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MASS SPECTRUM IN SUPERSIMMETRIC QCD AND PROBLEMS WITH THE SEIBERG DUALITY. EQUAL QUARK MASSES

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For the $\mathcal{N}=1$ SQCD with N_c colors and $N_c < N_F < 3N_c$ flavors with small but nonzero current quark masses $m_Q \neq 0$, the dynamic scenario is considered, in which quarks form the diquark-condensate phase. This means that colorless chiral quark pairs condense coherently in the vacuum, $\langle \overline{Q}Q\rangle \neq 0$, while quarks alone do not condense, $\langle Q\rangle = \langle \overline{Q}\rangle = 0$, and therefore the color is confined. Such condensation of quarks results in the formation of dynamic constituent masses $\mu_C \gg m_Q$ of quarks and the appearance of light "pions" (similarly to the case of QCD). The SQCD mass spectrum in this phase is described and compared with the Seiberg dual description. It is shown that the direct and dual theories are different (except, possibly, in the perturbative strictly superconformal regime).

1. INTRODUCTION

Because supersymmetric gauge theories are much more constrained than ordinary ones, it is easier to deal with them theoretically. Therefore, they can serve, at least, as useful models for elucidating the complicated strong-coupling gauge dynamics (not to speak of their potential relevance to the real world).

The closest to QCD is its supersymmetric extension, the $\mathcal{N}=1$ SQCD, which has been considered in many papers. We here consider SQCD in the nonperturbative region (or in the perturbative strong-coupling regime). Most impressive results here were obtained by Seiberg, who proposed describing this strongly coupled (and/or nonperturbative) SQCD in terms of the equivalent, but weakly coupled dual theory [1] (for reviews, see Refs. [2–4] and the references therein).

Our purpose in this paper is to introduce (in Sec. 3) the main dynamic assumption about the coherent diquark-condensate (DC) phase of SQCD, to describe its consequences for the behavior in the infrared region, the mass spectrum, etc., and to compare with predictions of the Seiberg dual theory.

The paper is organized as follows. In Secs. 2 and 4, we recall definitions of the direct and dual theories; some particular examples are considered in Sec. 2. Both direct and dual theories are considered in the conformal window $3N_c/2 < N_F < 3N_c$ in Secs. 3, 5, and 6 and for $N_c < N_F < 3N_c/2$ in Sec. 7. For completeness, the case $N_F > 3N_c$ is considered in Sec. 8. Finally, some conclusions are presented in Sec. 9 (and there is one appendix about the 't Hooft triangles).

2. DIRECT THEORY. DEFINITION AND SOME EXAMPLES

The fundamental Lagrangian of SQCD with N_c colors and N_F flavors (at a high scale $\mu \gg \Lambda_Q$) is given by

$$L = \int d^2\theta\, d^2\overline{\theta}\, \mathrm{Tr}\left(Q^\dagger e^V Q + \overline{Q}^\dagger e^{-V}\overline{Q}\right) + \\ + \int d^2\theta \left\{-\frac{2\pi}{\alpha(\mu)} S + m_Q(\mu)\,\mathrm{Tr}\,\overline{Q}Q\right\} + \mathrm{H.c.}, \quad (2.1)$$

where $\alpha(\mu)$ is the running gauge coupling (with its scale parameter Λ_Q independent of the quark masses), $m_Q(\mu)$ is the running current quark mass, $S = W_\alpha^2/32\pi^2$, W_α is the gluon field strength, and traces are taken over color and flavor indices. This theory has the exact $SU(N_c)$ gauge symmetry and, in the chiral limit $m_Q \to 0$, a global $SU(N_F)_L \times SU(N_F)_R \times U(1)_B \times U(1)_R$ symmetry. Un-

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der these symmetries, the quarks Q and $\overline{Q}$ transform as

$$Q:\ (N_c)_{col}\times(N_F)^{fl}_L\times(0)^{fl}_R\times(1)_B\times(N_F-N_c/N_F)_R,$$

$$\overline{Q}:\ (\overline{N}_c)_{col}\times(0)^{fl}_L\times(\overline{N}_F)^{fl}_R\times(-1)_B\times(N_F-N_c/N_F)_R.$$

The explicit dependence of the gluino condensate $\langle S\rangle$ on the current quark masses and Λ_Q can be found as follows.

a) We can start with $N_F < 3N_c$ and the heavy quarks, $m_Q^{pole}\equiv m_Q(\mu=m_Q^{pole})\gg\Lambda_Q$, such that the theory is UV-free and in the weak-coupling regime at sufficiently large energies.

b) We then integrate out all quarks directly in the perturbation theory at scales $\mu<\mu_H=m_Q^{pole}$, which yields a pure Yang–Mills theory with the scale factor Λ_{YM}. The value of Λ_{YM} can be found from the matching of the couplings $\alpha_+(\mu)$ and $\alpha_-(\mu)$ of the upper and lower theories at $\mu=\mu_H$: $\alpha_+(\mu_H)=\alpha_-(\mu_H)$. The upper theory is always the original one with N_c colors and N_F flavors, and the value of $\alpha_+(\mu_H)$ can be obtained starting with high $\mu\gg\mu_H$ and evolving down to $\mu=\mu_H$ through the standard perturbative renormgroup (RG) flow for the theory with N_c colors and N_F flavors of massless quarks[1)]. But instead, the same value $\alpha_+(\mu)$ can be obtained starting with $\mu\sim\Lambda_Q$ and going up to $\mu=\mu_H\gg\Lambda_Q$ with the same RG flow for massless quarks, that is ($g^2(\mu)=4\pi\alpha(\mu)$, $b_0=3N_c-N_F$),

$$\frac{2\pi}{\alpha_+(\mu_H)}=b_0\ln\Big(\frac{\mu_H}{\Lambda_Q}\Big)-N_F\ln\Big(\frac{1}{z_Q(\mu_H,\Lambda_Q)}\Big)+ + N_c\ln\Big(\frac{1}{g^2(\mu_H)}\Big)+C_+,\quad(2.2)$$

where $z_Q=z_Q(\mu_H,\Lambda_Q)\ll 1$ is the standard perturbative renormalization factor (logarithmic in this case) of massless quarks in the theory with N_c colors and N_F flavors. At the weak coupling $\alpha(\mu_H/\Lambda_Q)\ll 1$ (with $C_F=(N_c^2-1)/2N_c$),

$$z_Q(\mu_H,\Lambda_Q)=C_0\left(\frac{\alpha(\mu_H)}{\alpha(\Lambda_Q)}\right)^{2C_F/b_0}\times \times\left[1+O\left(\alpha\left(\frac{\mu_H}{\Lambda_Q}\right)\right)\right]\sim\Big(\frac{1}{\ln(\mu_H/\Lambda_Q)}\Big)^{2C_F/b_0}\ll 1,$$

where C_0 is a nonparametric constant $O(1)$.

As regards the lower theory, it is the Yang–Mills theory with N_c' colors and no quarks in all examples considered in this section. Its coupling can be written similarly as

$$\frac{2\pi}{\alpha_-(\mu_H)}=3N_c'\ln\Big(\frac{\mu_H}{\Lambda_{YM}}\Big)+ + N_c'\ln\Big(\frac{1}{g^2(\mu_H)}\Big)+C_-.\quad(2.3)$$

The $C_\pm$ in (2.2) and (2.3) are constants independent of the quark mass values. After introducing the Wilson coupling $\alpha_W(\mu)$ whose β-function is that of NSVZ for $\alpha(\mu)$ but without the denominator, $2\pi/\alpha_W(\mu)=2\pi/\alpha(\mu)-N_c\ln(1/g^2(\mu))$ [5], we have $C_+=2\pi/\alpha_W^+(\mu=\Lambda_Q)$ and $C_-= =2\pi/\alpha_W^-(\mu=\Lambda_{YM})$. In essence, the term $N_c\ln\Big(1/g^2(\mu_H)\Big)$ in (2.2) is the higher-loop perturbative renormalization factor of gluons, i.e., $N_c\ln\Big(z_g(\mu_H,\Lambda_Q)\Big)=N_c\ln\Big(\alpha_+(\mu_H)/\alpha_+(\mu=\Lambda_Q)\Big)$, and similarly in (2.3).

Our purpose here and below is to explicitly trace the dependence on the parameters like μ_H/Λ_Q, which are to be finally expressed through the universal parameter m_Q/Λ_Q, $m_Q\equiv m_Q(\mu=\Lambda_Q)$, which can be large, $m_Q/\Lambda_Q\gg 1$, or small, $m_Q/\Lambda_Q\ll 1$. Therefore, the constant terms like $C_\pm$ are omitted in what follows, because their effect is equivalent to a redefinition of Λ_Q by a constant factor.

In the case considered now, $N_c'=N_c$ and $\mu_H= =m_Q^{pole}\gg\Lambda_Q$, and it then follows from (2.2) and (2.3) that

$$\Lambda_{YM}=(\Lambda_Q^{b_0}\det m_Q)^{1/3N_c},\qquad m_Q\equiv z_Q^{-1}(m_Q^{pole},\Lambda_Q)\,m_Q^{pole}\gg m_Q^{pole}\gg\Lambda_Q.\quad(2.4)$$

c) Lowering the scale μ to $\mu<\Lambda_{YM}$ and integrating out all gauge degrees of freedom, except the field S itself, we can write the effective Lagrangian in the Veneziano–Yankielowicz (VY) form [6], from which we obtain the gluino condensate

$$\langle S\rangle=\Lambda_{YM}^3=(\Lambda_Q^{b_0}\det m_Q)^{1/N_c},\qquad m_Q=m_Q(\mu=\Lambda_Q).\quad(2.5)$$

Now, expression (2.5) can be continued in m_Q from large $m_Q\gg\Lambda_Q$ to small values, $m_Q\ll\Lambda_Q$. While m_Q is some formally defined parameter for $m_Q\gg\Lambda_Q$ (see (2.4), the physical quark mass is $m_Q^{pole}\gg\Lambda_Q$ and it does not run any more at $\mu<m_Q^{pole}$), it has a simple and direct meaning for $m_Q\ll\Lambda_Q$: $m_Q= =m_Q(\mu=\Lambda_Q)$.

Expression (2.5) for $\langle S\rangle$ has appeared many times in the literature, but to our knowledge, the exact definition of the parameter m_Q entering (2.5), i.e., its relation to $m_Q(\mu)$ in (1) that defines the theory, has not

1) In Eqs. (2.2) and (2.3) and everywhere below in the text, the perturbative NSVZ [5] β-function is used, corresponding to the Pauli–Villars scheme.

been given. Clearly, without this explicit relation, expression (2.5) is not very meaningful, because the quark mass parameter $m_Q(\mu)$ is running. For instance, if m_Q is understood as m_Q^{pole} in (2.5) for heavy quarks, than the relation $\langle S\rangle = (\Lambda_Q^{b_0}\det m_Q^{pole})^{1/N_c}$ would be erroneous. All this becomes especially important, in particular, in the conformal window $3N_c/2 < N_F < 3N_c$ and $m_Q \ll \Lambda_Q$, when $m_Q(\mu)$ runs at $\mu \ll \Lambda_Q$ in a power-like fashion: $m_Q(\mu_2) = (\mu_1/\mu_2)^{b_0/N_F}\, m_Q(\mu_1)$. Everywhere below, except in Sec. 8, only the case $m_Q \ll \Lambda_Q$ is considered.

d) From the Konishi anomaly equation [7]

$$\left\langle \left(\overline{Q}_{\bar j} Q^i\right)_\mu \right\rangle = \left(m_Q^{-1}(\mu)\right)^i_{\bar j}\langle S\rangle, \qquad (2.6)$$

we obtain the explicit value of the chiral condensate:

$$\begin{gathered}\langle(\overline{Q}_{\bar j} Q^i)_{\mu=\Lambda_Q}\rangle \equiv \mathcal{M}^2_{ch}\,\delta^i_{\bar j} = \frac{\langle S\rangle}{m_Q}\,\delta^i_{\bar j},\\ \mathcal{M}_{ch} = \left(\Lambda_Q^{b_0} m_Q^{\overline{N}_c}\right)^{1/2N_c}, \quad \overline{N}_c = N_F - N_c,\\ \langle S\rangle = \Lambda^3_{YM} = (\Lambda_Q^{b_0}\det m_Q)^{1/N_c}, \quad m_Q \ll \Lambda_Q.\end{gathered} \qquad (2.7)$$

Expression (2.5) can now be continued in N_F from the region $N_F < 3N_c$ to $N_F > 3N_c$ and, together with Konishi anomaly relation (2.6), these two then become the basic universal relations for any values of quark masses and any N_F.

To check this universal form of (2.5), we briefly consider (see Sec. 8 for more details) the case $N_F > 3N_c$ and $m_Q \ll \Lambda_Q$. In this case, $b_0 = 3N_c - N_F < 0$, and hence the theory is IR-free in the interval $\mu_H < \mu < \Lambda_Q$, where μ_H is the highest physical mass ($\Lambda_{YM} \ll \mu_H = m_Q^{pole} \ll \Lambda_Q$ in this example); its coupling, which is $O(1)$ at $\mu = \Lambda_Q$, becomes logarithmically small at $\mu \ll \Lambda_Q$. Besides, the parameter m_Q has now a direct physical meaning as the value of the running quark mass at $\mu = \Lambda_Q$, $m_Q \equiv m_Q(\mu = \Lambda_Q) \ll \Lambda_Q$. Therefore, starting with $\mu = \Lambda_Q$ and going down perturbatively to $\mu_H = m_Q^{pole} = m_Q(\mu = m_Q^{pole}) = z_Q^{-1}(\Lambda_Q, m_Q^{pole})\, m_Q \gg m_Q$ (where $z_Q(\Lambda_Q, m_Q^{pole}) \ll 1$ is the perturbative logarithmic renormalization factor of massless quarks), we can then integrate out all quarks as heavy ones. Writing the matching condition for two couplings α_+ and α_-, we obtain (2.2) and (2.3) with the only replacement:

$$z_Q(m_Q^{pole} \gg \Lambda_Q, \Lambda_Q) \to z_Q^{-1}(\Lambda_Q, m_Q^{pole} \ll \Lambda_Q),$$

and the same expression (2.5).

Another check can be performed for $N_F < N_c - 1$ and small quark masses, $m_Q \ll \Lambda_Q$. In this case, all quarks are Higgsed and the gauge symmetry $SU(N_c)$ is broken down to $SU(N'_c = N_c - N_F)$ at the high scale $\mu_H = \mu_{gl} \gg \Lambda_Q$:

$$\langle Q^i_a\rangle_{\mu=\mu_{gl}} = \delta^i_a \mathcal{M}_0, \quad \langle \overline{Q}^a_{\bar j}\rangle_{\mu=\mu_{gl}} = \delta^a_{\bar j}\mathcal{M}_0,$$

$$\mathcal{M}_0 \gg \Lambda_Q.$$

The $2N_cN_F - N_F^2$ gluons become massive, with the mass scale $\mu^2_{gl} = g^2_+\langle\hat\Pi\rangle = g^2_+\mathcal{M}^2_0$, where $g^2_+ = 4\pi\alpha_+(\mu = \mu_{gl}, \Lambda_Q) \ll 1$. The same number of quark degrees of freedom acquire the same masses and become the superpartners of massive gluons (in a sense, they can be considered as the heavy "constituent quarks"), and N_F^2 light complex pion fields $\hat\pi^i_{\bar j}$ remain:

$$\hat\Pi^i_{\bar j} = (\overline{Q}_{\bar j} Q^i)_{\mu=\mu_{gl}} = \mathcal{M}^2_0(\delta^i_{\bar j} + \hat\pi^i_{\bar j}/\mathcal{M}_0),$$

$$\langle\hat\Pi^i_{\bar j}\rangle = \delta^i_{\bar j}\mathcal{M}^2_0.$$

All heavy particles can be integrated out at scales $\mu < \mu_{gl}$. The couplings at $\mu_H = \mu_{gl}$: $\alpha_+(\mu = \mu_{gl}, \Lambda_Q)$ in (2.2), i. e., those of the original theory, with $\mu^2_{gl} = g^2_+\langle\hat\Pi\rangle = g^2_+\mathcal{M}^2_0$, $\hat\Pi = (\overline{Q}Q)_{\mu=\mu_{gl}}$, and $\alpha_-(\mu = \mu_{gl}, \Lambda_L)$ in (2.3) of the lower-energy pure Yang–Mills theory can be matched numerically similarly to the previous examples with heavy quarks. But in this case, we consider it more useful to write the explicit form of the $\hat\Pi$-dependence of the lower-energy coupling $\alpha_-(\mu < \mu_{gl}, \Lambda_L)$ multiplying the field strength squared of massless gluons, to see how the multiloop β-function reconciles with the holomorphic dependence of Λ_L on the chiral superfields $\hat\Pi$. We thus ibtain

$$\begin{gathered}\frac{2\pi}{\alpha_-\left(\mu < \mu_{gl}, \Lambda_L\right)} = \left\{3(N_c - N_F)\ln\left(\frac{\mu}{\Lambda_Q}\right) + \right.\\ \left. + (N_c - N_F)\ln\left(\frac{1}{g^2_-(\mu, \langle\Lambda_L\rangle)}\right)\right\} + \\ + \left\{\frac{3}{2}\ln\left(\frac{g_+^{2N_F}(\mu = \mu_{gl}, \Lambda_Q)\det\hat\Pi}{\Lambda_Q^{2N_F}}\right) + \right.\end{gathered}$$

$$+ N_F \ln\left(\frac{1}{g_+^2(\mu=\mu_{gl},\Lambda_Q)}\right)\Bigg\} - \\ - \left\{\frac{1}{2}\ln\left(\frac{g_+^{2N_F}(\mu=\mu_{gl},\Lambda_Q)\det\hat{\Pi}}{\Lambda_Q^{2N_F}}\right) + \\ + N_F\ln\left(\frac{1}{z_Q(\mu=\mu_{gl},\,\Lambda_Q)}\right)\right\}, \quad (2.8)$$

where three terms in curly brackets in (2.8) are the respective contributions of massless gluons, massive gluons, and Higgsed quarks, and $z_Q(\mu_{gl}\gg\Lambda_Q,\,\Lambda_Q)\ll 1$ is the standard perturbative logarithmic renormalization factor of massless quarks (see above).

