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CONDITIONAL PREPARATION OF $\chi^{(2)}$ MACROSCOPIC ENTANGLED STATES

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Two experimental arrangements consisting of coupled spontaneous parametric down-converters with type-I phase matching pumped simultaneously by a powerful optical field in a coherent state through the balanced beam splitter and linear optical elements are proposed for conditional preparation of macroscopic entangled states in output pumping modes of the studied system. Successful generation of the macroscopic entangled state in the pumping modes is unambiguously heralded by coincident detection of two photons in the generated signal and idler modes of the system. We calculate the amount of entanglement and success probabilities to observe the $\chi^{(2)}$ macroscopic entangled states in the total wave function. We show that the proposed schemes can be used to obtain a new type of macroscopic entangled states.

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1. INTRODUCTION

The theory of quantum computation promises to revolutionize the future of computer technology in factoring large integers [1] and combinational searches [2]. For quantum communication purposes, entangled states of light fields are of particular interest. Such states can also be used, for example, for quantum key distribution [3] and quantum teleportation [4]. The entangled states are useful for quantum processing, but they are hard to produce and tend to decohere fast. The spontaneous noncollinear parametric down-converter with type-II phase matching is well known to produce true entanglement along certain directions of propagation of the generated optical beams [5]. It is well known that at certain angles between the pump beam and the optical axis of the crystal, namely, along two intersection directions, the emitted light becomes unpolarized or entangled [5].

In recent years, the problem of physical production of entangled states has been intensively studied. But despite enormous progress in generating entangled states of photons [5], deterministic generation of the states remains an elusive entity [6]. Indeed, the majority of current experiments in optics is based on the use of spontaneous parametric down-conversion, which is inherently random. Consequently, we can determine whether a pair of photons has been generated only by postselection produced by detectors. In certain applications, for example, in testing Bell inequalities [7], the randomness of the generated pair is not essential. Nevertheless, the conditional preparation of entangled states is required in some applications of quantum information, for example, in experiments involving multiple photon pairs [8] and in construction of quantum controlled sign gate [9]. Therefore, it is important to study the problem of conditional preparation of entangled states by optical methods [10].

Here, we present the idea of conditional preparation of macroscopic entangled states in output pumping modes of the system of two spontaneous parametric down-converters with type-I phase matching (SPDCI) pumped simultaneously through a balanced beam splitter. We say that such states $\chi^{(2)}$ are macroscopic entangled states and use the symbol $\chi^{(2)}$ in regard to the second-order susceptibility of the crystal, to differentiate them from other macroscopic entangled states. Before we consider the problem of conditional preparation of the $\chi^{(2)}$ macroscopic entangled states, we develop a simplified theory of the SPDCI based on the three-mode Hamiltonian with a quantized pumping mode to take depletion of the pumping mode into account [11].

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Specifically, we propose two experimental setups both with and without beam splitters to project the output wave function of the coupled SPDCI onto one of the two $\chi^{(2)}$ macroscopic entangled states. In other words, we draw attention to nonclassical properties of the state in the pumping mode leaving the SPDCI. As was first noted in [11], the output state of the pumping mode can be squeezed after the SPDCI. In [11], numerical analysis has been done under the assumption of a large value of the product of the amplitude of the input coherent state and the coupling coefficient of the SPDCI. Here, we consider another possibility for the output pumping modes to manifest their nonclassical (more precisely, nonlocal) properties. We show that the produced states in the output modes are entangled and differ both from tensor product of two coherent states and from a vanishing state. We calculate both the probabilities for the macroscopic entangled states to be observed in the total wave function and the amount of the entanglement stored in the states.

We note that the generated $\chi^{(2)}$ entangled states resemble the well-known entangled Schrödinger cats formed by coherent states of light. A coherent field is a fundamental tool in quantum optics, and linear superposition of two coherent states may be considered as a realistic model of realizable macroscopic quantum systems [12]. Therefore, the entangled states of two coherent states are considered to be applicable in both quantum teleportation [13] and quantum computation [14]. The generation of the entangled Schrödinger cats requires the Kerr medium with high $\chi^{(3)}$ susceptibility [14]. But it is well known the $\chi^{(3)}$ susceptibility is weaker than the $\chi^{(2)}$ susceptibility ($\chi^{(2)} \gg \chi^{(3)}$). Thus, our proposal with the second-order susceptibility of the crystals allows realizing other more reliable resources of macroscopic entangled states, which can also be used in quantum information processing. Our analysis of the problem of conditional preparation of macroscopic entangled states by means of a medium with the $\chi^{(2)}$ susceptibility is simplified by use of special detectors discriminating between one- and multiphoton number states [15].

