\psi_1^* + \psi_2^*)\right) \times \\ \times \exp[-2X_1](z), \\ \tilde{\beta}_0^*(z) = \frac{1}{2}(\psi_1^* + \psi_2^*) \exp[-2X_1](z), \\ \tilde{b}_1(z) = \exp[X_2 - X_1](z), \\ \tilde{\beta}_1(z) = (\psi_2 - \psi_1) \exp[X_2 - X_1](z), \\ \tilde{b}_1^*(z) = (\partial X_2^* - (\psi_2 - \psi_1)\psi_2^*) \exp[X_1 - X_2](z), \\ \tilde{\beta}_1^*(z) = \psi_2^* \exp[X_1 - X_2](z). \end{gathered} \quad (47)$$

In view of the important property (40), these two "$bc\beta\gamma$" systems are related to each other like the coordinates of the standard covering of the total space of an $O(2)$ bundle over $\mathbb{P}^1$,

$$b_0 = \tilde{b}_0(\tilde{b}_1)^2, \quad b_1 = \tilde{b}_1^{-1}, \ldots \quad (48)$$

Therefore,

$$\begin{aligned} b_0(z) &\leftrightarrow \text{ coordinate } b_0 \text{ along the fiber}, \\ b_1(z) &\leftrightarrow \text{ coordinate } b_1 \text{ along the base} \end{aligned} \quad (49)$$

in the first open set of the standard covering. The tildaed fields are associated with the second open set. Thus, fields (46) and (47) generate sections of the chiral de Rham complex over open sets of the covering given by the fan $\sigma_1 \cup \sigma_2$. In the second substep, we calculate the cohomology of the Czech complex of the standard covering. It glues the sections of chiral de Rham complex over open sets into the chiral de Rham complex over the total space of the bundle. This finishes the first step of the cohomology calculation.

The differential Q^- at the second step of the cohomology calculation commutes with D_{orb} and survives the $\mathbb{Z}_4$ projection. It defines a function (potential) W on the total space of the $O(2)$ bundle, and the Q^--cohomology calculation restricts the chiral de Rham complex to the $dW = 0$ locus of the function. In terms of fields (46), the potential takes the form

$$W = b_0^2(1 + b_1^4). \quad (50)$$

The $dW = 0$ locus (Q^--cohomology) is given by the equations

$$b_0 = 0, \text{ when } b_1^4 \neq -1, \quad (b_0)^2 = 0, \text{ when } b_1^4 = -1. \tag{51}$$

The set of solutions is $\mathbb{P}^1$ with four marked points $b_1^4 = -1$, where the additional states are possible according to the last row in (51). Thus, we can think of $\mathbb{P}^1$ as a target space of the model, where the center of mass of the string is allowed to move.

This interpretation is not quite correct, however, because we did not resolve the orbifold singularity completely. It is easy to see from (15), (20), (46) or (47) that the subgroup

$$\mathbb{Z}_2 \subset \mathbb{Z}_4 \tag{52}$$

acts on sections of the chiral de Rham complex over each open set. But the action is nontrivial only along the fibers of the $O(2)$ bundle, and hence the base $\mathbb{P}^1$ is the fixed-point set of the action. Therefore, we should consider the target space of the model as two copies of $\mathbb{P}^1$ (except probably at the points $b_1^4 = -1$), where the second copy comes from the twisted sector. This picture is in agreement with the result in [11], where the chiral de Rham complex on orbifolds was introduced. It was shown there that twisted sectors of the chiral de Rham complex are the sheaves supported on fixed points of the orbifold group action.

Thus, the natural suggestion is that we reproduce the geometry of a 2-torus that is a double cover of $\mathbb{P}^1$ with ramification along the marked points $b_1^4 = -1$. This is confirmed by the Hodge number calculation based on (18):

$$h^{0,0} = h^{1,0} = h^{0,1} = h^{1,1} = 1.$$

Hence, adding fermionic screening charge (44), we blow up the orbifold singularity of the Gepner model and obtain a σ-model on the 2-torus that is a double cover of $\mathbb{P}^1$.

4.2. $K > 2$ generalization

In the general case, we deform the differential Q^+ in (35) by adding the screening charge

$$\begin{gathered} Q^+ \to Q^+ + D_{orb}, \\ D_{orb} = \oint dz \frac{1}{K}(\psi_1^* + \ldots + \psi_K^*) \times \\ \times \exp\left(\frac{1}{K}(X_1^* + \ldots + X_K^*)\right)(z), \end{gathered} \tag{53}$$

which comes from the spectral flow operator $\prod_{i=1}^K U_i$. Similarly to the $K = 2$ case, there are also other fermionic screening charges commuting with the $N = 2$ Virasoro currents as well as with the charges Q_i^-, but they do not appear as marginal operators of model (18).