It is worth noting that the dependence of the coupling $2\pi/\alpha_-$ on the quantum pion superfields $\hat{\pi}^i_{\bar{j}}/\mathcal{M}_0$ entering $\hat{\Pi}^i_{\bar{j}}$ originates only from the $\hat{\pi}/\mathcal{M}_0$-dependence of heavy-particle masses entering the "normal" one-loop contributions to the gluon vacuum polarization, while the "anomalous" higher-loop contributions [5] originating from the quark and gluon renormalization factors, z_Q and $z_g^{\pm}\sim g_{\pm}^2$, do not contain the quantum pion fields $\hat{\pi}/\mathcal{M}_0$ and enter (2.8) as pure neutral c-numbers. This is clear from the R-charge conservation (see footnote 2) and the holomorphic dependence of F-terms on chiral quantum superfields (the chiral superfields here are $\overline{Q}_{\bar{j}}Q^i(\mu_1)=z_Q(\mu_1,\,\mu_2)\,\overline{Q}_{\bar{j}}Q^i(\mu_2)$).

Therefore, the coupling $\alpha_-(\mu,\,\Lambda_L)$ of the lower-energy pure Yang–Mills theory at $\mu<\mu_{gl}$ and its scale factor Λ_L are given by

$$\frac{2\pi}{\alpha_-^W\Big(\mu<\mu_{gl},\Lambda_L\Big)} = \frac{2\pi}{\alpha_-\Big(\mu<\mu_{gl},\Lambda_L\Big)} - \\ - (N_c-N_F)\ln\frac{1}{g_-^2(\mu<\mu_{gl},\langle\Lambda_L\rangle)} = \\ = 3(N_c-N_F)\ln\Big(\frac{\mu}{\Lambda_L}\Big), \quad (2.9)$$

$$\Lambda_L^{3(N_c-N_F)} = \frac{\Lambda_Q^{b_0}}{z_Q^{N_F}(\mu_{gl},\,\Lambda_Q)\det\hat{\Pi}} \equiv \frac{\Lambda_Q^{b_0}}{\det\Pi} = \\ = \Lambda_{YM}^{3(N_c-N_F)}\left(\det\frac{\langle\Pi\rangle}{\Pi}\right), \quad (2.10)$$

where

$$\Pi\equiv z_Q(\mu_{gl},\,\Lambda_Q)\hat{\Pi},\quad \langle\Pi\rangle=\mathcal{M}_{ch}^2\ll\mathcal{M}_0^2,$$

$$\langle\Lambda_L\rangle=\Lambda_{YM}=\Big(\Lambda_Q^{b_0}\det m_Q\Big)^{1/3N_c},$$

and the Lagrangian at $\mu<\mu_{gl}$ takes the form[2)]

$$L=\int d^2\theta\, d^2\overline{\theta}\left\{2\,\mathrm{Tr}\sqrt{\hat{\Pi}^\dagger\hat{\Pi}}\right\} + \\ + \int d^2\theta\left\{-\frac{2\pi}{\alpha_-(\mu,\Lambda_L)}\hat{S}+\hat{m}_Q\mathrm{Tr}\hat{\Pi}\right\}, \quad (2.11)$$

where $\hat{S}=\hat{W}_\alpha^2/32\pi^2$ and $\hat{W}_\alpha$ are the gauge field strengths of the $(N_c-N_F)^2-1$ remaining massless gluon fields.

Lowering the scale μ to $\mu<\Lambda_{YM}$ and integrating out all gauge degrees of freedom except the field $\hat{S}$ itself (which leaves a large number of gluonia with masses $M_{gl}\sim\Lambda_{YM}$), we obtain the VY form

$$L=\int d^2\theta\,d^2\overline{\theta}\Bigg\{2\,\mathrm{Tr}\,\sqrt{\hat{\Pi}^\dagger\hat{\Pi}} + \\ + (D \text{ terms of the field } \hat{S})\Bigg\} + \\ + \int d^2\theta\Bigg\{-(N_c-N_F)\,\hat{S}\left(\ln\Big(\frac{\hat{S}}{\Lambda_L^3}\Big)-1\right) + \\ + \hat{m}_Q\,\mathrm{Tr}\,\hat{\Pi}\Bigg\},\quad \mu<\Lambda_{YM}. \quad (2.12)$$

It is worth noting that it is the first place where the nonperturbative effects were incorporated to obtain the VY form of the superpotential (nonperturbative effects introduce the infrared cutoff of the order of Λ_{YM}, and hence the explicit dependence on μ disappears at $\mu<\Lambda_{YM}$), while all previous calculations with this example were purely perturbative. From (2.12), we obtain the gluino vacuum condensate $\langle\hat{S}\rangle=\langle\Lambda_L^3\rangle=\Lambda_{YM}^3=\langle S\rangle=(\Lambda_Q^{b_0}\det m_Q)^{1/N_c}$.

Finally, integrating out the last gluonium field $\hat{S}$ (with its mass scale of the order of Λ_{YM}) at lower energies, we obtain the Lagrangian of pions

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2) Because the gluon fields are not yet integrated completely, there are the gluon regulator fields (implicit) whose contributions ensure the R-charge conservation in (2.11) (see also (2.12) below). In (2.11), we also neglected the additional dependence of the Kähler term on the quantum pion fields $\hat{\pi}/\mathcal{M}_0$ (originating from the dependence on $\hat{\pi}/\mathcal{M}_0$ of the quark renormalization factor $z_Q(\hat{\Pi}^\dagger,\hat{\Pi})$), because at weak coupling, this influences the pion mass values through logarithmically small corrections only.

$$L = \left[2\,\mathrm{Tr}\sqrt{\hat{\Pi}^\dagger\hat{\Pi}}\right]_D + \left[(N_c - N_F) \times \right.$$
$$\left. \times \left(\frac{\Lambda_Q^{b_0}}{z_Q(\mu_{gl},\,\Lambda_Q)\det\hat{\Pi}}\right)^{1/(N_c-N_F)} + \hat{m}_Q \mathrm{Tr}\,\hat{\Pi}\right]_F =$$
$$= \left[\frac{2}{z_Q(\mu_{gl},\,\Lambda_Q)}\,\mathrm{Tr}\sqrt{\Pi^\dagger\Pi}\right]_D +$$
$$+ \left[(N_c - N_F)\left(\frac{\Lambda_Q^{b_0}}{\det\,\Pi}\right)^{1/(N_c-N_F)} + \right.$$
$$\left. + m_Q \mathrm{Tr}\,\Pi\right]_F, \quad \mu \ll \Lambda_{YM}. \quad (2.13)$$

The superpotential $(N_c - N_F)(\Lambda_Q^{b_0}/\det\,\Pi)^{1/\overline{N}_c}$ appeared many times in the literature because, up to an absolute normalization of the field $\Pi(\mu) = \overline{Q}Q(\mu)$ (which is not RG-invariant by itself), this is the only possible form of the superpotential if it can be shown that the lowest-energy Lagrangian depends on N_F^2 pion superfields only. But it seems that the absolute normalization of all terms entering (2.13) has never been carefully specified (clearly, the absolute normalization makes sense only when both the superpotential and the Kähler terms are absolutely normalized simultaneously). Lagrangian (2.13) describes weakly interacting pions with small masses $M_\pi = 2\hat{m}_Q = 2z_Q(\mu_{gl},\,\Lambda_Q)m_Q \ll m_Q \ll \Lambda_{YM} \ll \Lambda_Q$.

On the whole, the mass spectrum in this case contains $2N_cN_F - N_F^2$ massive gluons and "constituent quarks" with the mass scale $\mu_{gl} = g_H\mathcal{M}_0 \gg \Lambda_Q$, a large number of gluonia with the mass scale of the order of $\Lambda_{YM} \ll \Lambda_Q$, and N_F^2 pions with small masses $M_\pi = 2\hat{m}_Q = 2m_Q(\mu = \mu_{gl}) \ll \Lambda_{YM}$.

Form (2.13) can be continued in N_F to the point $N_F = N_c - 1$ and it then predicts the form of the pion Lagrangian in this case. The whole gauge group is now Higgsed at the high scale $\mu_H = \mu_{gl} \gg \Lambda_Q$, and the direct way to obtain (2.13) is not through the VY procedure but through the calculation of the one-instanton contribution [8] (see also [9] and the references therein). The changes in the mass spectrum are evident and, most important, there is then no confinement and no particles with masses of the order of Λ_{YM} in the spectrum.

3. DIRECT THEORY. CONFORMAL WINDOW $3N_c/2 < N_F < 3N_c$

The superconformal behavior means the absence of the scale Λ_Q in the physical mass spectrum. In other words, there are no particles with masses of the order of Λ_Q, with all quarks and gluons remaining effectively massless for $\mu_H \ll \mu \ll \Lambda_Q$, where μ_H is the highest physical mass scale. Therefore, "nothing particularly interesting" happens when decreasing the scale μ from $\mu \gg \Lambda_Q$ to $\mu_H \ll \mu < \Lambda_Q$. Only the character of running of the coupling $\alpha(\mu)$ and the quark renormalization factor $z_Q(\mu)$ change. The slow logarithmic evolution in the weak-coupling region $\mu \gg \Lambda_Q$ is replaced with freezing of $\alpha(\mu)$ at $\mu < \Lambda_Q$: $\alpha(\mu) \to \alpha^*$, while $z_Q(\mu)$ acquires a power-low behavior:

$$z_Q(\Lambda_Q, \mu) = (\mu/\Lambda_Q)^{b_0/N_F} < 1.$$

As a result, the Green's functions of chiral superfields also behave in a power-like fashion, with dynamical dimensions determined by their R-charges: $D = 3|R|/2$. This conformal regime continues until μ reaches the highest physical mass scale $\mu_H \ll \Lambda_Q$, and then the conformal behavior breaks down.

There are three characteristic scales at $\mu = \Lambda_Q$ in the direct theory: the current quark mass m_Q, the scale $\mathcal{M}_{ch}$ of its chiral vacuum condensate, and the scale Λ_{YM} of the gluino condensate. It follows from (2.5)–(2.7) that in the whole region $N_c < N_F < 3N_c$, there is an hierarchy:

$$m_Q \ll \Lambda_{YM} \ll \mathcal{M}_{ch} \quad \text{for} \quad N_c < N_F < 3N_c. \quad (3.1)$$

By itself, this hierarchy has no direct physical consequences, until it is realized that some physical masses stay behind the above quantities. We show below that within the dynamical scenario considered, the above inequalities reflect a real hierarchy of physical masses: m_Q is the mass of lightest pions, Λ_{YM} is the mass scale of gluonia, and $\mathcal{M}_{ch}$ is the dynamical constituent mass of quarks.

The main idea of the dynamical scenario for SQCD considered in this paper, with $N_c < N_F < 3N_c$ and small equal quark masses, is that this theory is in the collective coherent (DC) phase. This means that quarks do not condense alone, $\langle Q^i\rangle = \langle\overline{Q}_{\bar{j}}\rangle = 0$ (because there are too many flavors at $N_f > N_c$). In other words, the theory is not Higgsed by quarks, all gluons remain massless at scales $\mu \gg \Lambda_{YM}$, and color is confined. But quarks condense in colorless chiral pairs $(\overline{Q}_{\bar{j}}Q^i)$, and these pairs form the coherent condensate (like the quark–antiquark pairs in the Nambu–Jona-Lasinio model and, more importantly, like QCD). And as a result of this coherent condensation, quarks acquire a large (in comparison with their pole mass $m_Q^{pole} = m_Q(\mu = m_Q^{pole})$) dynamical constituent mass $\mu_C^2 = \langle\Pi_2\rangle = \langle(\overline{Q}Q)_{\mu=\mu_2}\rangle$, ($\mu_2 = \mu_C/$(several), $\mu_C =$

$= \mathcal{M}_{ch}$, see below). This constituent quark mass $\mu_C = \mathcal{M}_{ch}$ is the highest physical mass μ_H and it stops the massless perturbative RG evolution at scales $\mu < \mu_C$. Simultaneously, the light composite pions $\pi^{i}_{\bar j}$ are formed, with masses $M_\pi \sim m_2 = m_Q(\mu = \mu_2)$, ($m_2 = m_Q$, see below)[3].

All this occurs in the "threshold region" $\mu_2 = \mu_C/(\text{several}) < \mu < \mu_1 = (\text{several})\,\mu_C$ around the scale μ_C of the constituent quark mass. In other words, the nonperturbative effects operate in this threshold region, such that they "turn on" at $\mu = \mu_1$ and "saturate" at $\mu = \mu_2$.

If this idea is accepted, the proposed effective Lagrangian at the scale μ_2 has the form

$$L = \int d^2\theta\, d^2\overline{\theta}\Biggl\{\mathrm{Tr}\sqrt{\Pi_2^\dagger \Pi_2} + + Z_2 \mathrm{Tr}\left(Q_2^\dagger e^V Q_2 + \overline{Q}_2{}^\dagger e^{-V}\overline{Q}_2\right) + \ldots\Biggr\} + + \int d^2\theta \left(W_g + W_Q\right) + \mathrm{H.c.}, \quad (3.2)$$

where

$$W_g = -\frac{2\pi}{\alpha(\mu_2)}\, S, \quad S = W_\alpha^2/32\pi^2,$$

$$W_Q = \left(\frac{\det \Pi_2}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} \mathrm{Tr}\left(\overline{Q}_2\, \Pi_2^{-1}\, Q_2\right) - - N_F \left(\frac{\det \Pi_2}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + m_2\, \mathrm{Tr}\,\Pi_2,$$

$$Z_2 = \frac{\Lambda_0}{\mu_C} = \left(\frac{\mu_C}{\Lambda_Q}\right)^{b_0/\overline{N}_c} = \frac{m_Q}{\mathcal{M}_{ch}},$$

$$\Lambda_0 = \frac{1}{\langle \Pi_2 \rangle}\left(\frac{\det \langle \Pi_2\rangle}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c}, \quad \overline{N}_c = N_F - N_c,$$

the field $(\Pi_2)^{i}_{\bar j} = \left(\overline{Q}_{\bar j}\, Q^i\right)^{(light)}_{\mu=\mu_2}$ represents the dynamically generated "one-particle light part" of the composite field $(\Pi_2)^{i}_{\bar j} = \mu_C^2\left(\delta^{i}_{\bar j} + \pi^{i}_{\bar j}/\mu_C\right)$; it contains the c-number vacuum part $\mu_C^2\, \delta^{i}_{\bar j} = \langle (\Pi_2)^{i}_{\bar j}\rangle = \langle\, \overline{Q}_{2,\bar j}\, Q_2^i\,\rangle \equiv \langle\, \overline{Q}_{\bar j}\, Q^i\,\rangle_{\mu=\mu_2}$ and the quantum fields $\pi^{i}_{\bar j}/\mu_C$ of light pions. The canonically normalized quark fields $C_2 = Z_2^{1/2} Q_2$ and $\overline{C}_2 = Z_2^{1/2}\, \overline{Q}_2$ have no c-number vacuum parts, $\langle C\rangle = \langle \overline{C}\rangle = 0$, and are the quantum fields of heavy constituent quarks with the "field masses" $(\mu_C)^{\bar j}_{i}$ and c-number masses μ_C:

$$(\mu_C)^{\bar j}_{i} = \frac{1}{Z_2}\left(\frac{\det \Pi_2}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c}\left(\Pi_2^{-1}\right)^{\bar j}_{i}, \quad \langle (\mu_C)^{\bar j}_{i}\rangle = \delta^{\bar j}_{i}\, \mu_C. \quad (3.3)$$

The nonzero vacuum condensate $\langle \overline{C}_{2,\bar j} C_2^i\rangle = Z_2 \langle\, \overline{Q}_{2,\bar j}\, Q_2^i\,\rangle = Z_2\, \mu_C^2\, \delta^{i}_{\bar j} = \left(\langle S\rangle/\mu_C\right) \delta^{i}_{\bar j}$ of these heavy constituent quarks is a pure quantum effect from the one-loop triangle diagram with the constituent quark fields C_2 and $\overline{C}_2$ contracted into their massive propagators with the masses μ_C and emitting two external gluino lines; this contribution realizes the Konishi anomaly.

Besides, by definition, all effects of evolution through the threshold region are already taken into account in (3.2), and hence the quark terms in the Lagrangian are needed in practical calculations with the valence heavy quarks only. And finally, the dots in (3.2) indicate other possible D-terms, which are supposed to play no significant role in what follows.