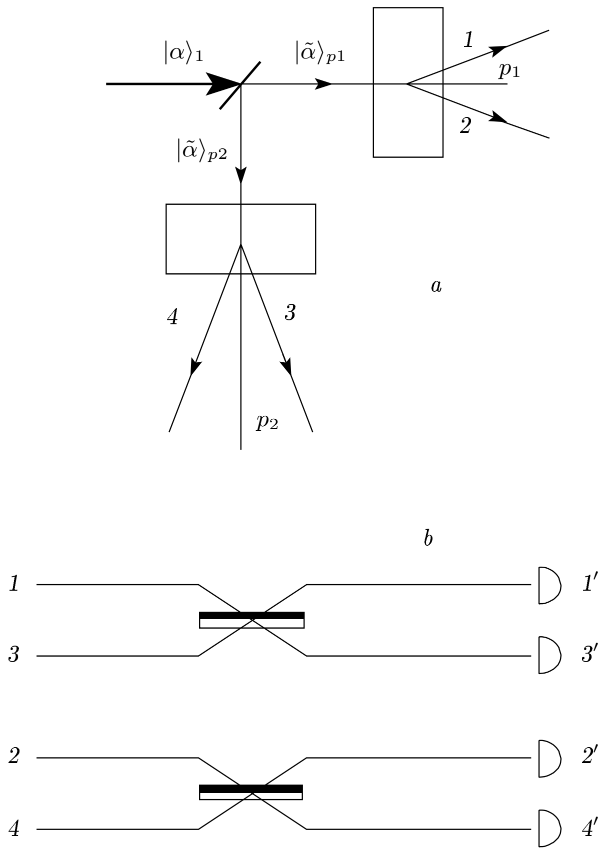

Fig. 1. *a* — Experimental arrangement to conditionally produce $\chi^{(2)}$ macroscopic entangled states in the pumping modes. The system involves two SPDCI simultaneously pumped by a powerful mode through the balanced beam splitter. *b* — The optical scheme with two Hadamard gates to distinguish the mode entangled states $|\Psi_1^{(1234)}\rangle$ and $|\Psi_2^{(1234)}\rangle$ from each other. Here, the white surface indicates the one from which a sign change occurs upon reflection. The modes *1*, *2*, *3*, and *4* are the input ones and the modes *1′*, *2′*, *3′*, and *4′* are the output ones

2. CONDITIONAL PREPARATION WITH THE HELP OF BEAM SPLITTERS

The proposed setup consists of two SPDCI (Fig. 1*a*) and a passive optical circuit (Fig. 1*b*) reducing the output state of the SPDCI to macroscopic entangled states. Before we start analyzing the optical projective systems in Fig. 1*b* and Fig. 2, we consider the system of two down-converted crystals pumped simultaneously by a powerful field in the coherent state as shown in Fig. 1*a*. We label the participating modes with the corresponding annihilation operators. Let $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, and $\hat{a}_4$ be the modes of down-converted photons. The operators $\hat{a}_1$ and $\hat{a}_2$ describe the modes of the first crystal (directions 1 and 2) and the operators $\hat{a}_3$ and $\hat{a}_4$ describe the modes of the second down-converted crystal (directions 3 and 4). The operators $\hat{a}_{p_1}$ and $\hat{a}_{p_2}$ are responsible for the modes of the powerful beams that simultaneously pump the first and second down-converted crystals through the balanced beam splitter. The responses of the two down-converted crystals in

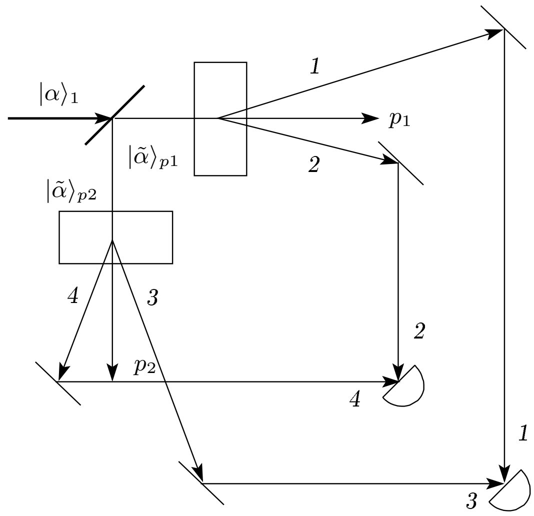

Fig. 2. The optical scheme is adjusted to conditionally produce $\chi^{(2)}$ macroscopic entangled states. A holographic scheme in auxiliary generated modes is used for a Bell-state measurement

Fig. 1*a* are considered to be identical to each other. If we take quantization and depletion of the pumping mode into account, neglect the multi-frequency structure of the pump, and use narrowband filters in order to choose only those generated modes that satisfy the phase matching condition, then the simplified three-mode Hamiltonian governing the down-converted processes in Fig. 1 *a* is given by [11]

$$H = H_1 + H_2 = \\ = \frac{i\hbar r}{2}\left(\hat{a}_1^+\hat{a}_2^+\hat{a}_{p_1} - \hat{a}_{p_1}^+\hat{a}_2\hat{a}_1 + \hat{a}_3^+\hat{a}_4^+\hat{a}_{p_2} - \hat{a}_{p_2}^+\hat{a}_4\hat{a}_3\right), \quad (1)$$

where the respective Hamiltonians H_1 and H_2 are referred to as the first (the first two terms) and the second (the two subsequent terms) SPDCI and the coupling coefficient r is related to the nonlinear second-order susceptibility tensor $\chi^{(2)}$. As the input condition to Hamiltonian (1), we take the coherent state $|\alpha\rangle_1|0\rangle_2$, where the subscripts «1» and «2» refer to the respective pumping modes. The state $|\alpha\rangle_1|0\rangle_2$ is transformed to the state

$$|\tilde{\Psi}_{in}\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{p_1}\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_{p_2}$$

after the balanced beam splitter, where the subscripts p_1 and p_2 refer to the first and second pumping modes. Applying $\pi/2$-phase shifter to the second pumping mode p_2, we get the state $|\alpha\rangle_{p_1}|\alpha\rangle_{p_2}$. The two SPDCI processes described by Hamiltonian (1) with the input condition

$$|\Psi_{in}\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{p_1}\left|\frac{\alpha}{\sqrt{2}}\right\rangle_{p_2}$$

are independent of each other and give rise to the output wave function

$$|\Psi_{out}\rangle = |\Psi_{12}\rangle|\Psi_{34}\rangle,$$

where $|\Psi_{12}\rangle$ and $|\Psi_{34}\rangle$ are the respective wave functions of the first and second SPDCI.