The set of screening charges $\{Q_1^+, \ldots, Q_K^+, D_{orb}\}$ defines the standard fan of the total space of an $O(K)$ bundle over $\mathbb{P}^{K-1}$. The top-dimension cones σ_i of the fan are labeled by the differentials

$$D_i = Q_1^+ + \ldots + Q_{i-1}^+ + D_{orb} + Q_{i+1}^+, \ldots, Q_K^+, \quad i = 1, \ldots, K, \tag{54}$$

where Q_i^+ is missing. In the standard basis $(e_1, \ldots, e_K)$ of $\mathbb{R}^K$, the cones are generated by the set of vectors

$$\Sigma_i = \left(s_1 = e_1, \ldots, s_{i-1} = e_{i-1}, s_i = \frac{1}{K} \times \right. \\ \left. \times (e_1 + \ldots + e_K), s_{i+1} = e_{i+1}, \ldots, s_K = e_K\right). \tag{55}$$

From the first substep of the cohomology calculation, we obtain a "$bc\beta\gamma$" system of fields associated with each differential D_i, and the space of states generated by this system is the set of sections of the chiral de Rham complex over the open set associated with the cone σ_i of the standard covering of the $O(K)$ bundle total space over $\mathbb{P}^{K-1}$. The analogue of formulas (46) can be written easily in terms of the dual basis $\check{\Sigma}_i$ to the Σ_i,

$$\check{\Sigma}_i = (w_{(i)1}, \ldots w_{(i)K}), \quad \langle w_{(i)j}, s_m \rangle = \delta_{jm}. \tag{56}$$

Then the cohomology of D_i is generated by

$$\begin{gathered} b_{(i)j}(z) = \exp[w_{(i)j} \cdot X](z), \\ \beta_{(i)j}(z) = w_{(i)j} \cdot \psi \exp[w_{(i)j} \cdot X](z), \\ b_{(i)j}^*(z) = (s_j \cdot \partial X^* - w_{(i)j} \cdot \psi s_j \cdot \psi^*) \times \\ \times \exp[-w_{(i)j} \cdot X](z), \\ \beta_{(i)j}^*(z) = s_j \cdot \psi^* \exp[-w_{(i)j} \cdot X](z), \end{gathered} \tag{57}$$

where

$$\begin{gathered} b_{(i)i}(z) \leftrightarrow \text{ coordinate } b_{(i)i} \text{ along the fiber}, \\ b_{(i)j}(z), \ j \neq i \leftrightarrow \text{ coordinate } b_{(i)j} \\ \text{along the base}. \end{gathered} \tag{58}$$

Global sections of the chiral de Rham complex on the $O(K)$ bundle total space are given by the Czech complex associated with the standard covering [3]. This finishes the first step of the cohomology calculation.

In terms of fields (57), the LG potential determined by the differential Q^- becomes

$$W = (b_{(i)i})^2 \left(1 + \sum_{j\neq i} (b_{(i)j})^{2K} \right). \quad (59)$$

The $dW = 0$ locus (Q^--cohomology) is given by the equations

$$\begin{aligned} b_{(i)i} = 0, \quad & \sum_{j\neq i} (b_{(i)j})^{2K} \neq -1, \\ (b_{(i)i})^2 = 0, \quad & \sum_{j\neq i} (b_{(i)j})^{2K} = -1. \end{aligned} \quad (60)$$

Hence, the set of solutions is $\mathbb{P}^{K-1}$ with a marked submanifold

$$\sum_{j\neq i} (b_{(i)j})^{2K} = -1, \quad (61)$$

where the additional states are possible according to the last row in (60).

Similarly to the case $K = 2$, we can see that only the fields of the fiber are charged with respect to the operator $J[0]$, and the subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_{2K}$ acts nontrivialy along the fibers. Therefore, the base $\mathbb{P}^{K-1}$ (considered as a zero section of the $O(K)$ bundle) is the fixed-point set of the $\mathbb{Z}_2$ action, and we conclude that the target space of the model is given by two copies of $\mathbb{P}^{K-1}$ (except the submanifold (61)), where the second copy comes from the twisted sector (see [11]).

Hence, it is natural to suggest that the geometry of the model is the $(K-1)$-dimensional CY manifold geometry that doubly covers $\mathbb{P}^{K-1}$ with ramification along submanifold (61). This is confirmed by the Hodge number calculation based on (18). For example, when $K = 3$,

$$h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1, \quad h^{1,1} = 20, \quad (62)$$

which are the Hodge numbers of $K3$. When $K = 4$, we find

$$\begin{gathered} h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = h^{1,1} = h^{2,2} = 1, \\ h^{1,2} = h^{2,1} = 149, \end{gathered} \quad (63)$$

which are the Hodge numbers of the known CY manifold that doubly covers $\mathbb{P}^3$. Hence, adding fermionic screening charge (53), we blow up the orbifold singularity of the Gepner model and obtain a σ-model on the CY manifold that is a double cover of $\mathbb{P}^{K-1}$.

It is important to note that in our free-field realization, the center of mass of the string is allowed to move on the $\mathbb{P}^{K-1}$, which can be considered as a target space, and hence we can interprate the model as a σ-model on $\mathbb{P}^{K-1}$. Although the target space $\mathbb{P}^{K-1}$ is not a CY manifold, we do have the $N = 2$ superconformal invariance. The possible solution of this puzzle is to consider these models as examples of flux compactification [14, 15]. It is interesting to note also that the models considered here are very close to the known examples of the weak-coupling limit of F-theory compactifications [16, 17]. The only difference is that they do not have the orientifolds planes. It would be interesting to know whether these models can be related with F-theory compactifications.

To conclude, we mention the question of the geometry of mirror models. It can be investigated by free-field "$bc\beta\gamma$" representations, but we leave it for the future.

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