To a large extent, the form of the Lagrangian in (3.2) is unique, once the main assumption about formation at the scale $\mu \sim \mu_C$ of massive constituent quarks with masses $\mu_C^2 = \langle \overline{Q}_2 Q_2\rangle = \langle \Pi_2\rangle$ and light pions with masses m_2 (and with all gluons remaining massless) is adopted. The only important nontrivial point may be the nonzero value of the coefficient $-N_F$ in front of the second term in the superpotential W_Q. It was determined from the requirement that, until quark and/or gauge degrees of freedom are integrated out, the vacuum value of the superpotential does not change, in comparison with its original value at higher scales $\mu \gg \mu_C$:

$$\langle W_Q\rangle = \sum_{flav} m_Q(\mu)\langle (Q\overline{Q})_\mu\rangle = N_F \langle S\rangle$$

(contributions of all three terms in W_Q in (3.2) to $\langle W_Q\rangle$ are equal to $N_F\langle S\rangle$ each, but the vacuum averages of the first and second terms in (3.2) cancel each other).

[3] This is unlike (our) QCD, where the value of the constituent quark mass μ_C is also determined by the coherent chiral quark condensate, $\mu_C^3 \sim \langle \psi\overline{\psi}\rangle$, but it is here $\mu_C \sim \Lambda_Q$, while $m_\pi \sim (m_Q\, \mu_C)^{1/2}$. The difference in the parametric dependence of m_π on the current quark mass m_Q between SQCD and QCD is because spin-1/2 quarks condense in QCD, and spin-zero quarks in SQCD. Besides, unlike the genuine spontaneous breaking of the chiral flavor symmetry in QCD with $\mu_C \sim \langle\psi\overline{\psi}\rangle^{1/3} \sim \Lambda_Q \neq 0$ as $m_Q \to 0$, $\mu_C \sim \langle \overline{Q}Q_{\mu=\Lambda_Q}\rangle^{1/2} = \mathcal{M}_{ch} \to 0$ as $m_Q \to 0$ in SQCD, see (2.7). Nevertheless, because the ratio $\mathcal{M}_{ch}/m_Q \gg 1$ is parametrically large at $m_Q \ll \Lambda_Q$, all qualitative features remain the same, and this can therefore be considered a "quasispontaneous breaking" of the chiral flavor symmetry.

The absolute value of $\langle \Pi_2 \rangle = \langle \overline{Q}_2 Q_2 \rangle$ can be determined from the Konishi anomaly:

$$\frac{1}{\langle \Pi_2 \rangle} \left(\frac{\det \langle \Pi_2 \rangle}{\Lambda_Q^{b_0}} \right)^{1/\overline{N}_c} \langle \overline{Q}_2 Q_2 \rangle = = \langle S \rangle = \left(\frac{\det \langle \Pi_2 \rangle}{\Lambda_Q^{b_0}} \right)^{1/\overline{N}_c}. \quad (3.4)$$

Together with $m_2 \langle \Pi_2 \rangle = \langle S \rangle$, Eq. (3.4) implies (see (2.5)–(2.7)) that $m_2 = m_Q \equiv m_Q(\mu = \Lambda_Q)$ and $\mu_C^2 = \langle \Pi_2 \rangle = \langle \overline{Q}_2 Q_2 \rangle = \langle \overline{Q} Q \rangle_{\mu = \Lambda_Q} \equiv \mathcal{M}_{ch}^2$ [4].

It is also useful to consider the evolution through the threshold region in more detail. At the scale $\mu = \mu_1$, there is no real distinction between the original light quarks $Q_1 = Q(\mu = \mu_1)$ and $\overline{Q}_1 = \overline{Q}(\mu = \mu_1)$ with the current masses $m_1 = m_Q(\mu = \mu_1)$ and the (heavy at scales $\mu < \mu_2$) constituent quarks $C_1 = C(\mu = \mu_1)$, $\overline{C}_1 = \overline{C}(\mu = \mu_1)$, because the large constituent quark mass μ_C "turns on and saturates" only after the evolution through the threshold region $\mu_2 < \mu < \mu_1$. Similarly, there is no real distinction between the light composite field $(Q\overline{Q})(\mu = \mu_1)$ with its mass scale of the order of m_1 and the pion field $\Pi_1 = \Pi(\mu = \mu_1)$ (this is the pion $\Pi_2 = \Pi(\mu = \mu_2)$ evolved back to $\mu = \mu_1$), with its mass m_2 at $\mu = \mu_2$ evolving back to the current quark mass m_1 at $\mu = \mu_1$. In essence, all these are the obvious matching conditions. They can also be used as an independent check that the form of W_Q in (3.2) is self-consistent. After evolving back from $\mu = \mu_2$ to $\mu = \mu_1$, the difference between the composite field $\overline{Q}Q$ of heavy constituent quarks and the field Π of the light pion disappears due to disappearance of the mass gap of the order of μ_C, such that two first terms in W_Q cancel each other, while the last term evolves back into the original quark mass term.

But then, at $\mu < \mu_1$, the colorless light composite pions and colored heavy constituent quarks evolve differently through the threshold region $\mu_2 \le \mu \le \mu_1$, and their Kähler terms acquire different renormalization factors. The renormalization factor Z_π of pions is as follows: from $\Pi_1 \sim (Q_1 \overline{Q}_1)$ with the mass m_1 at $\mu = \mu_1$ to $\Pi_2 = Z_\pi \Pi_1$, with the mass m_2 at $\mu = \mu_2$, i. e., $Z_\pi = m_1/m_2$. Similarly, the overall renormalization factor of quarks is from $({C_1}^\dagger C_1) \sim ({Q_1}^\dagger Q_1)$ with the mass m_1 at $\mu = \mu_1$ to $({C_2}^\dagger C_2) = Z_Q\,({C_1}^\dagger C_1)$, with the mass μ_C at $\mu = \mu_2$, i. e., $Z_Q = m_1/\mu_C$.

Independently of (3.4), the absolute values of m_2 (the parameter m_2 explicitly enters the lowest-energy Lagrangian and determines the observable pole masses of pions, $M_\pi \sim m_2$) and $\langle \Pi_2 \rangle = \mu_C^2$ can be obtained from the following reasoning. We rewrite, say, the second term in the quark superpotential in (3.2) in terms of the quark fields $(Q_1 \overline{Q}_1)$ normalized at $\mu = \mu_1$ and then, once more, in terms of $(Q_\mu \overline{Q}_\mu)$ normalized at the running $\mu > \mu_1$ [5]:

$$\left(\frac{\det \Pi_2}{\Lambda_Q^{b_0}} \right)^{1/\overline{N}_c} = Z_\pi^{N_F/\overline{N}_c} \left(\frac{\det (Q_1 \overline{Q}_1)}{\Lambda_Q^{b_0}} \right)^{1/\overline{N}_c} = \left(Z_\pi z_Q(\mu, \mu_1) \right)^{N_F/\overline{N}_c} \left(\frac{\det (Q_\mu \overline{Q}_\mu)}{\Lambda_Q^{b_0}} \right)^{1/\overline{N}_c} = \left(Z_\pi z_Q(\Lambda_Q, \mu_1) \right)^{N_F/\overline{N}_c} \times \left(\frac{\det (Q_{\Lambda_Q} \overline{Q}_{\Lambda_Q})}{\Lambda_Q^{b_0}} \right)^{1/\overline{N}_c}. \quad (3.5)$$

Clearly, for $\mu_1 \le \mu \le \Lambda_Q$, the coefficient in front of the field $(Q_\mu \overline{Q}_\mu)$ depends explicitly on the running scale μ through the quark perturbative renormalization factor $z_Q(\mu, \mu_1)$, while Z_π is independent of μ. Hence, to find the value of Z_π, we have to fix the normalization at some definite value of μ. The only distinguished point is $\mu = \Lambda_Q$, in the sense that this term in the superpotential, being expressed though the fields $(Q_{\mu=\Lambda_Q} \overline{Q}_{\mu=\Lambda_Q})$ normalized at Λ_Q, should have the coefficient that depends on Λ_Q only. From this, it follows that

$$Z_\pi = \frac{m_1 = m_Q(\mu = \mu_1)}{m_2} = z_Q^{-1}(\Lambda_Q, \mu_1) \equiv z_Q^{-1} \gg 1, \quad \mu_1 \sim \mu_C \ll \Lambda_Q,$$

$$\mu_C^2 = \langle \Pi_2 \rangle = \langle \overline{Q}_2 Q_2 \rangle = \langle \, \overline{Q} Q \, \rangle_{\mu = \Lambda_Q} \equiv \mathcal{M}_{ch}^2, \quad m_2 = m_Q(\mu = \Lambda_Q) \equiv m_Q, \quad (3.6)$$

[4] It is worth noting that the concrete form of the Kähler term K_π of quantum pion fields π^i_j in (3.2) should not be taken literally. Its only purpose is to show a typical scale of this Kähler term. For instance, it can be replaced with the contribution of the order of $\mathrm{Tr}\,({\mu_C}^\dagger \mu_C)$ from the loop of constituent quarks, where the field $(\mu_C)^{\overline{j}}_i$ is given in (3.3). Finally, to determine the values of pion masses up to nonparametrical factors of the order of 1, it is only important that both these forms of the pion Kähler term have the same scale $\langle K_\pi \rangle \sim \mathcal{M}_{ch}^2$. For similar reasons, we neglect a possible additional dependence of Z_2-factors entering the Kähler term of the constituent quark in (3.2) on the quantum pion fields $\pi/\mathcal{M}_{ch}$.

[5] It is worth noting that this is only a change of notation, not a real evolution to another scale.

$$Z_2 = \frac{\Lambda_0}{\mu_C} = \left(\frac{\mathcal{M}_{ch}}{\Lambda_Q}\right)^{b_0/\overline{N}_c} = \frac{m_Q}{\mathcal{M}_{ch}},$$

$$Z_Q = \frac{m_1}{\mu_C} = \frac{m_2}{\mu_C}\frac{m_1}{m_2} = \frac{m_Q}{\mathcal{M}_{ch}}\frac{m_1}{m_2} = Z_2\, Z_\pi = Z_2 z_Q^{-1},$$

where $z_Q(\Lambda_Q,\,\mu=\mu_1) \ll 1$ is the standard perturbative renormalization factor of the massless quark describing its evolution from $\mu=\Lambda_Q$ down to $\mu=\mu_1$ (in the conformal window, it is known explicitly: $z_Q = z_Q(\Lambda_Q,\,\mu_1) = (\mu_1/\Lambda_Q)^{b_0/N_F} \ll 1$).

On the whole, the evolution of the current quark mass in the interval $\mu_2 \le \mu \le \Lambda_Q$ looks as follows. At $\mu=\Lambda_Q$, the current quark mass is $m_Q \equiv \equiv m_Q(\mu=\Lambda_Q)$. At smaller $\mu_1 < \mu < \Lambda_Q$, it runs with the perturbative $z_Q(\Lambda_Q,\,\mu) = (\mu/\Lambda_Q)^{b_0/N_F} \ll 1$ factor, $m_Q(\mu) = z_Q^{-1}(\Lambda_Q,\,\mu) m_Q \gg m_Q$, such that $m_1 \equiv m_Q(\mu=\mu_1) = z_Q^{-1} m_Q$. In the threshold region $\mu_2<\mu<\mu_1$, it runs such that (at $\mu<\mu_1$, the current quark mass can be understood more properly as the pion mass) $m_1 \equiv m_Q(\mu=\mu_1) \to m_2 \equiv m_Q(\mu=\mu_2)$, $m_2 = Z_\pi^{-1} m_1$. And at $\mu \ll \mu_2$, the current quark mass m_2 does not run any more. With $Z_\pi = z_Q^{-1}$ from (3.6), it follows that evolving through the threshold region from $\mu=\mu_1$ down to $\mu=\mu_2$, the current quark mass returns to its value at $\mu=\Lambda_Q$: $m_2 = Z_\pi^{-1} m_1 = Z_\pi^{-1}(z_Q^{-1} m_Q) = m_Q$. As regards the constituent quark mass μ_C, it originates in the threshold region $\mu \sim \mu_C$ due to the existence of the coherent quark condensate, $\mu_C^2 = \langle \overline{Q}_2 Q_2\rangle = \mathcal{M}_{ch}^2$, and it stops the further RG evolution of the constituent quark and pion fields at $\mu < \mu_C = \mathcal{M}_{ch}$. The self-consistency of this scenario requires that $\mu_C = \mathcal{M}_{ch}$ be larger than m_Q^{pole}, because otherwise the massless conformal regime would stop earlier, at the point $\mu = m_Q^{pole}$, i. e., quarks would be in the heavy-quark (HQ) phase and the coherent quark condensate could not be formed in this case. In the case considered, with $3N_c/2 < N_F < 3N_c$,

$$\frac{m_Q^{pole}}{\Lambda_Q} \equiv \frac{m_Q(\mu=m_Q^{pole})}{\Lambda_Q} = \frac{m_Q}{\Lambda_Q}\left(\frac{\Lambda_Q}{m_Q^{pole}}\right)^{b_0/N_F} = \\ = \left(\frac{m_Q}{\Lambda_Q}\right)^{N_F/3N_c} = \frac{\Lambda_{YM}}{\Lambda_Q} \ll \frac{\mu_C}{\Lambda_Q} = \frac{\mathcal{M}_{ch}}{\Lambda_Q},$$

which is therefore self-consistent.

We now dwell on the evolution of the Wilson coupling $\alpha_W(\mu)$ in the interval $\mu_2 < \mu < \Lambda_Q$. We first recall its standard perturbative evolution in the interval $\mu_1 < \mu < \Lambda_Q$:

$$\delta\left(\frac{2\pi}{\alpha_W(\mu)}\right) = \left\{3N_c \ln\frac{\mu}{\Lambda_Q} - N_F \ln\frac{\mu}{\Lambda_Q}\right\} + \\ + \left\{N_F \ln\frac{1}{z_Q(\Lambda_Q,\,\mu)}\right\}, \quad (3.7)$$

where the first two terms are the one-loop contributions of massless gluons and quarks, and the last term describes higher-loop effects from massless quarks [5]. In the conformal window $3N_c/2 < N_F < 3N_c$, the explicit form of the perturbative quark renormalization factor $z_Q(\Lambda_Q,\,\mu)$ is known at $\mu < \Lambda_Q$: $z_Q(\Lambda_Q,\mu) = (\mu/\Lambda_Q)^{b_0/N_F} \ll 1$. Then, the above three parametrically large logarithmic terms in (3.7) cancel each other. This describes the standard effect that the perturbative coupling freezes in the conformal regime at $\alpha^* = O(1)$, i. e., it remains nearly the same as it was at $\mu=\Lambda_Q$, because $\alpha(\mu=\Lambda_Q)$ is already close to α^*, by the definition of Λ_Q.

This perturbative form (3.7) can be used down to $\mu > \mu_1$. Now, on account of additional contributions from the threshold region $\mu_2 < \mu < \mu_1$, the coupling $\alpha(\mu,\Lambda_L)$ at $\mu<\mu_2$ looks as follows (the number $2\pi/N_c\alpha(\mu=\Lambda_Q)$ is considered $O(1)$ and is neglected in comparison with the large logarithm):

$$\frac{2\pi}{\alpha_W(\mu<\mu_2,\Lambda_L)} = \left\{\frac{2\pi}{\alpha(\mu<\mu_2,\Lambda_L)} - \right. \\ \left. - N_c \ln\left(\frac{1}{g^2(\mu,\langle\Lambda_L\rangle)}\right)\right\} = \left\{3N_c \ln\frac{\mu}{\Lambda_Q} - \right. \\ \left. - \ln\left(\frac{\det(\mu_C)_i^{\overline{j}}}{\Lambda_Q^{N_F}}\right) + N_F\left(\ln\frac{1}{z_Q} + \ln\frac{1}{Z_Q}\right)\right\}. \quad (3.8)$$

Here, the first term in the curly brackets is due to contributions of massless gluons, and in the second term in the curly brackets, the one-loop term from colored quarks stops its evolution at their constituent mass $(\mu_C)_{\overline{j}}^{i}$ (see (3.3)), i. e., with surviving light pion fields $\pi_{\overline{j}}^{i}$ still living at lower energies; besides, in addition to the previous term $\ln(1/z_Q)$, $z_Q \equiv z_Q(\Lambda_Q,\mu_1)$, which describes the standard smooth perturbative evolution from $\mu=\Lambda_Q$ down to μ_1, the last term $\ln(1/Z_Q)$ appears due to the additional (nonstandard) evolution of the colored constituent quark in the threshold region $\mu_2 \le \mu \le \mu_1$.

Numerically (i. e., neglecting the quantum pion fields $\pi_{\overline{j}}^{i}/\mathcal{M}_{ch}$ and replacing $\det\Pi_2$ by its vacuum value $\mathcal{M}_{ch}^{2N_F}$), the first three terms in the right-hand side of (3.8) still cancel each other. Therefore, the parametrically large value of $1/\alpha_W(\mu < \mu_2)$ (i. e., the weak coupling) originates from the parametrically large $\ln(1/Z_Q)$ threshold contribution only. In other words, the strong evolution of the coupling $\alpha(\mu)$ in the threshold region $\mu_2 < \mu < \mu_1$ decreases it from the $O(1)$ value at $\mu = \mu_1$ to a logarithmically small value $\alpha(\mu_2) \sim \alpha_W(\mu_2) \sim 1/\ln(\Lambda_Q/\mathcal{M}_{ch})$ at $\mu = \mu_2$.