We now consider the output of one of the Hamiltonians (Eq. 1), for example, H_1. The wave function $|\Psi_{12}\rangle$ of the SPDCI is then given by

$$|\Psi_{12}\rangle = \sum_{n=0}^{\infty}|\Psi_{2n}\rangle, \quad (3a)$$

where the partial wave function $|\Psi_{2n}\rangle$ has the form

$$|\Psi_{2n}\rangle = \sum_{k=1}^{n+1} f_k^{(2n)}(s;\beta)|k-1\rangle_1|k-1\rangle_2|n-k+1\rangle_{p_1}. \quad (3b)$$

Here, the quantity $f_k^{(2n)}(s;\beta)$ is the wave amplitude of the corresponding tensor product of the photon number states in the first generated, second generated, and pumping modes. The wave amplitudes $f_k^{(2n)}(s;\beta)$ obey the system of linear differential equations [11]

$$\frac{df_k^{(2n)}}{ds} = \beta\left((k-1)\sqrt{n-k+2}\,f_{k-1}^{(2n)} - \\ - k\sqrt{n-k+1}\,f_{k+1}^{(2n)}\right), \quad (3c)$$

where $\beta = rL/2c$ is the coupling constant, $s = ct/L$ is the dimensionless distance along the crystal ($s \in [0;1]$), and L is the crystal length. Because we take the coherent state with the real amplitude α,

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

as the input to the SPDCI with the signal and the idler modes injected to the SPDCI in vacuum states, the input conditions

$$f_1^{(2n)}(0) = \exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}}, \quad f_k^{(2n)}(0) = 0, \\ k = 2, \ldots, n+1,$$

are imposed on the set of linear differential equations (3c). We note that the wave amplitudes take real values because we use a real value of the coherent state.

The output wave function $|\Psi_{12}\rangle$ of the first SPDCI (Eq. (3a)) can be rewritten as

$$|\Psi_{12}\rangle = \sum_{n=0}^{\infty} |n\rangle_1 |n\rangle_2 |\psi^{(n)}\rangle_{p_1}, \tag{4a}$$

where the partial wave functions $|\psi^{(n)}\rangle_{p_1}$ in the pumping mode are given by

$$|\psi^{(n)}\rangle_{p_1} = = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{m=0}^{\infty} \frac{\alpha^{m+n} f_{n+1}^{(2(m+n))}(\beta)}{\sqrt{(m+n)!}} |m\rangle_{p_1}, \tag{4b}$$

where the function $f_{n+1}^{(2(m+n))}(\beta)$ is the output wave amplitude,

$$f_{n+1}^{(2(m+n))}(\beta) = f_{n+1}^{(2(m+n))}(s=1;\beta).$$

We note that the wave functions $f_k^{(2n)}(s;\beta)$ used in Eq. (4b) evolve in accordance with Eqs. (3c), but now with the input conditions $f_1^{(2n)}(0) = 1$ and $f_k^{(2n)}(0) = 0$ for any numbers $k = 2, \dots, n+1$.

The treatment developed above allows writing the wave functions $|\Psi_{12}\rangle$ and $|\Psi_{34}\rangle$, each of which is a part of the output wave function $|\Psi_{out}\rangle$ (Eq. (2)), as

$$|\Psi_{12}\rangle = \sum_{n=0}^{\infty} |n\rangle_1 |n\rangle_2 |\psi^{(n)}\rangle_{p_1}, \tag{5a}$$

$$|\Psi_{34}\rangle = \sum_{n=0}^{\infty} |n\rangle_3 |n\rangle_4 |\psi^{(n)}\rangle_{p_2}, \tag{5b}$$

where $|\psi^{(n)}\rangle_{p_1}$ and $|\psi^{(n)}\rangle_{p_2}$ are the corresponding output states in the first and second pumping output ports. We use only those states in the output wave function $|\Psi_{out}\rangle$ (Eq. (2)) that have precisely two generated photons in the signal and idler modes. Such states can be identified by special detectors in the Bell-state measurement system shown in Fig. 1*b*. These detectors must be able to discriminate between a one-photon click and all other clicks caused by the states with the number of photons greater than one. In other words, the detectors must initiate different responses to one- and multi-photon states. Such detectors are known to exist only in a prototype form [15]. In other words, ideal detectors monitoring the generated modes and projecting them onto the one-photon Fock state $|1\rangle\langle 1|$ are supposed to be used. If we have the detectors that discriminate between one- and multi-photon number states, we can write the wave function of the system of coupled SPDCI as

$$|\Psi\rangle = \alpha\beta \Big(|1100\rangle_{1234} |\psi^{(1)}\rangle_{p_1} |\psi^{(0)}\rangle_{p_2} + +|0011\rangle_{1234} |\psi^{(0)}\rangle_{p_1} |\psi^{(1)}\rangle_{p_2} \Big), \tag{6a}$$

where now the states in the pumping modes are defined by

$$|\psi^{(0)}\rangle_{p_1,p_2} = \exp\left(-\frac{|\tilde{\alpha}|^2}{2}\right) \times \times \sum_{m=0}^{\infty} \frac{\tilde{\alpha}^m f_1^{(2m)}(\beta)}{\sqrt{m!}} |m\rangle_{p_1,p_2}, \tag{6b}$$

$$|\psi^{(1)}\rangle_{p_1,p_2} = \exp\left(-\frac{|\tilde{\alpha}|^2}{2}\right) \times \times \sum_{m=0}^{\infty} \frac{\tilde{\alpha}^m f_2^{(2(m+1))}(\beta)}{\beta\sqrt{(m+1)!}} |m\rangle_{p_1,p_2}, \tag{6c}$$

with $\tilde{\alpha} = \alpha/\sqrt{2}$. Henceforth, the subscripts of the states are related to the optical modes of photons [16]. For example, the state $|1100\rangle_{1234}$ in Eq. (6a) is a tensor product of four modes, where the modes 1 and 2 are occupied by two photons and the modes 3 and 4 have zero photons.

State (6a) is a superposition state consisting of two photons that simultaneously take four auxiliary modes (1–4) and two modified coherent states in the respective pumping modes p_1 and p_2. State (6a) can be rewritten as

$$|\Psi\rangle = \alpha\beta \Big(\sqrt{p_+}\, |\Psi_+^{(1234)}\rangle |\Delta_+^{(p_1,p_2)}\rangle + +\sqrt{p_-}\, |\Psi_-^{(1234)}\rangle |\Delta_-^{(p_1,p_2)}\rangle \Big), \tag{7}$$

where we introduce the normalized states for the signal and idler modes (modes 1–4) and the pumping modes (p_1 and p_2) as

$$|\Psi_\pm^{(1234)}\rangle = \frac{1}{\sqrt{2}} \left\{ |1100\rangle \pm |0011\rangle \right\}_{1234}, \tag{8a}$$

$$|\Delta_\pm^{(p_1,p_2)}\rangle = = \frac{1}{\sqrt{2p_\pm}} \Big\{ |\psi^{(1)}\rangle|\psi^{(0)}\rangle \pm |\psi^{(0)}\rangle|\psi^{(1)}\rangle \Big\}_{p_1 p_2}, \tag{8b}$$

and

$$p_\pm = \langle\psi^{(0)}|\psi^{(0)}\rangle\langle\psi^{(1)}|\psi^{(1)}\rangle \pm \left|\langle\psi^{(0)}|\psi^{(1)}\rangle\right|^2 \tag{8c}$$

are the normalized coefficients for the respective states $|\Delta_+^{(p_1p_2)}\rangle$ and $|\Delta_-^{(p_1p_2)}\rangle$. The states $|\Delta_\pm^{(p_1p_2)}\rangle$ (Eq. (8b)) can be expressed as

$$|\Delta_\pm^{(p_1p_2)}\rangle = N_\pm \left\{ |U\rangle|V\rangle \pm |V\rangle|U\rangle \right\}_{p_1p_2}, \tag{9a}$$

where the normalized wave functions $|U\rangle$ and $|V\rangle$ are given by

$$|U\rangle_{p_1,p_2} = \frac{1}{\sqrt{L_1}} |\psi^{(1)}\rangle_{p_1,p_2}, \tag{9b}$$

$$|V\rangle_{p_1,p_2} = \frac{1}{\sqrt{L_0}}\,|\psi^{(0)}\rangle_{p_1,p_2}, \tag{9c}$$

with the normalization coefficients

$$L_0 = \langle\psi^{(0)}|\psi^{(0)}\rangle, \quad L_1 = \langle\psi^{(1)}|\psi^{(1)}\rangle.$$

The normalization coefficient is given by

$$N_\pm = \frac{1}{\sqrt{2\,(1\pm|a|^2)}} = \frac{1}{\sqrt{2(1\pm M/N)}}\,, \tag{9d}$$

where

$$|a|^2 = |\langle U|V\rangle|^2$$

and the magnitudes N and M have the form

$$N = \langle\psi^{(1)}|\psi^{(1)}\rangle\langle\psi^{(0)}|\psi^{(0)}\rangle = \exp\left(-2|\tilde{\alpha}|^2\right) \times \times \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{|\tilde{\alpha}|^{2(m+k)} f_1^{(2m)2}(\beta) f_2^{(2(k+1))2}(\beta)}{\beta^2 m!(k+1)!}, \tag{9e}$$

$$M = \langle\psi^{(1)}|\psi^{(0)}\rangle\langle\psi^{(0)}|\psi^{(1)}\rangle = \exp\left(-2|\tilde{\alpha}|^2\right)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{|\tilde{\alpha}|^{2(m+k)} f_1^{(2m)}(\beta) f_1^{(2k)}(\beta) f_2^{(2(m+1))}(\beta) f_2^{(2(k+1))}(\beta)}{\beta^2\sqrt{m!k!(m+1)!(k+1)!}}. \tag{9f}$$

A note about the notation used in Eqs. (9e) and (9f) is in order. The expressions $f_1^{(2m)2}(\beta)$ and $f_2^{(2(k+1))2}(\beta)$ denote the quantities $f_1^{(2m)}(\beta)$ and $f_2^{(2(k+1))}(\beta)$ squared, and the expression $|\tilde{\alpha}|^{2(m+k)}$ is the amplitude $|\tilde{\alpha}|$ raised to the power $2(m+k)$.

We now consider conditional preparation of the $\chi^{(2)}$ macroscopic entangled states $|\Delta_\pm^{(p_1p_2)}\rangle$. For this, the auxiliary modes 1–4 must be subjected to the Bell-state measurement. For the conditional preparation of the $\chi^{(2)}$ macroscopic entangled states $|\Delta_\pm^{(p_1p_2)}\rangle$, we use the Bell-measurement setup presented in Fig. 1*b* to distinguish the mode-entangled states $|\Psi_+^{(1234)}\rangle$ and $|\Psi_-^{(1234)}\rangle$ from each other. According to Fig. 1*b*, the first and third beams are directed to the top beam splitter acting in Fig. 1*b* as an Hadamard gate,

$$H = \frac{(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|}{\sqrt{2}},$$

while the second and fourth beams are directed to the same lower Hadamard gate. Straightforward calculations show that

$$|\Psi_+^{(1234)}\rangle \to |\Psi_+^{(1'2'3'4')}\rangle = = \frac{1}{\sqrt{2}}\,\{|1100\rangle + |0011\rangle\}_{1'2'3'4'}\,, \tag{10a}$$

$$|\Psi_-^{(1234)}\rangle \to |\Psi_-^{(1'2'3'4')}\rangle = = \frac{1}{\sqrt{2}}\,\{|1001\rangle + |0110\rangle\}_{1'2'3'4'}\,, \tag{10b}$$

where the modes $1'$, $2'$, $3'$, and $4'$ are the output modes of the corresponding Hadamard gates in Fig. 1*b*. As can be seen from Eqs. (10a)–(10b), the states $|\Psi_\pm^{(1'2'3'4')}\rangle$ are identified by simultaneous clicks of different pairs of the registering detectors. As a consequence, the coincident detection of two photons by any pair of the detectors in Fig. 1*b* reduces the state $|\Psi\rangle$ (Eq. (7)) to either the state $|\Delta_+^{(p_1p_2)}\rangle$ with the success probability $\alpha^2\beta^2 p_+$ (Eq. (8c)) or the state $|\Delta_-^{(p_1p_2)}\rangle$ with the success probability $\alpha^2\beta^2 p_-$ (Eq. (8c)) depending on the outcome of the Bell-state measurement.