Substituting the value of Z_Q from (3.6) and $\det(\mu_C)_i^{\overline{j}}$ from (3.3) in (3.8), we can finally write the Yang–Mills coupling as

$$\frac{2\pi}{\alpha_W(\mu<\mu_2)} = \left\{\frac{2\pi}{\alpha(\mu,\Lambda_L)} - N_c \ln\frac{1}{g^2(\mu,\langle\Lambda_L\rangle)}\right\} = 3N_c\ln\frac{\mu}{\Lambda_L},$$
$$\Lambda_L = \left(\frac{\det\Pi_2}{\Lambda_Q^{b_0}}\right)^{1/3\overline{N}_c}, \tag{3.9}$$
$$\Lambda_{YM} \equiv \langle\Lambda_L\rangle = \left(\frac{\mathcal{M}_{ch}^{2N_F}}{\Lambda_Q^{b_0}}\right)^{1/3\overline{N}_c}.$$

We emphasize (this is to become important in Sec. 7) that the explicit value of the quark perturbative renormalization factor $z_Q = z_Q(\Lambda_Q, \mu_1 \sim \mathcal{M}_{ch})$ is not actually needed to obtain (3.9), because z_Q cancels exactly in (3.8), independently of its explicit form (and Z_2 does as well).

Now, at lower scales $\mu < \mu_2$, if we are not interested in calculations with the valence quarks, the fields of heavy constituent quarks can be integrated out. Because quarks are confined, this leaves behind a large number of heavy quarkonia, both mesons and baryons, with masses $M_{meson} \sim \mathcal{M}_{ch}$ and $M_{baryon} \sim N_c\mathcal{M}_{ch}$, built from nonrelativistic (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}_{ch}$) constituent quarks with masses $\mu_C = \mathcal{M}_{ch}$. Indeed, the characteristic distance between the nonrelativistic quarks in a bound state is the Bohr radius $R_B \sim 1/p_B$, where p_B is the Bohr momentum, $p_B \sim \alpha(\overline{\mu} \approx p_B)\,\mathcal{M}_{ch}$. Supposing that $p_B \ll \mathcal{M}_{ch}$, this requires $\alpha(\overline{\mu} \ll \mathcal{M}_{ch}) \ll 1$. But indeed (see above), in this region $\Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}$, the coupling is already logarithmically small, $\alpha(\mu) \sim 1/\ln(\mu/\Lambda_{YM}) \ll 1$. Therefore, the nonrelativistic regime is self-consistent ($\alpha(\mu)$ becomes $O(1)$ only at much smaller distances $R_{ch} \sim 1/\mathcal{M}_{ch} \ll R_B$, while confinement effects begin to be important only at much larger distances $R_{conf} \sim 1/\Lambda_{YM} \gg R_B$).

This results in simply omitting all terms containing the quark fields in (3.2) (we recall that the quark loop contributions to the gauge coupling have already been taken into account in (3.8)). Besides, the pion fields Π_2 (and masses m_2) do not evolve any more at $\mu < \mu_2$, and therefore $\mathcal{M}_{ch}$ in ${\mathcal{M}_{ch}}^2 = \langle\Pi_2\rangle$ and m_2 become the low-energy constant observables at $\mu \ll \mathcal{M}_{ch}$ (the pion pole mass is of the order of m_2, and $\mathcal{M}_{ch} = \langle S\rangle/m_2$, or $\langle S\rangle$ itself, is related to the tension of BPS domain walls between different vacua [10]).Therefore, the only remaining evolution in the interval $\Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}$ is the standard (weak coupling) perturbative logarithmic evolution of massless gluons, and hence in this range of scales, the Lagrangian takes the form (from now on, to simplify the notation, we substitute $\Pi_2 \equiv \Pi$, and $m_2 = m_Q \equiv m_Q(\mu = \Lambda_Q)$; see also footnote 2 about the R-charge)

$$L = \int d^2\theta d^2\overline{\theta}\left\{\mathrm{Tr}\sqrt{\Pi^\dagger\Pi}\right\} + \int d^2\theta\left\{-\frac{2\pi}{\alpha(\mu,\Lambda_L)}S - N_F\left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + m_Q\mathrm{Tr}\,\Pi\right\}, \tag{3.10}$$

$$\Lambda_L = \left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/3\overline{N}_c}, \quad \Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}. \tag{3.11}$$

Lowering the scale μ to $\mu < \Lambda_{YM}$ and integrating out all gauge degrees of freedom except the field S itself (this leaves behind a large number of gluonia with masses $M_{gl} \sim \Lambda_{YM}$), we obtain the VY form

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{\mathrm{Tr}\sqrt{\Pi^\dagger\Pi}\right\} + (D \text{ terms of the field } S) + \int d^2\theta\left\{-N_c\,S\left(\ln\frac{S}{\Lambda_L^3} - 1\right) - N_F\left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + m_Q\,\mathrm{Tr}\,\Pi\right\}, \quad \mu < \Lambda_{YM}. \tag{3.12}$$

Finally, at lower energies $\mu \ll \Lambda_{YM}$, after integrating out the last gluonium field S (with its mass scale of the order of Λ_{YM}), we obtain the Lagrangian of pions

$$L = \int d^2\theta\, d^2\overline{\theta}\left\{\mathrm{Tr}\sqrt{\Pi^\dagger\Pi}\right\} + \int d^2\theta\left\{-\overline{N}_c\left(\frac{\det\Pi}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + m_Q\mathrm{Tr}\,\Pi\right\}, \quad \mu \ll \Lambda_{YM}. \tag{3.13}$$

This describes weakly interacting pions with the smallest masses $m_\pi \sim m_Q$. The vacuum value $\langle \Pi^i_j \rangle = \mathcal{M}^2_{ch}\, \delta^i_j$ remembers the scale $\mu_C = \mathcal{M}_{ch}$ at which the pions were formed and thus determines their "internal hardness", i.e., the scale up to which they behave as pointlike particles[6)].

This concludes our analysis of the direct theory in the conformal window.

4. DUAL THEORY. THE DEFINITION

The Lagrangian of the dual theory (at the scale $\mu \sim \Lambda_Q$) is taken in the form [1]

$$\overline{L} = \int d^2\theta\, d^2\overline{\theta} \left\{ \mathrm{Tr}\left(q^\dagger e^{\overline{V}} q + \overline{q}^\dagger e^{-\overline{V}} \overline{q} \right) + \right. \\ \left. + \frac{1}{(\mu_q')^2} \mathrm{Tr}\left(M^\dagger M \right) \right\} + \int d^2\theta \left\{ -\frac{2\pi}{\overline{\alpha}(\mu, \Lambda_q)}\, \overline{s} + \right. \\ \left. + \frac{1}{\mu_q} \mathrm{Tr}\left(\overline{q}\, M\, q \right) + \overline{m}_Q(\mu)\, \mathrm{Tr}\, M \right\} + \mathrm{H.c.}, \quad (4.1)$$

where $\overline{s} = \overline{w}^2_\alpha/32\pi^2$, $\overline{a}(\mu) = \overline{N}_c \overline{\alpha}(\mu)/2\pi$ is the running dual coupling (with its scale parameter Λ_q), $a_f(\mu) = N_F f^2(\mu)/4\pi$ is its running Yukawa coupling (with its scale parameter Λ_f) with $f(\mu = \Lambda_Q) \sim \mu_q'/\mu_q$, and $\overline{w}_\alpha$ is the dual gluon field strength. This theory has the exact $SU(\overline{N}_c = N_F - N_c)$ gauge symmetry, while in the chiral limit $\overline{m}_Q \to 0$, the global symmetries are the same as in the direct theory. Under these symmetries, the dual quarks and mesons M (mions) transform as

$$\begin{aligned} q &: (\overline{N}_F)^{fl}_L \times (0)^{fl}_R \times (N_c/\overline{N}_c)_B \times (N_c/N_F)_R, \\ \overline{q} &: (0)^{fl}_L \times (N_F)^{fl}_R \times (-N_c/\overline{N}_c)_B \times (N_c/N_F)_R, \quad (4.2) \\ M &: (N_F)^{fl}_L \times (\overline{N}_F)^{fl}_R \times (0)_B \times (2\overline{N}_c/N_F)_R. \end{aligned}$$

The mion fields M^i_j in (4.1) are defined as pointlike ones. This is unlike the pion fields Π^i_j of the direct theory, which appear as light pointlike fields only at energies below the scale of chiral flavor symmetry breaking, $\mu < \mu_C = \mathcal{M}_{ch}$. At higher scales $\mu \gg \mathcal{M}_{ch}$, strictly speaking, they cannot be used at all (or, at best, can be resolved as composite fields of two current quarks).

To match the parameters of the direct and dual theories (see below), the normalizations at $\mu = \Lambda_Q$ are taken as

$$\begin{aligned} \langle M^i_j \rangle_{\mu=\Lambda_Q} &= \mathcal{M}^2_{ch}\, \delta^i_j, \\ \overline{m}_Q(\mu = \Lambda_Q) &= m_Q(\mu = \Lambda_Q) \equiv m_Q. \end{aligned} \quad (4.3)$$

6) A short discussion of external anomalies (the 't Hooft triangles) is transferred to Appendix.

In addition, to match the values of gluino condensates, the scale parameter Λ_q has to be taken as [3]

$$\begin{aligned} \Lambda_q^{\overline{b}_0} &= (-1)^{\overline{N}_c} \mu_q^{N_F}/\Lambda_Q^{b_0} \to \langle S \rangle = \langle -\overline{s} \rangle, \\ \overline{b}_0 &= 3\overline{N}_c - N_F. \end{aligned} \quad (4.4)$$

5. DUAL THEORY WITH $\mu_q = \Lambda_Q$. CONFORMAL WINDOW

With $\mu_q = \Lambda_Q$, it follows that $|\Lambda_q| = \Lambda_Q$ (see (4.4)). In essence, this is the only natural value for μ_q from the viewpoint of the direct theory. For $\mu_q \ll \Lambda_Q$, the value of $|\Lambda_q|$ is either artificially small (at $N_F > 3N_c/2$), or artificially large (at $N_F < 3N_c/2$) (see (4.4)). At $\Lambda_f \sim |\Lambda_q| = \Lambda_Q$ ($\mu_q' \sim \mu_q$), the dual theory (which, self-consistent by itself, is considered to be in the UV-free logarithmic regime at $\mu \gg \Lambda_Q$, with $a_f(\mu) < \overline{a}(\mu)$ at $\mu \gg \Lambda_Q$), simultaneously with the direct one, enters the superconformal regime at $\mu \sim \Lambda_Q$, with frozen couplings $\overline{a}(\mu) \to \overline{a}^*$ and $a_f(\mu) \to a_f^*$. The dynamic dimensions of chiral superfields are here determined by their R-charges, $D = 3|R|/2$, such that, for instance, the distance dependence of the two-point correlators $\langle \{\overline{Q}_{\overline{j}} Q^i(x)\}^\dagger, \overline{Q}_{\overline{l}} Q^k(0) \rangle$ and $\langle \{M^i_j\}^\dagger(x), M^k_l(0) \rangle$ is the same [1]. In addition, all 't Hooft triangles are matched [1]. At present, no indication of possible differences between the direct and dual theories is known in this perturbative superconformal regime. We therefore pass to lower energies, where the physical scales originating from the chiral symmetry breaking begin to reveal themselves. What happens in the direct theory when reaching its highest physical scale $\mu_H \sim \mu_C = \mathcal{M}_{ch}$ was described in Sec. 3.

In the dual theory and in the case considered, the highest physical scale μ_H is determined by the constituent mass $\overline{\mu}_C$ of dual quarks, i.e., by the value of their coherent condensate $\mu_H = \overline{\mu}_C = |\langle \overline{q} q \rangle|^{1/2}_{\mu=\Lambda_Q} = (m_Q \Lambda_Q)^{1/2}$ because this value is parametrically larger in the conformal window $3N_c/2 < N_F < 3N_c$ than the pole mass m_q^{pole} of dual quarks ($m_q(\mu)$ is the running current mass of dual quarks, $m_q = m_q(\mu = \Lambda_Q) = \mathcal{M}^2_{ch}/\Lambda_Q$, $\gamma_q = \overline{b}_0/N_F = (3\overline{N}_c - N_F)/N_F$, $\Lambda_{YM} = (\Lambda_Q^{b_0} \det m_Q)^{1/3N_c}$):

$$\frac{\overline{\mu}_C}{\Lambda_Q} = \left(\frac{m_Q}{\Lambda_Q} \right)^{1/2} \gg \frac{m_q^{pole}}{\Lambda_Q},$$

$$\frac{m_q^{pole}}{\Lambda_Q} = \frac{m_q(\mu = m_q^{pole})}{\Lambda_Q} = \frac{\mathcal{M}^2_{ch}}{\Lambda_Q^2} \left(\frac{\Lambda_Q}{m_q^{pole}} \right)^{\gamma_q} = \frac{\Lambda_{YM}}{\Lambda_Q}.$$

This shows that similarly to the direct theory, the dual theory is also in the same (dual) DC phase here, with the appearance of N_F^2 dual pions $N_i^{\bar j}$ (nions) and the large constituent masses $\overline{\mu}_C = (m_Q\Lambda_Q)^{1/2}$ of dual quarks when μ crosses the corresponding threshold region: $\overline{\mu}_2 = \overline{\mu}_C/(\text{several}) \le \mu \le \overline{\mu}_1 = (\text{several})\,\overline{\mu}_C$. Similarly, all dual gluons also remain massless simultaneously. Therefore, the pattern of evolution through the threshold region is universal if either the direct or the dual theory is in the same DC phase. Hence, arguing as in Sec. 3 and making some simple substitutions of direct parameters by dual ones, we obtain the effective dual Lagrangian at $\mu = \overline{\mu}_2$ (with the meson and quark fields normalized at $\mu = \Lambda_Q$) in the form

$$\overline{L} = \int d^2\theta\, d^2\overline{\theta} \left\{ \frac{z_M}{\Lambda_Q^2} \operatorname{Tr}\left(M^\dagger M\right) + \right.$$
$$\left. + \operatorname{Tr}\sqrt{N^\dagger N} + \overline{Z}_2 \operatorname{Tr}\left(q^\dagger e^{\overline{V}} q + \overline{q}^\dagger e^{-\overline{V}}\overline{q}\right)\right\} +$$
$$+ \int d^2\theta \left\{ -\frac{2\pi}{\overline{\alpha}(\overline{\mu}_2)}\,\overline{s} + W_q \right\},$$
$$W_q = \frac{1}{\Lambda_Q}\operatorname{Tr}\left(MN\right) + m_Q \operatorname{Tr} M +$$
$$+ \left(\frac{\det N}{\Lambda_q^{\overline{b}_0}}\right)^{1/N_c} \left[\operatorname{Tr}\left(\overline{q}\,N^{-1} q\right) - N_F\right],$$
$$\overline{Z}_2 = \left(\frac{\overline{\mu}_C}{\Lambda_Q}\right)^{\overline{b}_0/N_c} = \left(\frac{m_q}{\overline{\mu}_C}\right), \quad (5.1)$$
$$\langle M_{\bar j}^{i}\rangle = \mathcal{M}_{ch}^2\,\delta_{\bar j}^{i},$$
$$\langle N_{i}^{\bar j}\rangle = \langle \overline{q}^{\bar j} q_i\rangle = -\overline{\mu}_C^2\,\delta_i^{\bar j} = -m_Q\Lambda_Q\,\delta_i^{\bar j},$$
$$m_q = \mathcal{M}_{ch}^2/\Lambda_Q.$$

The factor $z_M \equiv z_M(\Lambda_Q, \overline{\mu}_1) \gg 1$ in (5.1) is the standard perturbative renormalization factor of mion fields M in the interval $\overline{\mu}_1 < \mu < \Lambda_Q$ (the fields M and N and the dual quarks are frozen and do not evolve any more at $\mu < \overline{\mu}_2$; besides, like the gluon fields, the mion fields M have no nonstandard evolution in the threshold region; and finally, here and everywhere below, as in Secs. 2 and 3, we neglect the dependence of the renormalization factors z_M and $\overline{Z}_2$ on the quantum mion and nion fields $m/\mathcal{M}_{ch}$ and $n/\overline{\mu}_C$ because that would affect the particle mass values by nonparametric factors of the order of 1 only; see also footnote 4):

$$z_M \equiv z_M(\Lambda_Q, \overline{\mu}_1) = \left(\frac{\overline{\mu}_1}{\Lambda_Q}\right)^{\gamma_M} = \frac{1}{z_q^2},$$
$$z_q \equiv z_q(\Lambda_Q, \overline{\mu}_1) = \left(\frac{\overline{\mu}_1}{\Lambda_Q}\right)^{\overline{b}_0/N_F} \ll 1, \quad (5.2)$$

where z_q is the renormalization factor of the massless dual quarks due to the standard perturbative evolution from $\mu = \Lambda_Q$ down to $\overline{\mu}_1 = (\text{several})\,\overline{\mu}_C$.

Similarly to the direct theory, the factor $\overline{Z}_2$ in (5.1) is the overall renormalization factor of the dual quark due to its evolution from $\mu = \Lambda_Q$ down to $\mu = \overline{\mu}_2 = \overline{\mu}_C/(\text{several})$. It can be written in the form $\overline{Z}_2 = z_q\,\overline{Z}_q$, where z_q is due to the standard perturbative evolution in the interval $\overline{\mu}_1 < \mu < \Lambda_Q$ and $\overline{Z}_q$ is due to the additional nonstandard evolution in the threshold region $\overline{\mu}_2 = \overline{\mu}_C/(\text{several}) < \mu < \overline{\mu}_1 = (\text{several})\,\overline{\mu}_C$.