3. ASYMPTOTIC DECOMPOSITION OF THE WAVE AMPLITUDES

We now show that the macroscopic states $|\Delta_+^{(p_1p_2)}\rangle$ and $|\Delta_-^{(p_1p_2)}\rangle$ (Eqs. (8b)) are actually entangled states. For this, we are use the asymptotic decomposition of the wave amplitudes $f_k^{(2n)}(s;\beta)$ that directly follows from Eqs. (3c). Some particular analytic solutions for the output wave amplitudes $f_k^{(2n)}(\beta)$ with the number of photons $n = 1$–4 are presented in the Appendix. We note that the parameter β always takes very small values in real experiments. The smallness of the parameter $\beta \ll 1$ allows decomposing the wave amplitudes $f_k^{(2n)}(\beta)$ into asymptotic series in β. We first restrict the decomposition of the wave amplitudes by the first term. We then have from Eq. (3c) that

$$f_1^{(2n)}(s;\beta) \approx 1, \quad f_2^{(2n)}(s;\beta) \approx s\beta\sqrt{n},$$
$$f_3^{(2n)}(s;\beta) \approx (s\beta)^2\sqrt{n(n-1)}, \ldots,$$
$$f_m^{(2n)}(s;\beta) \approx (s\beta)^{m-1}\sqrt{n(n-1)\ldots(n-m+2)},$$

and so on for any $m \le n+1$. Taking $s = 1$, we deal with the output wave amplitudes:

$$f_{n+1}^{(2(m+n))}(\beta) \propto \beta^n\sqrt{(m+n)(m+n-1)\ldots(m+1)}\,.$$

Substituting the asymptotic wave amplitudes $f_{n+1}^{(2(m+n))}(\beta)$ in Eqs. (4a) and (4b), we obtain the output states

$$|\psi^{(n)}\rangle_{p_1} = |\alpha\rangle_{p_1}$$

in the pumping mode for any number n. In other words, the output wave function $|\Psi_{12}\rangle$ of the SPDCI with the first term of the asymptotic decomposition in the parameter $\beta \ll 1$ taken into account is given by

$$|\Psi_{12}\rangle = \sum_{n=0}^{\infty} (\alpha\beta)^n |n\rangle_1 |n\rangle_2 |\alpha\rangle_{p_1}. \tag{11}$$

It can be shown that the output wave function $|\Psi_{12}\rangle$ (Eq. (11)) obeys a geometrical distribution with the norm

$$\langle \Psi_{12}|\Psi_{12}\rangle \approx \frac{1}{1-\delta} \approx 1+\delta,$$

where

$$\delta = |\alpha\beta|^2$$

in the classical-pump approximation. We note that the obtained wave function $|\Psi_{12}\rangle$ (Eq. (10)) is similar to the wave function

$$|\Psi_{12}^{(cl)}\rangle = \frac{1}{\operatorname{ch}\chi t} \sum_{n=0}^{\infty} \left(\operatorname{th}(\chi t)\right)^n |n\rangle_1 |n\rangle_2$$

stemming from the classical-pump approximation [17], where the parameter χt in $|\Psi_{12}^{(cl)}\rangle$ [17] plays the same part as the parameter $\alpha\beta$ in our case.

The wave function $|\Delta_+^{(p_1 p_2)}\rangle$ becomes simply a tensor product of two coherent states

$$|\Delta_+^{(p_1 p_2)}\rangle = |\tilde{\alpha}\rangle_{p_1} |\tilde{\alpha}\rangle_{p_2},$$

while

$$|\Delta_-^{(p_1 p_2)}\rangle = 0$$

and the parameter M/N becomes equal to 1 in the classical-pump approximation (Eq. (11)). But from the physical standpoint, this is evidently wrong. The system of two coupled SPDCI presented in Fig. 1*a* makes its own contribution to the distribution of the photon number states in the coherent states $|\tilde{\alpha}\rangle_{p_1,p_2}$. We can consider the output pumping modes remaining in the coherent states only in the classical-pump approximation [17]. The difference between the output state of the SPDCI and a coherent state can be observed in the second term of the asymptotic expansion of the wave amplitudes $f_k^{(2n)}(\beta)$. Because we deal with the output wave functions $|\psi^{(0)}\rangle_{p_1,p_2}$ and $|\psi^{(1)}\rangle_{p_1,p_2}$ (Eqs. (6b) and (6c)), we present only the first two terms of the asymptotic series for the wave amplitudes $f_1^{(2m)}(\beta)$ and $f_2^{(2(m+1))}(\beta)$:

$$f_1^{(2m)}(\beta) \approx 1 - \frac{m\beta^2}{2}, \tag{12a}$$

$$f_2^{(2(m+1))}(\beta) \approx \beta\sqrt{m+1} - \frac{\beta^3\sqrt{m+1}\,(5m+1)}{6}. \tag{12b}$$

From the asymptotic expansion in (12a) and (12b), we have the nonnormalized modified coherent states

$$|\psi^{(0)}\rangle_{p_1,p_2} = \left\{|\tilde{\alpha}\rangle - \frac{\beta^2}{2}|\tau_1\rangle\right\}_{p_1,p_2}, \tag{13a}$$

$$|\psi^{(1)}\rangle_{p_1,p_2} = \left\{|\tilde{\alpha}\rangle - \frac{\beta^2}{2}|\tau_2\rangle\right\}_{p_1,p_2}, \tag{13b}$$

where

$$|\tau_1\rangle = \exp\left(-\frac{|\tilde{\alpha}|^2}{2}\right) \sum_{m=0}^{\infty} \frac{\tilde{\alpha}^m}{\sqrt{m!}} |m\rangle, \tag{13c}$$

$$|\tau_2\rangle = \frac{1}{3}\exp\left(-\frac{|\tilde{\alpha}|^2}{2}\right) \sum_{m=0}^{\infty} \frac{\tilde{\alpha}^m (5m+1)}{\sqrt{m!}} |m\rangle. \tag{13d}$$

Using Eqs. (13a)–(13d), we obtain the probabilities $P_\pm$ (Eq. (8c)) for the macroscopic entangled states $|\Delta_\pm^{(p_1 p_2)}\rangle$ in the leading order in β as

$$P_+ \approx 2\alpha^2\beta^2, \tag{14a}$$

$$P_- \approx \frac{\alpha^4\beta^6}{4}. \tag{14b}$$

The probability to generate the state $|\Delta_+^{(p_1 p_2)}\rangle$ is higher than the probability to generate the state $|\Delta_-^{(p_1 p_2)}\rangle$.