The heavy constituent dual quarks decouple at $\mu < \overline{\mu}_2$, and the mions M and nions N and the pure gauge $SU(\overline{N}_c)$ dual theory remain. For its inverse coupling $1/\overline{\alpha}(\mu)$, we obtain, similarly to the direct theory, that it increases from its frozen value $1/\overline{\alpha}^* = O(1)$ at $\mu = \overline{\mu}_1$ to a logarithmically large value at $\mu = \overline{\mu}_2$ due to the additional large renormalization factor $\overline{Z}_q$ of constituent dual quarks. The whole evolution from $\mu = |\Lambda_q|$ down to $\mu < \overline{\mu}_2$ results in the expression

$$\frac{2\pi}{\overline{\alpha}(\mu < \overline{\mu}_2, \overline{\Lambda}_L)} =$$
$$= \left\{ 3\overline{N}_c \ln\frac{\mu}{\Lambda_q} + \overline{N}_c \ln\frac{1}{\overline{g}^2(\mu, \langle\overline{\Lambda}_L\rangle)} \right\} -$$
$$- \left\{ \ln\left(\frac{\det\left(\overline{\mu}_C\right)^{i}_{\bar j}}{\Lambda_q^{N_F}}\right) - N_F \ln\frac{1}{\overline{Z}_2} \right\}, \quad (5.3)$$
$$\left(\overline{\mu}_C\right)^{i}_{\bar j} = \frac{1}{\overline{Z}_2}\left(\frac{\det N}{\Lambda_q^{\overline{b}_0}}\right)^{1/N_c}\left(N^{-1}\right)^{i}_{\bar j},$$

where $\left(\overline{\mu}_C\right)^{i}_{\bar j}$ is the constituent mass of dual quarks (see (5.1)). Therefore, it follows from (5.3) that the scale parameter $\overline{\Lambda}_L$ of $\overline{\alpha}(\mu, \overline{\Lambda}_L)$ is

$$\overline{\Lambda}_L = \left(\frac{\det N}{\Lambda_q^{\overline{b}_0}}\right)^{1/3N_c}, \quad |\langle\overline{\Lambda}_L\rangle| = \Lambda_{YM}. \quad (5.4)$$

Lowering the scale to $\mu < \Lambda_{YM}$ and integrating out all gauge degrees of freedom through the VY procedure, we obtain the lowest-energy Lagrangian

$$\overline{L} = \int d^2\theta\, d^2\overline{\theta} \left\{ \frac{z_M}{\Lambda_Q^2} \operatorname{Tr}\left(M^\dagger M \right) + \operatorname{Tr}\sqrt{N^\dagger N} \right\} + \\ + \int d^2\theta \left\{ \frac{1}{\Lambda_Q} \operatorname{Tr}\left(MN \right) + (\overline{N}_c - N_F) \left(\frac{\det N}{\Lambda_q^{\overline{b}_0}} \right)^{1/N_c} + \\ + m_Q \operatorname{Tr} M \right\}. \quad (5.5)$$

Substituting Λ_q from (4.4) and changing $N \to -N$, we can write the superpotential in the more convenient form

$$W = \frac{1}{\Lambda_Q} \operatorname{Tr}\left(-MN \right) + \\ + N_c \left(\frac{\det N}{\Lambda_Q^{\overline{b}_0}} \right)^{1/N_c} + m_Q \operatorname{Tr} M. \quad (5.6)$$

Therefore, the masses of mions M and nions N are (see (5.2))

$$\mu_M \sim \mu_N \sim \left(\frac{\overline{\mu}_C^2}{z_M} \right)^{1/2} \sim \left(\frac{m_Q \Lambda_Q}{z_M} \right)^{1/2} = \\ = \Lambda_Q \left(\frac{m_Q}{\Lambda_Q} \right)^{3\overline{N}_c/2N_F} \ll \Lambda_{YM}. \quad (5.7)$$

On the whole, the mass spectrum looks here as follows: a) there is a large number of hadrons made of nonrelativistic (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \overline{\mu}_C$) dual quarks, with their dynamic constituent masses $\overline{\mu}_C = (m_Q \Lambda_Q)^{1/2} \ll \Lambda_Q$; b) there is a large number of gluonia with their universal mass scale of the order of Λ_{YM}; c) the lightest are N_F^2 mions M and N_F^2 dual pions N (nions) with masses $\mu_M \sim \mu_N \sim \Lambda_Q \Big(m_Q/\Lambda_Q \Big)^{3\overline{N}_c/2N_F} \ll \Lambda_{YM}$.

Comparing the mass spectra of the direct and dual theories shows that they are very different.

6. DUAL THEORY WITH $\mu_q = \mathcal{M}_{ch}$. CONFORMAL WINDOW

We now consider the choice $\mu_q = \mathcal{M}_{ch}$ of parameters in (4.1). As is to be shown below, this choice results in a much more close similarity of the mass spectra of the direct and dual theories.

But we first note that in this case, it follows from (4.4) that $|\Lambda_q| = (\mathcal{M}_{ch}^{N_F}/\Lambda_Q^{b_0})^{1/\overline{b}_0} \ll \Lambda_Q$, i.e., the scale parameter of the dual gauge coupling $\overline{\alpha}(\mu, \Lambda_q)$ is parametrically smaller than those of the direct theory. Moreover, it is parametrically smaller than even $\mathcal{M}_{ch}$: $(|\Lambda_q|/\mathcal{M}_{ch}) = (\mathcal{M}_{ch}/\Lambda_Q)^{b_0/\overline{b}_0} \ll 1$. But this means that these two theories are clearly distinct in the perturbative interval $\mathcal{M}_{ch} < \mu < \Lambda_Q$. Indeed, the direct theory entered the perturbative conformal regime already at $\mu < \Lambda_Q$, and therefore its coupling is frozen at the value α^* and does not run.

As regards the dual theory, the most natural boundary condition at $\mu = \Lambda_Q$ is obtained by setting the scale factor Λ_f of the Yukawa coupling $\Lambda_f \sim \Lambda_q$, which allows self-consistently considering the dual theory UV free by itself (but nothing changes essentially at $\mu < \Lambda_Q$ with $\Lambda_f \sim \Lambda_Q$ either; the Yukawa coupling is then $O(1)$ at $\mu \sim \Lambda_Q$ and decreases logarithmically with decreasing $\mu < \Lambda_Q$; the problems with such a theory will arise in the region $\mu > \Lambda_Q$). With this choice,

$$a_f^{-1}(\mu = \Lambda_Q) = 2\pi/N_F \alpha_f(\mu = \Lambda_Q) \sim \\ \sim (\overline{a})^{-1}(\mu = \Lambda_Q) = 2\pi/\overline{N}_c \overline{\alpha}(\mu = \Lambda_Q) \approx \\ \approx \overline{b}_0 \ln(\Lambda_Q/\Lambda_q) \gg 1.$$

Then, with decreasing $\mu < \Lambda_Q$, both couplings of the dual theory increase logarithmically but remain $\ll 1$ for $|\Lambda_q| \ll \mathcal{M}_{ch} < \mu < \Lambda_Q$. Hence, the dual theory is in the weak-coupling logarithmic regime for $\mathcal{M}_{ch} \ll \mu \ll \Lambda_Q$. Therefore, while correlators of the direct theory already behave in a power-like fashion, those of the dual theory acquire only slowly varying logarithmic renormalization factors. (Indeed, with so small a value of $|\Lambda_q| \ll \mathcal{M}_{ch}$, the dual theory never enters the conformal regime, see below.) Unfortunately, this is the price for a better similarity of both theories at lower scales $\mu < \mathcal{M}_{ch}$[7].

The current mass of dual quarks is now $m_q = \mathcal{M}_{ch}$, and it is much larger than the scale of their condensate: $|\langle q\,\overline{q}\rangle|^{1/2} = (m_Q\, \mathcal{M}_{ch})^{1/2}$. Hence, they cannot be in the collective coherent condensate phase, because their quantum fields are short-range and fluctuate independently locally. Therefore, they can be treated simply as heavy quarks (because their mass $\mathcal{M}_{ch}$ is also much larger than $|\Lambda_q|$). (Their non-zero vacuum condensate is now a pure quantum effect induced by the one-loop triangle diagram: $\langle \overline{q}q(\mu = \mathcal{M}_{ch}) \rangle = \langle \overline{s} \rangle/\mathcal{M}_{ch}$, where $\langle \overline{s} \rangle$ is the vacuum condensate of dual gluinos and $\mathcal{M}_{ch} \gg \Lambda_{YM}$ is the large current mass of dual quarks. This realizes the Konishi anomaly.)

[7] From now on, to simplify all expressions, whenever the dual theory is in the weak-coupling perturbative logarithmic regime, we ignore the logarithmic renormalization factors z_q and z_M in calculations of mass spectra. In any case, because these nonleading effects from $z_q \neq 1$ and $z_M \neq 1$ are only logarithmic, taking them into account would not violate any power hierarchies and, besides, they are not of great importance for numerical values of masses.

At lower scales $\mu \ll \mathcal{M}_{ch}$, they can be integrated out directly as heavy particles. (Because the dual quarks are confined, this leaves behind a large number of mesons and baryons (with the mass scale of the order of $\mathcal{M}_{ch}$, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}_{ch}$) made of weakly interacting nonrelativistic heavy dual quarks with the current masses $\mathcal{M}_{ch}$.)

What remains then is the $SU(\overline{N}_c)$ Yang–Mills theory (plus the mions M) with the scale parameter $\overline{\Lambda}_L$ of its coupling $\overline{\alpha}(\mu)$:

$$\frac{2\pi}{\overline{\alpha}(\mu, \overline{\Lambda}_L)} = 3\overline{N}_c \ln \frac{\mu}{\overline{\Lambda}_L} + \overline{N}_c \ln \frac{1}{\overline{g}^2(\mu, \langle\overline{\Lambda}_L\rangle)}, \quad -\overline{\Lambda}_L^3 = (\det M/\Lambda_Q^{b_0})^{1/\overline{N}_c}, \quad |\langle\overline{\Lambda}_L\rangle| = \Lambda_{YM}. \tag{6.1}$$

Therefore, for $\Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}$, the effective dual Lagrangian takes the form (see footnote 7)

$$\overline{L} = \int d^2\theta\, d^2\overline{\theta} \left\{ \frac{1}{\mathcal{M}_{ch}^2} \mathrm{Tr}\left(M^\dagger M\right) \right\} + \int d^2\theta \left\{ -\frac{2\pi}{\overline{\alpha}(\mu, \overline{\Lambda}_L)}\, \overline{s} + m_Q \mathrm{Tr}\, M \right\}. \tag{6.2}$$

Finally, at scales $\mu < \Lambda_{YM}$, using the VY procedure for integrating dual gluons, we obtain the lowest-energy Lagrangian of mions:

$$\overline{L} = \int d^2\theta\, d^2\overline{\theta} \left\{ \frac{1}{\mathcal{M}_{ch}^2} \mathrm{Tr}\left(M^\dagger M\right) \right\} + \int d^2\theta \left\{ -\overline{N}_c \left(\frac{\det\, M}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + m_Q \mathrm{Tr}\, M \right\}, \quad \mu \ll \Lambda_{YM}. \tag{6.3}$$

This describes the mions M with masses of the order of m_Q, interacting weakly through the standard superpotential.

We compare the direct and dual theories in the case considered.

a) As was pointed out above, they are clearly different in the region $\mathcal{M}_{ch} < \mu < \Lambda_Q$.

b) There is a large number of colorless hadrons, heavy mesons (quasistable, decaying into light pions or mions) and baryons (those of the lowest mass at least being stable) in both theories, made of heavy nonrelativistic (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}_{ch}$) constituents. In the direct theory, these are the constituent quarks with the dynamically generated masses $\mu_C = \mathcal{M}_{ch}$, while in the dual theory, these are simply the dual quarks themselves with the same (but now current) masses $\mathcal{M}_{ch}$. It seems that the mesons are indistinguishable in both theories, but the baryons are different because they know about the number of colors and their masses are different:

$$M_{baryon} = N_c \mathcal{M}_{ch} \neq \overline{M}_{baryon} = \overline{N}_c \mathcal{M}_{ch}.$$

c) The remaining light particles in both theories for $\Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}$ are gauge particles, with respectively N_c and $\overline{N}_c$ colors, and pions (or mions). It is important that both the direct and the dual Yang–Mills theories are at weak couplings in this interval of scales, but have different numbers of colors. Therefore, they are clearly different here. For instance, we consider two-point correlators of the energy–momentum tensors in both theories. Because both gauge couplings are small and the contributions from pion or mion interactions are already power-suppressed at $\mu \ll \mathcal{M}_{ch}$, these correlators are dominated by the lowest-order one-loop diagrams. The contributions of pions and mions are the same, but the contributions of gauge particles are different because $N_c^2 \neq \overline{N}_c^2$.

d) There is a large number of (strongly coupled and quasistable due to decays into pions or mions) gluonia in both theories, all with masses determined the same scale Λ_{YM}. Hence, it seems, they look indistinguishable.

e) Finally, there are N_F^2 light pions (mions) with masses of the order of m_Q in both theories, weakly interacting at low energies $\mu \ll \Lambda_{YM}$ through the same universal chiral superpotential. Nevertheless, the interactions of pions and mions with gluons at $\Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}$ are different in (3.10) and (6.2).

On the whole, it follows that (with the logarithmic accuracy; see footnote 7) the mass spectra look very (but not completely) similar in both theories in this case. But in many other respects (see above), the direct and dual theories are clearly different.

7. THE REGION $N_c < N_F < 3N_c/2$

There are two possible ways to interpret the meaning of the Seiberg dual theories at $N_c < N_F < 3N_c/2$.

a) The first variant is similar to the one that is the only possibility in the conformal window $3N_c/2 < N_F < 3N_c$. That is the description of all light degrees of freedom of the direct theory in terms of massless quarks $Q, \overline{Q}$, and gluons remains adequate in the interval of scales $\mu_H \ll \mu \leq \Lambda_Q$, where $\mu_H \ll \Lambda_Q$ is the highest physical mass scale due to $m_Q \neq 0$, and there are no massive particles with masses of the order of Λ_Q in the spectrum at $m_Q \ll \Lambda_Q$. In comparison with the conformal behavior, the difference is not qualitative but only quantitative: the strong coupling does not approach the constant value α^* at $\mu \ll \Lambda_Q$ but continues to increase. Nevertheless, the nonperturbative contributions are power-suppressed until $\mu \gg \mu_H$, and the correct answers for all Green's functions follow by resummation of the standard perturbative series with massless quarks and gluons. The dual theory is then interpreted as a possible alternative but equivalent (weak coupling) description. This variant can be regarded as some formal "algebraic duality", i.e., something like "a generalized change of variables".

b) The second variant is qualitatively different (it is sometimes referred to as "confinement without chiral symmetry breaking", i.e., due only to $\Lambda_Q \neq 0$ as $m_Q \to 0$). It implies that, unlike in variant **a**, the nonperturbative contributions already become essential at $\mu \sim \Lambda_Q$, resulting in a high-scale confinement with the string tension $\sqrt{\sigma} \sim \Lambda_Q$ which binds direct quarks and gluons into colorless hadron states with masses of the order of Λ_Q. This can be understood, for instance, as follows. At N_F close to $3N_c$, the value of $a^* = N_c\alpha^*/2\pi$ is small. As N_F decreases, a^* increases and $a^* \approx 1$ for N_F close to $3N_c/2$. When $N_F < 3N_c/2$, the coupling $a(\mu)$ exceeds some critical value $a^{crit} = O(1)$ already at $\mu \sim \Lambda_Q$; it is therefore assumed that the theory is then in another phase. The strong nonperturbative confining gauge interactions begin to operate at the scale $\sim \Lambda_Q$, resulting in the appearance of a large number of colorless hadrons with masses of the order of Λ_Q. Hence, the use of old massless quark and gluon fields for the description of light degrees of freedom at $\mu \ll \Lambda_Q$ becomes totally inadequate. (This is especially visible at $N_F = N_c + 1$ where, for instance, the gauge degrees of freedom are not present at all amongst light ones in the dual theory.)

Instead, the new (special solitonic?) light degrees of freedom are formed at the scale of the order of Λ_Q as a result of these strong nonperturbative effects. These are the dual quarks and gluons and dual mesons M (mions), with their sizes of the order of $1/\Lambda_Q$ and the internal hardness scale $\sim \Lambda_Q$ (i.e., they appear pointlike at $\mu < \Lambda_Q$). These new light particles are described by fields of the dual theory. Hence, this variant **b** can be regarded as the "physical duality", in the sense that the dual theory is indeed the low-energy description of the original theory at $\mu < \Lambda_Q$.

We now present arguments against variant **b**. The above-described scenario of "confinement without chiral symmetry breaking" implies that even as $m_Q \to 0$, there is a large number of massive (with masses of the order of Λ_Q) colorless hadrons H_n in the spectrum, both nonchiral made of $(Q^\dagger, Q)$ or $(\overline{Q}^\dagger, \overline{Q})$ quarks and chiral made of $(\overline{Q}, Q)$ quarks.

For instance, we consider the action of the simplest colorless chiral superfield $\overline{Q}_{\bar{j}} Q^i$ on the vacuum state: $\overline{Q}_{\bar{j}} Q^i |0\rangle$ (or any other colorless spin-zero or higher-spin chiral superfield composed in some way from Q^i, $\overline{Q}_{\bar{j}}$, and the gauge field strength W_α, for instance, $(\overline{Q}_{\bar{j}} T^a Q^i)\, W^a_\alpha$). From the vacuum, this operator excites not only, say, the massless one-mion state $|M^{\bar{i}}_{\bar{j}}\rangle$ but also many one-particle states of massive chiral hadrons $|\Psi_n\rangle$. Let $\Psi^{\bar{i}}_{\bar{j}}$ be the regular chiral superfield of any such a hadron. Then in the effective Lagrangian describing the theory at the scale $\mu \sim \Lambda_Q$, there should be a superpotential term that describes the nonzero mass $\sim \Lambda_Q$ of this chiral hadron. But the standard regular term $\Lambda_Q \mathrm{Tr}(\Psi\,\Psi)$ is not allowed because it explicitly breaks the chiral flavor $SU(N_F)_L \times SU(N_F)_R$ symmetry (and R-charge), and it seems impossible to write the appropriate regular mass terms for massive chiral hadron superfields with masses of the order of Λ_Q in the superpotential as $m_Q \to 0$.

We could try to "improve" the situation by multiplying the regular chiral superfield $\Psi^{\bar{i}}_{\bar{j}}$ by the chiral superfields $(\overline{Q}_{\bar{j}} Q^i/\Lambda_Q^2)^{-1}$ and $(\det \overline{Q}_{\bar{j}} Q^i/\Lambda_Q^{2N_F})^{1/\Delta}$ to build up the term in the superpotential with appropriate quantum numbers, but all such terms are singular at $\langle 0|\overline{Q}_{\bar{j}} Q^i|0\rangle \to 0$, and hence all this would not result in obtaining the genuine regular mass term for this hadron. Trying to use the dual quark fields q and $\overline{q}$ together with Ψ does not help either because $\langle \overline{q} q\rangle \to 0$ as $m_Q \to 0$.

We could also consider variant **b** when the direct color is not confined. Then, the absence of the confinement at $m_Q = 0$ implies that the individual quarks Q^i and $\overline{Q}_{\bar{j}}$ would be present in the spectrum and would be massive, with masses $\sim \Lambda_Q$ (because there are no such light fields in the dual theory). And we would face the same problem that it is impossible to write the right regular mass term for these quarks in the superpotential.

From our standpoint, the absence of confinement is the only realistic variant in the chiral limit $m_Q = 0$, because (at least in SQCD) the strong coupling $a(\mu \sim \Lambda_Q) \gtrsim 1$ does not actually mean that the scale of confining forces is $\sim \Lambda_Q$ (in other words, that the string tension is $\sqrt{\sigma} \sim \Lambda_Q$). The underlying reason is that the role of the order parameter for the confinement is played not by Λ_Q itself but by the scale of the gluino condensate, i.e., $\sqrt{\sigma} \sim \Lambda_{YM} \sim \langle\lambda\lambda\rangle^{1/3}$. But $\langle\lambda\lambda\rangle \to 0$ as $m_Q \to 0$. Hence, there is no confinement at all in the chiral limit $m_Q = 0$, and the regimes at $m_Q = 0$ and $N_c < N_F < 3N_c$ can be more adequately called the "pure perturbative massless regimes with neither confinement nor chiral symmetry breaking", down to $\mu \to 0$. They are the conformal regime at $3N_c/2 < N_F < 3N_c$, and the strong coupling regime at $N_c < N_F < 3N_c/2$ (see (7.4) below).

In other words, the appearance in the spectrum of massive chiral flavored (and R-charged) particles with masses of the order of Λ_Q as $m_Q \to 0$ seems impossible without the spontaneous breaking of $SU(N_F)_L \times SU(N_F)_R$ (and R-charge) symmetry.

If the symmetry is broken spontaneously, there should then be the appropriate noninvariant (elementary or composite) chiral superfield(s) ϕ_k that condense in the vacuum with a large value: $\langle 0|\phi_k|0\rangle = \phi_k^{(0)} \sim \Lambda_Q$. In principle, this condensate can then give, the masses of the order of Λ_Q to chiral hadron superfields. But this basic condensate $\phi_k^{(0)}$ should then occur explicitly in the low-energy Lagrangian, from which its numerical value in a chosen vacuum should be determined. The dual theory claims that it gives the correct description at low energies. But no large chiral vacuum condensate $\phi_k^{(0)} \sim \Lambda_Q$ appear either in the dual theory or in the direct one. We conclude that, indeed, the chiral flavor $SU(N_F)_L \times SU(N_F)_R$ and R-charge symmetries are not broken spontaneously at $m_Q \to 0$.

Hence, the above considerations imply that variant **b** is incompatible with unbroken $SU(N_F)_L \times SU(N_F)_R$ (and R-charge) symmetries at $m_Q/\Lambda_Q \to 0$.

Below, we therefore consider variant **a** only, in which the nonzero particle masses arise only because of the breaking of the $SU(N_F)_L \times SU(N_F)_R$ and R-charge symmetries due to $m_Q \neq 0$, and all these masses are much smaller than Λ_Q for $m_Q \ll \Lambda_Q$. Because the spectrum of light (i. e., with masses much smaller than Λ_Q) particles is known in this variant in both the direct and dual theories, it becomes possible, in addition to the 't Hooft triangles, to also compare the values of some special correlators in the perturbative range of energies where all particles can still be considered massless ($\mu_H \ll \mu \ll \Lambda_Q$, where μ_H is the highest physical scale due to $m_Q \neq 0$). These are the two-point correlators of external conserved currents, e. g., the baryon and $SU(N_F)$ flavor currents, because these can be computed in the perturbation theory even in the strong-coupling region. Actually, it is more convenient to couple these conserved currents to the external vector fields and to consider the corresponding external β_{ext} functions. Such β_{ext} functions have the form (see, e. g., [11])

$$\frac{d}{d\ln\mu}\,\frac{2\pi}{\alpha_{ext}} = \sum_i T_i\left(1+\gamma_i\right), \tag{7.1}$$

where the sum ranges over all fields that can be considered massless at a given scale μ, the unity in the brackets is due to one-loop contributions, and the anomalous dimensions γ_i of fields represent all higher-loop effects.

We then equate the values of such β_{ext} functions in the direct and dual theories at scales $\mu_H \ll \mu \ll \Lambda_Q$. The light particles in the direct theory are the original quarks Q, $\overline{Q}$, and gluons, and in the dual theory, these are the dual quarks q, $\overline{q}$, the dual gluons, and the mions M. For the baryon currents, we obtain

$$N_F N_c\left(B_Q = 1\right)^2(1+\gamma_Q) = \\ = N_F\overline{N}_c\left(B_q = \frac{N_c}{\overline{N}_c}\right)^2(1+\gamma_q), \tag{7.2}$$

and for the $SU(N_F)_L$ (or $SU(N_F)_R$) flavor currents,

$$N_c\,(1+\gamma_Q) = \overline{N}_c\,(1+\gamma_q) + N_F\,(1+\gamma_M). \tag{7.3}$$

Here, the left-hand sides are from the direct theory and the right-hand sides are from the dual theory, γ_Q is the anomalous dimension of the quark Q, and γ_q and γ_M are the anomalous dimensions of the dual quark q and the mion M.

Now, for $\mu_H \ll \mu \ll \Lambda_Q$, the dual theory is IR-free and both of its couplings are small in this range of energies, $\overline{a}(\mu) \ll 1$ and $a_f(\mu) \ll 1$. Hence, $\gamma_q(\mu) \ll 1$ and $\gamma_M(\mu) \ll 1$ are both also logarithmically small at $\mu \ll \Lambda_Q$. It then follows that (7.2) and (7.3) are incompatible with each other because they predict different values for the infrared limit of γ_Q. We conclude that both correlators cannot be equal simultaneously in the direct and dual theories, and hence these two theories are different.

Taking the IR value $\gamma_Q \to (N_c/\overline{N}_c - 1) = (2N_c - N_F)/(N_F - N_c)$ from (7.2) as a concrete example, and using the perturbative NSVZ β-function [5], we obtain the perturbative IR behavior of the strong coupling $\alpha(\mu)$:

$$\begin{gathered} \frac{da(\mu)}{d\ln\mu} \equiv \beta(a) = -\frac{a^2}{1-a}\,\frac{b_0 - N_F\gamma_Q}{N_c}, \\ a(\mu) \equiv \frac{N_c\alpha(\mu)}{2\pi}, \quad b_0 = 3N_c - N_F, \\ \gamma_Q \equiv \frac{d\ln z_Q}{d\ln\mu}, \quad z_Q(\Lambda_Q,\mu) = \left(\frac{\mu}{\Lambda_Q}\right)^{\gamma_Q} \ll 1, \\ \gamma_Q = \frac{2N_c - N_F}{N_F - N_c}, \quad a(\mu) = \left(\frac{\Lambda_Q}{\mu}\right)^{\nu} \gg 1, \\ \nu = \frac{3N_c - 2N_F}{N_F - N_c}, \quad \frac{\mu}{\Lambda_Q} \ll 1. \end{gathered} \tag{7.4}$$

In this case, the behavior of $a(\mu/\Lambda_Q)$ looks as follows. As $z = \mu/\Lambda_Q$ decreases from large values, $a(z)$ increases first in a standard way as $1/\ln z$. At $z = z_0 \sim 1$, $a(z)$ crosses unity. At this point, γ_Q crosses the value $b_0/N_F = (3N_c - N_f)/N_F$. As a result, the β-function is smooth, it has neither pole nor zero at this point and remains negative all the way from the UV region $z \gg 1$ to the IR region $z \ll 1$, while $a(z)$ increases in the infrared region in a power-like fashion (see (7.4)). On the other hand, it is not difficult to see that the IR value of γ_Q obtained from (7.3) with $\gamma_q \to 0$ and $\gamma_M \to 0$ is incompatible with the NSVZ β-function.

Nevertheless, it is interesting to compare the mass spectra of the direct and dual theories that reveal themselves at lower energies.

It was argued above that the qualitative properties of the direct theory do not differ much from those described for the conformal window. The main quantitative difference is that the gauge coupling $\alpha(\mu)$ does not freeze at $\mu \ll \Lambda_Q$ but continues to increase (for instance, as in (7.4)) until μ reaches the dynamic chiral symmetry breaking scale $\mu \sim \mu_C = \mathcal{M}_{ch}$. But after crossing the threshold region $\mu_2 = \mathcal{M}_{ch}/(\text{several}) < \mu < \mu_1 = (\text{several})\,\mathcal{M}_{ch}$, the coupling also becomes logarithmically small, and the effective Lagrangian has the same form as in (3.2). Indeed, as was emphasized in Sec. 3, this is independent of the explicit form of the quark perturbative renormalization factor $z_Q(\Lambda_Q, \mu_1)$ entering the evolution of the coupling $\alpha^{-1}(\mu)$ in the region $\mu_1 < \mu < \Lambda_Q$, because this last cancels in (3.8) independently of its explicit form. The only restriction is that the dynamic scenario has to be self-consistent, i. e., the constituent mass μ_C of quarks has to be larger than their perturbative pole mass, $\mu_C = \mathcal{M}_{ch} > m_Q^{pole}$, so as to stop the perturbative massless RG evolution before this is done by m_Q^{pole}. It is not difficult to verify that this is fulfilled with $\gamma_Q = (2N_c - N_F)/(N_F - N_c)$ in (7.2):

$$\frac{m_Q^{pole}}{\Lambda_Q} = \frac{m_Q}{\Lambda_Q}\left(\frac{\Lambda_Q}{m_Q^{pole}}\right)^{\gamma_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F-N_c)/N_c} \ll \ll \frac{\mathcal{M}_{ch}}{\Lambda_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F-N_c)/2N_c}.$$

Hence, below the threshold region $\mu < \mu_2$, all equations and all qualitative properties of the direct theory described above for the conformal window remain the same also in the region $N_c < N_F < 3N_c/2$.

As regards the dual theory, we also consider two variants for the scale parameter μ_q in (4.1), $\mu_q = \Lambda_Q$ and $\mu_q = \mathcal{M}_{ch}$.

1. $\mu_q' \sim \mu_q = \Lambda_Q$. In this case, the scale parameter Λ_q of the dual gauge coupling $\overline{\alpha}(\mu)$ is $|\Lambda_q| \sim \Lambda_f \sim \Lambda_Q$ (see (4.4)), both couplings $\overline{a}(\mu)$ and $a_f(\mu)$ are $\lesssim 1$ at $\mu = \Lambda_Q$ and both decrease logarithmically when μ decreases from $\mu \sim \Lambda_Q$ to $\mu_H \sim \mathcal{M}^2_{ch}/\Lambda_Q \ll \Lambda_Q$.

In the case considered, the current mass of dual quarks is (see footnote 7):

$$m_q = \langle M\rangle/\mu_q = \mathcal{M}^2_{ch}/\Lambda_Q, \quad m_q \gg |\langle\overline{q}\,q\rangle|^{1/2} = (m_Q\Lambda_Q)^{1/2}, \tag{7.5}$$

which is much larger than the scale of their condensate, and therefore the dual theory is here in the HQ phase described in Sec. 6. Therefore, at lower scales, all quarks can simply be integrated out as heavy (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}^2_{ch}/\Lambda_Q$) particles, leaving a large number of hadrons with masses of the order of $\mathcal{M}^2_{ch}/\Lambda_Q$ composed of nonrelativistic dual quarks. After this, we obtain the effective Lagrangian in the form

$$\overline{L} = \left\{\frac{1}{\Lambda_Q^2}\mathrm{Tr}\left(M^\dagger M\right)\right\}_D + \left\{-\overline{N}_c\,\overline{s}\left[3\ln\frac{\mu}{\overline{\Lambda}_L} + \ln\frac{1}{\overline{g}^2(\mu/\Lambda_{YM})}\right] + m_Q\mathrm{Tr}\,M\right\}_F, \tag{7.6}$$
$$\overline{\Lambda}_L^3 = -(\det M/\Lambda_Q^{b_0})^{1/\overline{N}_c},$$
$$|\langle\overline{\Lambda}_L\rangle| = \Lambda_{YM}, \quad \Lambda_{YM} \ll \mu \ll \mathcal{M}^2_{ch}/\Lambda_Q.$$

Going down in energy and integrating out all gluonia (with masses of the order of Λ_{YM}) via the VY procedure, we finally obtain:

$$\overline{L} = \left\{\frac{1}{\Lambda_Q^2}\,\mathrm{Tr}\left(M^\dagger M\right)\right\}_D + \left\{-\overline{N}_c\left(\frac{\det M}{\Lambda_Q^{b_0}}\right)^{1/\overline{N}_c} + m_Q\mathrm{Tr}\,M\right\}_F, \quad \mu \ll \Lambda_{YM}. \tag{7.7}$$

This describes the mions M with the masses

$$\mu_M \sim m_Q\left(\frac{\Lambda_Q^2}{\mathcal{M}^2_{ch}}\right) \sim m_Q\left(\frac{\Lambda_Q}{m_Q}\right)^{\overline{N}_c/N_c}, \quad m_Q \ll \mu_M \ll \Lambda_{YM}, \tag{7.8}$$

interacting weakly through the standard superpotential.

Thus, comparing the mass spectra of the direct and dual theories, we see that they are very different.

2. $\mu_q = \mathcal{M}_{ch}$. With $\Lambda_f \sim |\Lambda_q|$, both dual scale factors become very large with this choice of μ_q (see (4.4)):

$$|\Lambda_q| = \left(\frac{\Lambda_Q^{b_0}}{\mathcal{M}_{ch}^{N_F}}\right)^{(-1/\overline{b}_0)} \gg \Lambda_Q. \tag{7.9}$$

But we can ignore the high-energy region $\mu > |\Lambda_q|$ where the dual theory is strongly coupled, and start directly with $\mu \lesssim \Lambda_Q \ll |\Lambda_q|$, where both couplings are already logarithmically small: $2\pi/\alpha_f(\mu = \Lambda_Q) \sim \sim 2\pi/\overline{\alpha}(\mu = \Lambda_Q) \approx \overline{b}_0 \ln(|\Lambda_q|/\Lambda_Q) \gg 1$, and both continue to decrease logarithmically with decreasing μ for $\mathcal{M}_{ch} < \mu < |\Lambda_Q|$. The region $\mathcal{M}_{ch} < \mu < \Lambda_Q$ was discussed above (see (7.2) and (7.3)). We therefore consider $\mu < \mathcal{M}_{ch}$.

The regime in this case **2** is qualitatively the same as in case **1** (see also footnote 7), i. e., there are now even heavier dual quarks with the current mass $m_q = \mathcal{M}_{ch}$ (and even smaller condensate), the intermediate-mass gluonia and smallest-mass mions M. The Lagrangian in (7.6) and (7.7) remains essentially the same, only the factor $1/\Lambda_Q^2$ in the meson Kähler term is now replaced with $1/\mathcal{M}_{ch}^2$. Hence, the masses μ_M of mions become

$$m_q \sim \mathcal{M}_{ch} \gg M_{gl} \sim \Lambda_{YM} \gg \mu_M \sim m_Q. \qquad (7.10)$$

Thus, in this case, the mass spectra of the direct and dual theories (with the logarithmic accuracy) are much more similar, as it was in Sec. 6 in the conformal window. But all the differences (at scales $\mu < \mathcal{M}_{ch}$) described in Sec. 6 also remain.

The case $N_F = N_c + 1$. As regards the direct theory, this point is not special and all equations and results described above remain without changes[8]. But this point is somewhat special for the dual theory because its field content then amounts to light mesons $M^i_{\bar{j}}$ and baryons B_i, $\overline{B}^{\bar{j}}$ only [1].

The dual Lagrangian at $\mu \ll \Lambda_Q$ is supposed to have the form [1]

$$\overline{L} = \int d^2\theta\, d^2\overline{\theta} \left\{ \frac{M^\dagger M}{\mu_M^2} + \frac{B^\dagger B + \overline{B}^\dagger \overline{B}}{\mu_B^{2(N_c-1)}} \right\} + \int d^2\theta \left\{ \frac{Tr(\overline{B} M B) - \det M}{\Lambda_Q^{b_0}} + m_Q \mathrm{Tr}\, M \right\}, \qquad (7.11)$$
$$\mu \ll \Lambda_Q.$$

Here, the scale factors μ_M and μ_B in the Kähler terms are due to noncanonical dimensions of meson and baryon fields ($M \to \overline{Q}Q$, $B \to Q^{N_c}$).