We now apply expressions (13a)–(13d) to the calculation of some parameters of the states $|\Delta_\pm^{(p_1 p_2)}\rangle$. Calculations of the quantity M/N characterizing the degree of orthogonality of the states $|U\rangle_{p_1}$ and $|V\rangle_{p_1}$ ($p_i = p_1, p_2$) give the result

$$\frac{M}{N} = \frac{1-\beta^2/3-8|\tilde{\alpha}|^2\beta^2/3+47|\tilde{\alpha}|^4\beta^4/18+13|\tilde{\alpha}|^2\beta^4/9}{1-\beta^2/3-8|\tilde{\alpha}|^2\beta^2/3+47|\tilde{\alpha}|^4\beta^4/18+14|\tilde{\alpha}|^2\beta^4/9} \approx 1 - \frac{|\tilde{\alpha}|\beta^4}{9}, \tag{15}$$

where we neglect higher powers of the parameter $\tilde{\alpha}\beta$. Because $M/N < 1$ in accordance with (15), the macroscopic states $|U\rangle_{p_1}$ and $|V\rangle_{p_1}$ are not equal to each other. We note that the quantity M/N can take values in the range from 0 to 1. If $M/N = 0$, then we can talk about the full orthogonality,

$${}_{p_i}\langle U/V\rangle_{p_i} = 0,$$

of the macroscopic states $|U\rangle_{p_i}$ and $|V\rangle_{p_i}$ in Eqs. (9b) and (9c). If $M/N = 1$, then we deal with the opposite case, where

$$|U\rangle_{p_i} = |V\rangle_{p_i}.$$

Because the states $|\Delta_+^{(p_1p_2)}\rangle$ are not pairwise orthogonal, i.e.,

$$ {}_{p_i}\langle U/V\rangle_{p_i} \neq 0 $$

for any $\tilde{\alpha}$ and β, we should calculate the amount of entanglement stored in the generated macroscopic entangled states $|\Delta_\pm^{(p_1p_2)}\rangle$. To quantify the entanglement degree of the macroscopic entangled state, we consider concurrency of the state. The concurrence for a pure state $|\psi\rangle$ in a tensor product $H_A \otimes H_B$ of two (finite-dimensional) Hilbert spaces H_A and H_B for two systems A and B is defined by

$$ C(|\psi\rangle) = \sqrt{2(1 - \mathrm{Sp}(p_A^2))}\,, $$

where the reduced density matrix ρ_A is obtained by taking the trace over the subsystem B. We note that the concurrence of a separable state is equal to zero, while the maximally entangled state has unit concurrence. We first construct an orthonormal basis

$$ \{|\tilde{0}\rangle_{p_i}, |\tilde{1}\rangle_{p_i}\} $$

for each pumping mode p_i as

$$ |\tilde{0}\rangle_{p_i} = |U\rangle_{p_i}, \tag{16a} $$

$$ |\tilde{1}\rangle_{p_i} = \frac{|V\rangle_{p_i} - a|U\rangle_{p_i}}{\sqrt{1-a^2}}. \tag{16b} $$

The state $|\Delta_+^{(p_1p_2)}\rangle$ can then be represented in terms of the basis states as

$$ |\Delta_+^{(p_1p_2)}\rangle = = N_+ \left\{2a|\tilde{0}\rangle|\tilde{0}\rangle + \sqrt{1-a^2}\left(|\tilde{0}\rangle|\tilde{1}\rangle + |\tilde{1}\rangle|\tilde{0}\rangle\right)\right\}_{p_1p_2}. \tag{16c} $$

Finally, the concurrence of the state $|\Delta_+^{(p_1p_2)}\rangle$ is given by

$$ C\left(|\Delta_+^{(p_1p_2)}\rangle\right) = 2N_+^2(1-a^2) = = \frac{1-a^2}{1+a^2} = \frac{1-M/N}{1+M/N}. \tag{16d} $$

Again using the estimate for the parameter M/N in Eq. (15), we obtain an approximate expression for the concurrence of the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ as

$$ C\left(|\Delta_+^{(p_1p_2)}\rangle\right) = \frac{|\tilde{\alpha}|^2\beta^4}{9}. $$

The other macroscopic entangled state $|\Delta_-^{(p_1p_2)}\rangle$ in orthonormal basis (16a), (16b) has the form

$$ |\Delta_-^{(p_1p_2)}\rangle = \frac{1}{\sqrt{2}}\,\{|\tilde{0}\rangle|\tilde{1}\rangle - |\tilde{1}\rangle|\tilde{0}\rangle\}_{p_1p_2}. \tag{17} $$

It is evident from expression (17) that

$$ C\left(|\Delta_-^{(p_1p_2)}\rangle\right) = 1 $$

irrespective of the values of α and β.

Thus, the proposed scheme in Fig. 1*a,b* allows us to conditionally generate two $\chi^{(2)}$ macroscopic entangled states in the output pumping modes provided that we have special detectors discriminating between one- and multi-photon number states. The studied scheme enables conditionally producing the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ with a very small amount of entanglement (Eq. (16d)) but with a sufficiently large success probability (Eq. (14a)). The problem under investigation is to develop some methods to decrease the parameter M/N to make it close to zero in order to increase the amount of entanglement of the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$. A natural way to do this by enhancement of the parameter $|\tilde{\alpha}|\beta$ may be restricted by experimental conditions in practice. Nevertheless, there may exist other ways to solve the problem, such as the use of the output of the system in Fig. 1*a* as an input to the next system of the coupled SPDCI, and so on. From the physical standpoint, such a procedure may be naturally considered as an increase in the parameter $|\tilde{\alpha}|\beta$ and, as a consequence, it may lead to a decrease in the parameter M/N. The same system provides us with the possibility to conditionally obtain the macroscopic entangled state $|\Delta_-^{(p_1p_2)}\rangle$ with the concurrence, albeit with a very small success probability (Eq. (14b)). We estimate the range of values for the probabilities of the generated states $|\Delta_+^{(p_1p_2)}\rangle$ and $|\Delta_-^{(p_1p_2)}\rangle$ in Eqs. (8b) and the value of the concurrence of the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$. We consider a down-converter with the standard value of the second-order susceptibility. We then estimate the value of the probability P_+ in Eq. (14a) as $\alpha^2\beta^2 \approx 10^{-2}$–$10^{-4}$. This value of the probability to generate the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ is comparable with the probability to observe two-photon mode entangled states $|\Psi_+^{(1234)}\rangle$ or $|\Psi_-^{(1234)}\rangle$ (Eq. (8a)) at the output of the system. The concurrence of the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ is β^2 times less than the probability P_+ and takes small values. The order of $\alpha\beta$ can be estimated as $\alpha^4\beta^4 \approx 10^{-4}$–$10^{-8}$. The probability P_- in (14b) to observe the maximally entangled macroscopic state $|\Delta_-^{(p_1p_2)}\rangle$ take a value β^2 times less than the quantity $\alpha^4\beta^4$. It is supposed that the probability P_- can be enhanced if we deal with resonance nonlinear three-photon processes leading to huge values of $\chi^{(2)}$ and take the absolute value of the coherent state sufficiently large.

4. CONDITIONAL PREPARATION WITHOUT BEAM SPLITTERS

We now discuss another possibility to conditionally prepare one of the two $\chi^{(2)}$ macroscopic entangled states $|\Delta_+^{(p_1p_2)}\rangle$ or $|\Delta_-^{(p_1p_2)}\rangle$ (Eqs. (8b)), without the balanced beam splitters. For this, the linear optical circuit as shown in Fig. 2 is placed after the system of coupled SPDCI. The two optical beams converge in one of the two detectors in Fig. 2. The detected modes are the sum of the generated signal and idler modes given by

$$\hat{c} = \frac{\hat{a}_1 + \hat{a}_3}{\sqrt{2}}$$

and

$$\hat{d} = \frac{\hat{a}_2 + \hat{a}_4}{\sqrt{2}},$$

respectively, where the factor $1/\sqrt{2}$ in the quantum operators of the detected modes is introduced to satisfy the commutation relations. As a consequence of this geometry of experiment, the coincidence count rate becomes

$$\langle\Psi_1^{(1234)}|\hat{d}^+\hat{c}^+\hat{c}\hat{d}|\Psi_1^{(1234)}|\rangle = 1,$$

which means projection of the total state $|\Psi\rangle$ (Eq. (7)) onto the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ in the pumping modes after the registration of two simultaneous clicks by two detectors. The same coincidence count rate for the state $|\Psi_-^{(1234)}\rangle$ is

$$\langle\Psi_2^{(1234)}|\hat{d}^+\hat{c}^+\hat{c}\hat{d}|\Psi_2^{(1234)}|\rangle = 0.$$

This is essentially the destructive two-photon interference effect in registering detectors, first observed in [18].

The experimental arrangement in Fig. 2 can also be adjusted for the opposite case to conditionally produce the other macroscopic entangled state $|\Delta_-^{(p_1p_2)}\rangle$ in the pumping modes. In this case, the auxiliary optical scheme must be supplied by the π-phase shifter in one of the four auxiliary modes to change the sign of state $|\Psi_+^{(1234)}\rangle$ to the opposite and vice versa. Then, detecting two photons by trigger detectors in Fig. 2 would provide a priori information that the $\chi^{(2)}$ macroscopic entangled state $|\Delta_-^{(p_1p_2)}\rangle$ is generated in the pumping modes. Information about the other $\chi^{(2)}$ macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ is erased due to the interference effect in auxiliary modes [18]. Therefore, the Bell-state measurement scheme in Fig. 2 also enables conditionally preparing one of the two $\chi^{(2)}$ macroscopic entangled states, either $|\Delta_+^{(p_1p_2)}\rangle$ or $|\Delta_-^{(p_1p_2)}\rangle$. We note that the optical scheme for conditional preparation of the $\chi^{(2)}$ macroscopic entangled state presented in Fig. 2 requires special detectors discriminating between one- and multi-photon number states [15].

5. CONCLUSION AND DISCUSSION

We have proposed two optical schemes consisting of a system of two spontaneous parametric down-converters with type-I phase matching combined with Bell-state measurement arrangement in the generated modes to conditionally produce the $\chi^{(2)}$ macroscopic entangled states. One of the proposed schemes for the Bell-state measurement uses a pair of the ancillary photons in the signal and idler modes to direct them to two Hadamard gates. A pair of Hadamard gates constructed on the base of the beam splitters is used in identification of the outcome of the states in auxiliary modes and is therefore applicable to the identification of the $\chi^{(2)}$ macroscopic states in the output pumping modes. The other projection scheme is based on a «holographic» type of coincidence counting of photons and can work without the Hadamard gates. Deleterious contribution of one of the two projected states vanishes due to the well-known destructive two-photon interference effect [18]. Our analysis has been done under the assumption of the presence of ideal detectors able to distinguish one-photon clicks from all other ones.