In the interval of energies above the highest physical scale μ_H, $\mu_H \ll \mu \ll \Lambda_Q$, Eqs. (7.2) and (7.3) still hold, with the substitution $\overline{N}_c = 1$, $\gamma_q \to \gamma_B$, and γ_M, $\gamma_B \to 0$. They therefore remain incompatible.

[8] Truly special is the point $N_F = N_c$ because $\mathcal{M}_{ch} = \Lambda_Q$ in this case, even in the chiral limit $m_Q \to 0$ (see (2.7)). We do not consider this case here.

At lower energies, the meson and baryon masses can be obtained directly from Lagrangian (7.11):

$$M_M \sim m_Q \left(\frac{\mu_M}{\mathcal{M}_{ch}}\right)^2, \qquad M_B = M_{\overline{B}} \sim \frac{\mathcal{M}_{ch}^2\, \mu_B^{2(N_c-1)}}{\Lambda_Q^{b_0}}. \qquad (7.12)$$

Therefore, at $\mu_M \sim \mu_B \sim \Lambda_Q$, we have

$$M_M \sim \Lambda_Q \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_c-1)/N_c}, \quad M_B \sim \Lambda_Q \left(\frac{m_Q}{\Lambda_Q}\right)^{1/N_c}.$$

We recall (see above) that the mass spectrum of the direct theory consists here also of a large number of flavored hadrons with the mass scale $\sim \mathcal{M}_{ch} \sim \sim \Lambda_Q(m_Q/\Lambda_Q)^{1/2N_c}$, a large number of gluonia with masses $M_{gl} \sim \Lambda_{YM} \sim \Lambda_Q(m_Q/\Lambda_Q)^{(N_c+1)/3N_c}$, and N_F^2 light pions with masses of the order of m_Q.

8. THE REGION $N_F > 3N_c$

For completeness, we also consider this region. The direct theory is IR-free in this region ($b_0 < 0$) for $m_Q^{pole} < \mu < \Lambda_Q$, and in a sense, is therefore very "simple" at $\mu \gg \Lambda_{YM}$ (but at the price that it is now, at best, strongly coupled in the UV region $\mu \gg \Lambda_Q$ and, at worst, cannot be defined self-consistently in the UV range and needs an UV completion).

The current quark mass $m_Q = m_Q(\mu = \Lambda_Q) \ll \Lambda_Q$ is now much larger than the scale of its chiral condensate $\mathcal{M}_{ch} \ll m_Q$ (see (2.7)), and this power hierarchy persists at lower energies because the RG evolution is here only logarithmic for $\Lambda_{YM} \ll \mu < \Lambda_Q$. Therefore, the direct theory is at $N_F > 3N_c$ in the HQ phase, such that there is a standard weak-coupling slow logarithmic evolution in the region $m_Q^{pole} \ll \mu \ll \Lambda_Q$, $m_Q^{pole} \equiv \equiv m_Q(\mu = m_Q^{pole}) = z_Q^{-1}(\Lambda_Q, \mu = m_Q^{pole})\, m_Q \gg m_Q$, where $z_Q(\Lambda_Q, \mu = m_Q^{pole}) \ll 1$ is the standard perturbative logarithmic renormalization factor of massless quarks, and the highest physical scale is $\mu_H = m_Q^{pole} \gg \gg \Lambda_{YM} \gg \mathcal{M}_{ch}$. At $\mu \ll m_Q^{pole}$, all quarks can be integrated out as heavy (and weakly confined, the string tension being $\sqrt{\sigma} \sim \Lambda_{YM} \ll m_Q^{pole}$, and their vacuum condensate $\langle \overline{Q}Q(\mu = m_Q^{pole})\rangle = \langle S \rangle / m_Q^{pole}$ is due to a simple quantum one-loop contribution) nonrelativistic particles, leaving behind a large number of mesons and baryons made of these nonrelativistic quarks, with masses $M_{meson} \sim m_Q^{pole}$ and $M_{baryon} \sim N_c\, m_Q^{pole}$. Evidently, there are no additional lighter pions now.

Using (2.2) and (2.3) to match couplings at $\mu = m_Q^{pole}$, we obtain the Yang–Mills Lagrangian with the scale factor of its gauge coupling $\Lambda_{YM} = (\Lambda_Q^{b_0} m_Q^{N_F})^{1/3N_c} \ll m_Q$ at lower energies $\mu < m_Q^{pole}$, such that this Yang–Mills theory is in the weak-coupling regime at $\Lambda_{YM} \ll \mu < m_Q^{pole}$. It describes strongly coupled gluonia with masses $M_{gl} \sim \Lambda_{YM} \ll m_Q^{pole}$, and these are the lightest particles in this case. This concludes our brief analysis of the direct theory.

In the dual theory, as before, the mass spectrum depends on the value of μ_q.

8.1. Dual theory with $\mu'_q \sim \mu_q = \Lambda_Q$

We have $\Lambda_f \sim |\Lambda_q| \sim \Lambda_Q$ (see (4.4)), but there are no particles with masses of the order of Λ_Q, similarly to the case of the direct theory in Sec. 7. The dual theory is taken to be UV-free and it enters the strong coupling perturbative regime at $\mu_H < \mu < \Lambda_Q$. For definiteness, we use the values of the dual quark and mion anomalous dimensions in (7.2) and (7.3) with $\gamma_Q \to 0$ for $\mu \ll \Lambda_Q$:

$$\gamma_q = \frac{\overline{N}_c}{N_c} - 1, \quad \gamma_M = -\frac{\overline{N}_c}{N_c}. \tag{8.1}$$

Now, it follows that the dynamic constituent mass of dual quarks $\overline{\mu}_C$ is parametrically larger than their pole mass m_q^{pole}:

$$\overline{\mu}_C = (m_Q \Lambda_Q)^{1/2} \gg m_q^{pole},$$

$$m_q^{pole} = \frac{\mathcal{M}_{ch}^2}{\Lambda_Q}\Big(\frac{\Lambda_Q}{m_q^{pole}}\Big)^{\gamma_q} = \Lambda_Q\Big(\frac{\mathcal{M}_{ch}^2}{\Lambda_Q^2}\Big)^{1/(1+\gamma_q)} = m_Q.$$

Hence, $\mu_H = \overline{\mu}_C$ and the dual quarks are in the (dual) DC phase. The Lagrangian has the same form (5.1), all equations (5.3)–(5.6) remain the same and, instead of (5.7), the masses of mions and nions are now given by

$$\frac{\mu_M}{\Lambda_Q} \sim \frac{\mu_N}{\Lambda_Q} \sim \left(\frac{\overline{\mu}_C^2}{z_M \Lambda_Q^2}\right)^{1/2} = \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F+N_c)/4N_c}, \tag{8.2}$$

$$z_M = \Big(\frac{\overline{\mu}_C}{\Lambda_Q}\Big)^{\gamma_M} \gg 1.$$

On the whole, the mass spectrum of the dual theory includes: a) a large number of flavored hadrons with their mass scale $\sim \overline{\mu}_C$, made of dual quarks with the constituent masses $\overline{\mu}_C = (m_Q\Lambda_Q)^{1/2} \ll \Lambda_Q$; b) N_F^2 mions and N_F^2 nions with masses $\mu_M \sim \mu_N \sim \Lambda_Q\Big(m_Q/\Lambda_Q\Big)^{(N_F+N_c)/4N_c} \ll \overline{\mu}_C$; c) a large number of gluonia with the mass scale $\sim \Lambda_{YM} \ll \mu_M \sim \mu_N$.

8.2. Dual theory with $\mu_q = \mathcal{M}_{ch}$

With the choice $|\Lambda_q| \sim \Lambda_f$, both values are very small (see (4.4)). The dual theory is also taken to be UV-free at $\mu > |\Lambda_q|$ in this case, and it also enters the strong-coupling perturbative regime at $\mu_H < \mu < |\Lambda_q|$. The boundary conditions for the dual gauge coupling $\overline{a} = \overline{N}_c\overline{\alpha}/2\pi$ and the Yukawa coupling $a_f = N_F f^2/2\pi$ at $\mu = \Lambda_Q$ are $\overline{a}(\mu = \Lambda_Q) \sim a_f(\mu = \Lambda_Q) \sim 1/\ln(\Lambda_Q/|\Lambda_q|) \ll 1$. In the perturbative regions $\Lambda_{YM} \ll \mu \ll \Lambda_Q$ for the direct theory and $\Lambda_{YM} \ll |\Lambda_q| \ll \mu \ll \Lambda_Q$ for the dual one, both theories are now in the weak-coupling logarithmic regime. For the direct theory, this is so because it is IR-free at $|\Lambda_q| \ll m_Q^{pole} \ll \mu \ll \Lambda_Q$, while its coupling $a(\mu)$ increases logarithmically at $\Lambda_{YM} \ll \mu \ll m_Q^{pole}$ but is still small, and for the dual theory because $|\Lambda_q| \sim \Lambda_f$ are so small, and both its couplings $\overline{a}(\mu)$ and $a_f(\mu)$ increase logarithmically with decreasing $\mu < \Lambda_Q$ but still remain small at $\mu \gg |\Lambda_q|$. Hence, at $m_Q^{pole} \ll \mu \ll \Lambda_Q$, both the direct and dual theories are in the weak-coupling logarithmic perturbative massless regime, Eqs. (7.2) and (7.3) can be used with all γ_Q, γ_q, $\gamma_M \ll 1$, and they are incompatible.

For $\mu \ll |\Lambda_q|$, the dual theory is in the strong-coupling regime $\overline{a}(\mu) \gg 1$, $a_f(\mu) \gg 1$, and we use values (8.1) for the anomalous dimensions γ_q and γ_M.

The hierarchies in the dual theory at $\mu = \Lambda_Q \gg |\Lambda_q|$ are given by

$$m_q = \mathcal{M}_{ch} \ll \overline{\mu}_C = |\langle \overline{q}q\rangle|^{1/2} = (m_Q\mathcal{M}_{ch})^{1/2} \ll |\Lambda_q|,$$

$$\frac{|\Lambda_q|}{\Lambda_Q} = \Big(\frac{\mathcal{M}_{ch}}{\Lambda_Q}\Big)^{N_F/\overline{b}_0} \ll 1,$$

where m_q is the current quark mass and $\overline{\mu}_C$ is its (possible) constituent mass. The evolution in the interval $|\Lambda_q| < \mu < \Lambda_Q$ is only logarithmic (all logarithmic effects are neglected in what follows) and the hierarchies at $\mu \sim |\Lambda_q|$ remain the same. The dual quarks are in the DC phase with the constituent mass $\overline{\mu}_C = |\langle\overline{q}q\rangle|^{1/2} = (m_Q\mathcal{M}_{ch})^{1/2} \ll |\Lambda_q|$ if $\overline{\mu}_C \gg m_q^{pole}$. This condition is indeed satisfied (see (8.1)):

$$\frac{m_q^{pole}}{|\Lambda_q|} = \Big(\frac{m_q}{|\Lambda_q|}\Big)^{1/(1+\gamma_q)} = \Big(\frac{\overline{\mu}_C}{|\Lambda_q|}\Big)^2 \ll \frac{\overline{\mu}_C}{|\Lambda_q|} \ll 1,$$

$$\Lambda_{YM} \ll \overline{\mu}_C \ll |\Lambda_q|.$$

Therefore, the Lagrangian of mions and nions has form (5.5), (5.6), with the only replacement $\Lambda_Q \to \mathcal{M}_{ch}$ in the mion Kähler term and in the first term of the superpotential. Hence, instead of (8.2), the masses of mions and nions (with the logarithmic accuracy) are now given by

$$\frac{\mu_M}{|\Lambda_q|} \sim \frac{\mu_N}{|\Lambda_q|} \sim \left(\frac{\overline{\mu}_C^2}{z_M|\Lambda_q|^2}\right)^{1/2} = \left(\frac{\overline{\mu}_C}{|\Lambda_q|}\right)^{(N_F+N_c)/2N_c}, \quad z_M = \left(\frac{\overline{\mu}_C}{|\Lambda_q|}\right)^{\gamma_M} \gg 1. \tag{8.3}$$

8.3. Direct theory with $m_Q \gg \Lambda_Q$

We finish this section with a short discussion of a possible behavior of the direct theory in the case $m_Q \gg \Lambda_Q$. We then have to start with the UV region $\mu = M_0$, regarding this theory as the effective low-energy theory with the UV cutoff M_0.

We use (2.7) for $N_F > 3N_c$. It follows that the hierarchy of the standard scale parameters at $\mu = \Lambda_Q$ and $N_F > 3N_c$, $m_Q \gg \Lambda_Q$ remains the same as it was at $N_F < 3N_c$ and $m_Q \ll \Lambda_Q$, i.e., $\mathcal{M}_{ch} \gg \Lambda_{YM} \gg m_Q$. But what is actually the highest physical scale μ_H depends on a competition between $\mathcal{M}_{ch}$ and the quark pole mass m_Q^{pole}. The value of this last depends on the value of the quark anomalous dimension γ_Q. If $\mathcal{M}_{ch} > m_Q^{pole}$, then the theory is in the DC phase, and if $m_Q^{pole} > \mathcal{M}_{ch}$, then it is in the HQ phase.

For definiteness, we use the same value of γ_Q as in (7.2) with $\gamma_q \to 0$:

$$\gamma_Q = (2N_c - N_F)/(N_F - N_c) < 0 \quad \text{at} \quad N_F > 3N_c. \tag{8.4}$$

Then

$$\frac{m_Q^{pole}}{\Lambda_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{1/(1+\gamma_Q)} = \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F-N_c)/N_c} \gg \frac{\mathcal{M}_{ch}}{\Lambda_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{(N_F-N_c)/2N_c}, \quad \frac{m_Q}{\Lambda_Q} \gg 1. \tag{8.5}$$

Therefore, with this value of γ_Q, when going from high UV $\mu = M_0 \gg m_Q^{pole}$ to lower energies, the highest physical scale encountered is $\mu_H = m_Q^{pole}$. The quarks are in the HQ phase.

After integrating out all quarks as heavy ones, we are left with the pure Yang–Mills theory, but now in the strong-coupling regime, $a_- = N_c\alpha(\mu = m_Q^{pole})/2\pi \gg 1$. Hence, the matching of couplings at $\mu = m_Q^{pole}$ is now as follows. The coupling of the higher energy theory is (see (7.4))

$$a_+ = \left(\frac{m_Q^{pole}}{\Lambda_Q}\right)^{-\nu=(2N_F-3N_c)/(N_F-N_c)} = \left(\frac{m_Q}{\Lambda_Q}\right)^{(2N_F-3N_c)/N_c} \gg 1. \tag{8.6}$$

It follows from the perturbative NSVZ β-function [5] that the coupling of the lower-energy Yang–Mills theory in the strong-coupling regime is $a_-(\mu \gg \lambda_{YM}) = (\mu/\lambda_{YM})^3$. Therefore,

$$a_- = \left(\frac{m_Q^{pole}}{\lambda_{YM}}\right)^3 = a_+ \to \lambda_{YM} = \left(\Lambda_Q^{b_0} \det m_Q\right)^{1/3N_c} = \Lambda_{YM} \gg \Lambda_Q. \tag{8.7}$$

We now have the Yang–Mills theory in the strong-coupling perturbative regime at $\Lambda_{YM} \ll \mu < m_Q^{pole}$, with its coupling decreasing with μ as $a(\mu) = (\mu/\Lambda_{YM})^3$ until it becomes $O(1)$ at $\mu \sim \Lambda_{YM}$, where the nonperturbative effects become essential. Therefore, at $\mu < \Lambda_{YM}$, integrating all gauge degrees of freedom except the field $S \sim W_\alpha^2$ and using the VY form for the superpotential of S [6], we obtain the correct value of the gluino condensate $\langle S\rangle = \Lambda_{YM}^3$ (and a large number of gluonia with the mass scale $\sim \Lambda_{YM}$).

On the whole, the mass spectrum includes only two mass scales in this case: a large number of heavy flavored quarkonia with the mass scale $\sim m_Q^{pole} \gg \Lambda_{YM}$ and a large number of gluonia with the universal mass scale $\sim \Lambda_{YM} \gg \Lambda_Q$.

9. CONCLUSIONS

As was described above, within the dynamical scenario considered in this paper, the direct SQCD theory is in the DC (diquark-condensate) phase at $N_c < N_F < 3N_c$. In this case, its properties and the mass spectrum were described and compared with those of the dual theory. It was shown that the direct and dual theories are different, in general. The only region where no difference has been found, is the case where both theories are in the perturbative superconformal regime. All this can be significant in a wider aspect, as a hint that many of the various dualities considered in the literature can also be strictly valid, at best, in the superconformal regime only.