We have shown that it is possible to observe the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ with a sufficiently large success probability P_+ but with a small amount of entanglement. The nonlinear effect, although comparable with the effect of generating the signal–idler pair, is typically too weak to generate the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ with better entanglement with current technology. The problem of generating the macroscopic entangled state $|\Delta_+^{(p_1p_2)}\rangle$ with a larger value of stored entanglement requires further study. The same optical scheme allows obtaining the macroscopic maximally entangled state $|\Delta_-^{(p_1p_2)}\rangle$ with unit concurrence. Performance of the optical schemes presented in Figs. 1*a*,*b*, and 2 is plausible with current technologies.

APPENDIX

Some particular analytic solutions for the wave probabilities

The probability $P_1^{(0)}(x;\beta)$, $x = |\alpha^2|$, is entirely determined by the first term of the coherent state distribution

$$P_1^{(0)}(x;\beta) = \exp(-x). \tag{A.1}$$

The probabilities to find the corresponding tensor products consisting of either one pumping or two signals and idler photons in superposition state (3b) are given by

$$P_1^{(2)}(x;\beta) = x\exp(-x)\cos^2\beta, \tag{A.2a}$$

$$P_2^{(2)}(x;\beta) = x\exp(-x)\sin^2\beta. \tag{A.2b}$$

The next probabilities with $n = 2$ are given by

$$P_1^{(4)}(x;\beta) = \frac{2x^2\exp(-x)}{9}\left(1 + \frac{\cos\left(\sqrt{6}\,\beta\right)}{2}\right)^2, \tag{A.3a}$$

$$P_2^{(4)}(x;\beta) = \frac{x^2\exp(-x)}{6}\sin^2\left(\sqrt{6}\,\beta\right), \tag{A.3b}$$

$$P_3^{(4)}(x;\beta) = \frac{x^2\exp(-x)}{9}\left(1 - \cos\left(\sqrt{6}\,\beta\right)\right)^2. \tag{A.3c}$$

The probabilities for the states forming the partial wave function $|\Psi_6\rangle$ are given by

$$P_1^{(6)}(x;\beta) = \frac{x^3\exp(-x)}{1752}\left\{\left(\sqrt{73}+7\right)\times \cos\left(\sqrt{10-\sqrt{73}}\,\beta\right) + \left(\sqrt{73}-7\right)\times \cos\left(\sqrt{10+\sqrt{73}}\,\beta\right)\right\}^2, \tag{A.4a}$$

$$P_2^{(6)}(x;\beta) = \frac{x^3\exp(-x)}{5256}\left\{\left(\sqrt{73}+7\right)\sqrt{10-\sqrt{73}}\times \sin\left(\sqrt{10-\sqrt{73}}\,\beta\right) + \left(\sqrt{73}-7\right)\sqrt{10+\sqrt{73}}\times \sin\left(\sqrt{10+\sqrt{73}}\,\beta\right)\right\}^2, \tag{A.4b}$$

$$P_3^{(6)}(x;\beta) = \frac{x^3\exp(-x)}{73}\left\{\cos\left(\sqrt{10-\sqrt{73}}\,\beta\right) - \cos\left(\sqrt{10+\sqrt{73}}\,\beta\right)\right\}^2, \tag{A.4c}$$

$$P_4^{(6)}(x;\beta) = \frac{9x^3\exp(-x)}{73}\times \left\{\frac{\sin\left(\sqrt{10-\sqrt{73}}\,\beta\right)}{\sqrt{10-\sqrt{73}}} - \frac{\sin\left(\sqrt{10+\sqrt{73}}\,\beta\right)}{\sqrt{10+\sqrt{73}}}\right\}^2. \tag{A.4d}$$

The probabilities with $n = 4$ pumping photons are

$$P_1^{(8)}(x;\beta) = \frac{x^4\exp(-x)}{47928672}\left\{\left(17\sqrt{297}+261\right)\times \cos\left(\sqrt{25-\sqrt{297}}\,\beta\right) + \left(17\sqrt{297}-261\right)\times \cos\left(\sqrt{25+\sqrt{297}}\,\beta\right) + 48\sqrt{297}\right\}^2, \tag{A.5a}$$

$$P_2^{(8)}(x;\beta) = \frac{x^4\exp(-x)}{191714688}\times \left\{\left(17\sqrt{297}+261\right)\sqrt{25-\sqrt{297}}\times \sin\left(\sqrt{25-\sqrt{297}}\,\beta\right) + \left(17\sqrt{297}-261\right)\times \sqrt{25+\sqrt{297}}\sin\left(\sqrt{25+\sqrt{297}}\,\beta\right)\right\}^2, \tag{A.5b}$$

$$P_3^{(8)}(x;\beta) = \frac{x^4\exp(-x)}{3994056}\left\{\left(4\sqrt{297}+18\right)\times \cos\left(\sqrt{25-\sqrt{297}}\,\beta\right) + \left(4\sqrt{297}-18\right)\times \cos\left(\sqrt{25+\sqrt{297}}\,\beta\right) - \frac{8}{3}\sqrt{\frac{99}{2}}\right\}^2, \tag{A.5c}$$

$$P_4^{(8)}(x;\beta) = \frac{x^4\exp(-x)}{3550272}\left\{\left(25+\sqrt{297}\right)\times \sqrt{25-\sqrt{297}}\sin\left(\sqrt{25-\sqrt{297}}\,\beta\right) - \left(25-\sqrt{297}\right)\times \sqrt{25+\sqrt{297}}\sin\left(\sqrt{25+\sqrt{297}}\,\beta\right)\right\}^2, \tag{A.5d}$$

$$P_5^{(8)}(x;\beta) = \frac{x^4\exp(-x)}{221892}\left\{\left(25+\sqrt{297}\right)\times \cos\left(\sqrt{25-\sqrt{297}}\,\beta\right) - \left(25-\sqrt{297}\right)\times \cos\left(\sqrt{25+\sqrt{297}}\,\beta\right) - 2\sqrt{297}\right\}^2. \tag{A.5e}$$

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