Here, we do not repeat the above-described results in detail. Instead, we compare the major features of SQCD and ordinary QCD. In many respects, the above-described properties of SQCD at $N_c < N_F < 3N_c$ resemble those of QCD[9]. For example, there is simultaneously confinement and chiral flavor symmetry breaking, with the formation of heavy constituent quarks and light pions. In addition, both theories have a large number of (quasi)stable heavy quarkonia and gluonia. The main difference is in the parametric dependence of different observable masses in the spectrum on the fundamental parameters of the Lagrangians: Λ_Q and the current quark masses $m_Q = m_Q(\mu = \Lambda_Q)$, when $m_Q \ll \Lambda_Q$.

a) The scale of the chiral symmetry breaking Λ_{ch} (and hence the masses of constituent quarks) is $\Lambda_{ch}^{QCD} \sim \Lambda_Q$ in QCD, while it is parametrically smaller in SQCD: $\Lambda_{ch}^{SQCD} \sim \mathcal{M}_{ch} = (\Lambda_Q^{b_0} m_Q^{\overline{N}_c})^{1/2N_c} \ll \Lambda_Q$.

b) The confinement scale (i.e., the string tension $\sqrt{\sigma}$) is $\Lambda_{conf}^{QCD} = (\sigma_{QCD})^{1/2} \sim \Lambda_Q \sim \Lambda_{ch}^{QCD}$ in QCD, while it is parametrically smaller than even Λ_{ch}^{SQCD} in SQCD: $\Lambda_{conf}^{SQCD} = (\sigma_{SQCD})^{1/2} \sim \Lambda_{YM} = (\Lambda_Q^{b_0} m_Q^{N_F})^{1/3N_c} \ll \Lambda_{ch}^{SQCD} \sim \mathcal{M}_{ch} \ll \Lambda_Q$.

c) Therefore, the masses of heavy quarkonia (meson and baryon) are also parametrically different: $M_{meson}^{QCD} \sim (\Lambda_{ch}^{QCD} + \Lambda_{conf}^{QCD}) \sim \Lambda_Q$, and $M_{baryon}^{QCD} \sim N_c \Lambda_{ch}^{QCD} \sim N_c \Lambda_Q$ in QCD, while $M_{meson}^{SQCD} \sim \mathcal{M}_{ch} \ll \Lambda_Q$, and $M_{baryon}^{SQCD} \sim N_c \mathcal{M}_{ch}$ in SQCD.

d) The masses of gluonia are $M_{gl}^{QCD} \sim \Lambda_{conf}^{QCD} \sim \Lambda_Q$ in QCD, and $M_{gl}^{SQCD} \sim \Lambda_{conf}^{SQCD} \sim \Lambda_{YM} \ll \mathcal{M}_{ch} \ll \Lambda_Q$ in SQCD.

e) The smallest pion masses are $M_\pi^{QCD} \sim (m_Q \Lambda_{ch}^{QCD})^{1/2} \sim (m_Q \Lambda_Q)^{1/2} \gg m_Q$ in QCD, while they are not of the order of $(m_Q \mathcal{M}_{ch})^{1/2}$, but $M_\pi^{SQCD} \sim m_Q$ in SQCD (this last difference is because the spin-1/2 quarks are condensed in QCD, while these are spin-zero quarks in SQCD).

We now briefly comment on the N_c-dependence of various quantities that appeared above. The standard N_c-counting rules predict that the gluino and quark condensates $\langle S \rangle$ and $\langle \overline{Q}^{\bar{j}} Q_i \rangle$ are not $O(1)$ at $N_c \gg 1$, $N_F/N_c = \text{const}$, as in the text, but N_c times larger, $O(N_c)$, and this agrees with explicit calculations, see, e.g., [9]. Besides, this can be seen from the example with $N_F < N_c$, when quarks are Higgsed (see Sec. 2). The gluon masses $\mu_{gl}^2 \sim \alpha(\mu = \mu_{gl}) \mathcal{M}_0^2$ are $O(1)$. Because $\alpha = O(1/N_c)$, $\mathcal{M}_0^2$ is $O(N_c)$ and $\langle S \rangle = \hat{m}_Q \mathcal{M}_0^2 = O(N_c)$[10]. The correct dependence on N_c can easily be restored throughout the text by simple substitutions, for instance, $\Lambda_Q^{b_0} \to N_c^{N_c} \Lambda_Q^{b_0}$ in (3.13), etc.

Finally, we comment about the spontaneously SUSY-breaking metastable local vacuum in SQCD with $N_c + 1 < N_F < 3N_c/2$, $m_Q \neq 0$, $m_Q \ll \Lambda_Q$. The arguments for the existence of such a state in the dual theory are presented in [12].

Recalling general arguments in Sec. 7 (see (7.2)–(7.4)); it is also worth recalling that these arguments are not connected with the use of the dynamic scenario with the diquark condensate) that the direct and dual theories are not equivalent in the infrared region, it becomes insufficient to show such a state in the dual theory, because this does not automatically imply that this state also exists in the direct theory. We therefore try to identify this state in the direct theory.

In terms of the direct-theory fields, this state is characterized by all N_F^2 components $\langle M_{\bar{j}}^i \rangle = \langle \overline{Q}_{\bar{j}} Q^i \rangle = 0$, while $\langle B \rangle = \text{const} \cdot \langle b \rangle \neq 0$ (and $\langle \overline{B} \rangle$ the same), $B \to Q^{N_c}$, $b \to q^{\overline{N}_c}$. Unfortunately, no simple possibility for a local vacuum with these properties is seen in the direct theory. For instance, the dynamics underlying the appearance of the above basic nonzero baryon condensates looks obscure. If these baryon condensates were, for instance, due to Higgsed quarks $\langle Q^i \rangle = \langle \overline{Q}_{\bar{j}} \rangle \neq 0$, with $i, \bar{j} = 1, \ldots, N_c$, such that $\langle B \rangle = \langle \overline{B} \rangle \sim \langle Q^i \rangle^{N_c} \neq 0$, then no reason is seen for all components of $\langle M_{\bar{j}}^i \rangle = \langle \overline{Q}_{\bar{j}} Q^i \rangle$ to be exactly zero. Rather, $\langle M_{\bar{j}}^i \rangle$ with $i = \bar{j} = 1, \ldots, N_c$ is of the order of $\langle Q^i \rangle \langle \overline{Q}_i \rangle \neq 0$. Besides, looking at the Lagrangian in (3.2), we see that it becomes singular as $\mathcal{M}_{ch} \to 0$. Hence, it seems impossible that the local vacuum with the above properties can appear here.

However, this is not the whole story because (3.2) is a local Lagrangian, i.e., it is valid only locally in the field space, not too far from the genuine SUSY vacuum. This implies that in general, besides $M_{\bar{j}}^i$, additional fields can be involved to correctly describe the vicinity of the above metastable vacuum. We therefore make, in addition, an attempt from another side, using some specific properties of the above metastable state of the dual theory. We also consider the lightest excitations around this vacuum. As was argued in [12], all excita-

[9] QCD means here our QCD with $N_c = 3$ and with $N_F \approx 3$ light flavors.

[10] Connected with this, there is an inherent ambiguity in the VY procedure for the pure Yang–Mills theory: we can replace $\ln(\mu^3/\Lambda^3)$ with $\ln(S/C_0\Lambda^3) - 1$, where C_0 is some constant. The value $C_0 = 1$ was used everywhere in the text, while μ^3 is definitely N_c-independent, and therefore a better replacement is $\ln(\mu^3/\Lambda^3) \to \ln(S/N_c\Lambda^3) - 1$, resulting in $\langle S \rangle = N_c \Lambda^3$.

tions have masses of the order of $(m_Q \Lambda_Q)^{1/2}$, except for some massless modes of the baryon and $M^i_{\bar j} = (\overline{Q}_{\bar j} Q^i)$ fields (and the basic vacuum condensates of baryons). We therefore take the scale $\mu \ll (m_Q \Lambda_Q)^{1/2}$ and try to write by hand an effective superpotential made of these meson and baryon fields only. The simplest form is

$$W_{eff} = -\overline{N}_c \left\{ \frac{\det M - \mathrm{Tr}\,(\overline{B}\, M^{\overline{N}_c} B)}{\Lambda_Q^{b_0}} \right\}^{1/\overline{N}_c} + \\ + m_Q \mathrm{Tr}\, M. \quad (9.1)$$

For $\overline{N}_c \geq 2$, no possibility is seen to obtain a nonsingular expansion in quantum fluctuations around the state with $\langle M \rangle = 0$, $\langle B \rangle = \langle \overline{B} \rangle \neq 0$ from (9.1)[11]. Only the case $\overline{N}_c = 1$ is nonsingular in (9.1). But even in this case, it must then be shown how to obtain (9.1) starting with (2.1) and expanding self-consistently around this metastable vacuum. This is likely to be problematic.

Finally, the absence of the above metastable spontaneously SUSY-breaking state in the direct theory may be not so surprising if we recall all arguments given above that the direct and dual theories are not equivalent.

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APPENDIX

The purpose of this appendix is to briefly comment on a situation with anomalous divergences of external currents (the 't Hooft triangles) in SQCD, within the dynamic scenario considered in this paper.

In our ordinary QCD, at the scale $\mu_{ch} \sim \Lambda_Q$ and at $m_Q \to 0$, there is a genuine spontaneous breaking of the flavor symmetry: $SU(N_F)_L \times SU(N_F)_R \to \to SU(N_F)_{L+R}$, while the baryon symmetry $U(1)_B$ remains unbroken. Therefore, the quarks acquire the constituent masses $\mu_C \sim \mu_{ch}$ and decouple at $\mu < \mu_{ch}$ (together with all gluons, which acquire either electric or magnetic masses of the order of Λ_Q due to nonperturbative confining interactions, such that the lower-energy theory contains only $N_F^2 - 1$ light pions). If the quarks are exactly massless, the pions are also massless, but if the chiral symmetry $SU(N_F)_L \times SU(N_F)_R$ is explicitly broken down to $SU(N_F)_{L+R}$ by parametrically small quark masses $0 < m_Q \ll \Lambda_Q$, then the pions become the pseudo-Goldstone bosons with parametrically small masses $m_\pi \sim (m_Q \Lambda_Q)^{1/2} \ll \mu_{ch}$.

In SQCD with $N_F < N_c$ and with small explicit breaking of chiral flavor symmetry and R-charge by quark masses $0 < m_Q \ll \Lambda_Q$ (see Sec. 2), the scalar quarks are Higgsed at the high scale $\mu_{ch} = = \mu_{gl} \gg \Lambda_Q$ ($\mu_{gl} \approx \mathcal{M}_{ch}$, with the logarithmic accuracy) and acquire the large "constituent masses" $\mu_C = = \mu_{gl}$. The color symmetry $SU(N_c)$ is broken down to $SU(N_c - N_F)$, and the $2N_c N_F - N_F^2$ gluons become massive by absorbing the Goldstone bosons. Hence, all this can be considered as a quasispontaneous symmetry breaking $SU(N_c)_C \times SU(N_F)_L \times SU(N_F)_R \times U(1)_R \times \times U(1)_B \to SU(N_c - N_F)_C \times SU(N_F)_{C+L+R} \times U(1)_B$, because the "constituent masses" $\mu_C \sim \mathcal{M}_{ch}$ are parametrically larger than the pion masses $m_\pi \sim m_Q$ (with the logarithmic accuracy). As a result, there appear N_F^2 pseudo-Goldstone pions (together with their superpartners). Therefore, the lower-energy theory at $\mu < \mu_{gl}$ includes the superfields of light $(N_c - N_F)^2 - 1$ gluons and N_F^2 pions.

In SQCD with $N_F > N_c$ and $m_Q \ll \Lambda_Q$ (in the dynamic scenario considered in this paper), all quarks acquire the constituent masses $\mu_C = \mathcal{M}_{ch} \ll \Lambda_Q$ in the threshold region $\mu \sim \mu_{ch} = \mathcal{M}_{ch}$, and there appear N_F^2 light pions, while all gluons remain massless. This can also be regarded as the quasi-spontaneous symmetry breaking $SU(N_F)_L \times SU(N_F)_R \times U(1)_R \times \times U(1)_B \to SU(N_F)_{L+R} \times U(1)_B$, because the constituent quark masses μ_C are parametrically larger than the pion masses $m_\pi \sim m_Q \ll \mathcal{M}_{ch}$. The lower-energy theory at $\mu < \mathcal{M}_{ch}$ includes the superfields of light $N_c^2 - 1$ gluons and N_F^2 pions.

We now recall some important and well-known properties of the lower-energy theory at $\mu < \mu_{ch}$.

1) After integrating out all heavy fields (and all Fourier components of light fields with $k > \mu_{ch}$), the Lagrangian of the lower energy theory at $\mu < \mu_{ch}$ is local, just because all the integrated modes were hard (it is always implied that this integration is performed in a way that respects all symmetries).

2) The external global symmetries can be gauged by introducing external vector fields and adding the appropriate set of massless "leptons", such that all anomalous divergences of external currents originating from the quark–gluon sector are canceled by those originating from the lepton sector.

3) After all this, because the symmetry breaking in the quark–gluon sector was quasi-spontaneous, the lower-energy Lagrangian preserves all previous symmetries, both internal and external. Therefore, because nothing happens to leptons when crossing the scale

[11] Formally, we can multiply the first term in the right-hand side of (9.1) with a function $f(z)$, $z = \det M / \mathrm{Tr}\,(\overline{B}\, M^{\overline{N}_c} B)$, but this does not help avoid singularities.

$\mu = \mu_{ch}$, the anomalous divergences originating from the quark–gluon sector also remain the same [13].

Hence, there is no question of whether the lower-energy theory behaves properly under symmetry transformations, both internal and external, or whether the anomalous divergences of external currents originating from the quark–gluon sector remain the same in the lower-energy theory[12)] as they were in the higher-energy theory at $\mu > \mu_{ch}$ — this is automatic. The only relevant questions are: a) what is the explicit form of the lower energy Lagrangian? b) in what way, explicitly, the anomalous divergences of external currents originating from the quark–gluon sector are saturated by fields of the lower energy theory?

As regards the first question if the dynamics of the theory is under the full control, the explicit form of the lower-energy Lagrangian is obtained by the direct integration described above. As is well known, Wess–Zumino-like terms appear in addition to the "standard terms" [14, 15].

Now, a few words about the second question within the dynamic scenario for SQCD considered in this paper. First, as for pions, it is worth noting that because contributions of pion loops are power suppressed at scales $\mu < \mu_{ch}$, these loops give only small power corrections to the contributions of tree diagrams to the amplitudes with low-energy external pions and/or external gauge fields.

There appear one-pion terms $J_\nu^{ext} \sim iF_\pi \partial_\nu \pi + \ldots$ in those external currents that correspond to quasi-spontaneously broken generators, with the pion decay constant $F_\pi \sim \mathcal{M}_0$ for $N_F < N_c$ and $F_\pi \sim \mathcal{M}_{ch}$ for $N_c < N_F < 3N_c$. Besides, among many others, there occur the well-known terms of the order of $F_\pi^{-1}\mathrm{Tr}(\pi F_{\mu\nu}\tilde{F}_{\mu\nu})$ in the Wess–Zumino part of the Lagrangian (here, $F_{\mu\nu}$ is the field strength of the external vector fields, W_L or W_R bosons, or the R-photon A^R), with the appropriate coefficients. As a result, the anomalous divergences of all such currents are automatically saturated by the sum of three contributions: a) the one intermediate pion exchange; b) the direct contributions of fermionic pion superpartners to the triangles; and c) additional direct contributions of gluinos to the R and R^3 triangles.

For instance, for all $N_c < N_F < 3N_c$ (with the logarithmic accuracy for $N_F < N_c$), the decay width of the pion π_R into two photons is then given by

$$\Gamma(\pi_R \to 2\gamma) \sim \alpha_{ext}^2 m_\pi^3 / F_\pi^2 \sim \alpha_{ext}^2 \frac{m_Q^3}{\mathcal{M}_{ch}^2} \sim \\ \sim \alpha_{ext}^2 \Lambda_Q \left(\frac{m_Q}{\Lambda_Q}\right)^\Delta$$

with $\Delta = (4N_c - N_F)/N_c$.

Those external currents (e.g., the baryon one) that correspond to the unbroken generators do not contain the one-pion term (because there is no corresponding pion), and their anomalous divergences, like $\langle W_L|\partial_\nu J_\nu^B|W_L\rangle$, are then directly saturated by the point-like terms of the order of $(\epsilon_{\nu\lambda\sigma\tau} A_\nu^B W_\lambda^L \partial_\sigma W_\tau^L + \ldots)$ in the Wess–Zumino part of the Lagrangian.

We did not explicitly write the Wess–Zumino-like terms in the main text because this is not a simple matter to find their explicit form, and they are irrelevant for the main purpose of this paper, which is to calculate the mass spectrum of the theory.

12) That is, at scales $\mu_{expl} < \mu < \mu_{ch}$, where $\mu_{expl} \sim \sim m_\pi \sim m_Q \ll \mu_{ch}$ is the scale of the explicit global chiral symmetry breaking, because an explicitly broken global symmetry is incompatible with gauging this symmetry, and μ_{expl} can be neglected only at scales $\mu > \mu_{expl}$. Formally, to avoid this problem, we can replace the quark masses $m_Q(\mu)$ in (2.1) with the N_F^2 quantum fields $m_i^{\bar{j}}$ and add the term $(-N_c\Lambda_{YM}^3(m)) = -N_c(\Lambda_Q^{b_0}\det m)^{1/N_c}$ to the superpotential. By the dimensional counting, this term is irrelevant at high energies. The genuine global symmetry of the Lagrangian is then $SU(N_F)_L \times SU(N_F)_R \times U(1)_R \times U(1)_B$. Then, after integrating out all the N_F^2 pion fields $\Pi_{\bar{j}}^{i}$ as heavy ones at low energies $\mu < m_Q$, the $m_i^{\bar{j}}$ are massless fields with zero superpotential, and choosing the vacuum with $\langle m_i^{\bar{j}}\rangle = \delta_i^{\bar{j}} m_Q$, $0 < m_Q \ll \Lambda_Q$, we then have a genuine spontaneous breaking of global symmetries (in this theory by itself).

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