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NON-MARKOVIAN STOCHASTIC LIOUVILLE EQUATION AND ANOMALOUS QUANTUM RELAXATION KINETICS

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The kinetics of phase and population relaxation in quantum systems induced by noise with the anomalously slowly decaying correlation function $P(t) \propto (wt)^{-\alpha}$, where $0 < \alpha < 1$, is analyzed within the continuous-time random walk approach. The relaxation kinetics is shown to be anomalously slow. Moreover, for $\alpha < 1$, in the limit of short characteristic time of fluctuations w^{-1}, the kinetics is independent of w. As $\alpha \to 1$, the relaxation regime changes from the static limit to fluctuation narrowing. Simple analytical expressions are obtained describing the specific features of the kinetics.

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1. INTRODUCTION

The noise-induced relaxation in quantum systems is a very important process observed in magnetic resonance [1], quantum optics and nonlinear spectroscopy [2], etc. These processes are often analyzed assuming conventional stochastic properties of the noise: fast decay of correlation functions and a short correlation time τ_c [1]. In the absence of memory, the relaxation is described by very popular Bloch-type equations. The memory effects are also discussed (within the Zwanzig projection operator approach [3]), but either in the lowest orders in the fluctuating interaction V that induces the relaxation or by approximate summation of terms of different orders in V [4].

Recently, much attention was drawn to the processes governed by noises with anomalously slowly decaying correlation functions $P(t) \propto t^{-\alpha}$ with $\alpha < 1$. They are discussed in relation to spectroscopic studies of quantum dots (see [5, 6] and the references therein). Similar problems are analyzed in the theory of stochastic resonances [7].

Such anomalous processes cannot be properly described by methods based on expansion in powers of V. The goal of this paper is to analyze the corresponding anomalous relaxation within the continuous-time random walk approach (CTRWA) [8] with the use of the recently derived non-Markovian stochastic Liouville equation (SLE) [9], which enables one to describe relaxation kinetics without the above-mentioned approximations (expansions in V), although assumes the classical nature of the noise. In some physically reasonable models, it allows describing the phase and population relaxation kinetics in the analytical form even for multilevel systems. In particular, the kinetics is shown to be strongly nonexponential.

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2. GENERAL FORMULATION

We consider noise-induced relaxation in the quantum system whose dynamical evolution is governed by the Hamiltonian

$$H(t) = H_s + V(t), \tag{1}$$

where H_s is the term independent of time and $V(t)$ is the fluctuating interaction, which models effects of the noise. The evolution is described by the density matrix $\rho(t)$ satisfying the Liouville equation ($\hbar = 1$)

$$\begin{aligned} \dot{\rho} &= -i\hat{H}(t)\rho, \\ \hat{H}\rho &= [H, \rho] = [H\rho - \rho H]. \end{aligned} \tag{2}$$

$V(t)$-fluctuations are assumed to be symmetric ($\langle V \rangle = 0$) and to result from stochastic jumps between the states $|x_\nu\rangle$ in the (discrete or continuum) space

$\{x_\nu\} \equiv \{x\}$ with different $V = V_\nu$ and $H = H_\nu$ (i.e., different $\hat{V} = \hat{V}_\nu \equiv [V_\nu, \ldots]$ and $\hat{H} = \hat{H}_\nu$):

$$\hat{\mathcal{V}} = \sum_\nu |x_\nu\rangle \hat{V}_\nu \langle x_\nu|,$$
$$\hat{\mathcal{H}} = \sum_\nu |x_\nu\rangle \hat{H}_\nu \langle x_\nu|. \tag{3}$$

We use the bra–ket notation $|k\rangle$ and $|kk'\rangle \equiv |k\rangle\langle k'|$ for eigenstates of H (in the original space) and $\hat{H}$ (in the Liouville space), respectively, and the notation $|x\rangle$ for states in the $\{x\}$-space.

The macroscopic evolution of the system under study is determined by the evolution operator $\hat{\mathcal{R}}(t)$ in the Liouville space averaged over $V(t)$-fluctuations,

$$\rho(t) = \hat{\mathcal{R}}(t)\rho_i,$$
$$\hat{\mathcal{R}}(t) = \sum_{x,x_i} \hat{\mathcal{G}}(x, x_i|t) P_e(x_i), \tag{4}$$

where $\hat{\mathcal{G}}(x, x'|t)$ is the averaged evolution operator and $P_e(x)$ is the equilibrium distribution in the $\{x\}$-space.

Non-Markovian $V(t)$-fluctuations are described with the use of the CTRWA (which leads to the non-Markovian SLE [9] for $\hat{\mathcal{G}}(t)$). It treats fluctuations as a sequence of sudden changes of $\hat{V}$. The onset of any particular change labeled by j is described by the matrix $\hat{P}_{j-1}$ (in the $\{x\}$-space) of the probabilities not to have any change during time t and its derivative

$$\hat{W}_{j-1}(t) = -\frac{d\hat{P}_{j-1}(t)}{dt}.$$

These matrices are diagonal and independent of j:

$$\hat{P}_{j-1}(t) = \hat{P}(t),\ \hat{W}_{j-1}(t) = \hat{W}(t) = -\frac{d\hat{P}(t)}{dt}, \quad j > 1,$$

except

$$\hat{P}_0(t) \equiv \hat{P}_i(t), \quad \hat{W}_0(t) \equiv \hat{W}_i(t) = -\frac{d\hat{P}_i(t)}{dt}$$

depending on the problem considered. For nonstationary (n) and stationary (s) fluctuations [8],

$$\hat{W}_i(t) = \hat{W}_n(t) = \hat{W}(t),$$
$$\hat{W}_i(t) = \hat{W}_s(t) = \hat{t}_w^{-1} \int_t^\infty d\tau\, \hat{W}(\tau),$$

respectively, where

$$\hat{t}_w = \int_0^\infty d\tau\, \tau \hat{W}(\tau)$$

is the matrix of average times of waiting for the change [8].

In what follows, we mainly operate with the Laplace transforms denoted as

$$\widetilde{Z}(\epsilon) = \int_0^\infty dt\, Z(t) e^{-\epsilon t}$$

for any function $Z(t)$. In particular, noteworthy is the relation

$$\hat{\widetilde{P}}_j(\epsilon) = \frac{1 - \hat{\widetilde{W}}_j(\epsilon)}{\epsilon}$$

and suitable representations

$$\hat{\widetilde{W}}(\epsilon) = [1 + \hat{\Phi}(\epsilon)]^{-1},$$
$$\hat{\widetilde{P}}(\epsilon) = [\epsilon + \epsilon/\hat{\Phi}(\epsilon)]^{-1} \tag{5}$$

in terms of a diagonal matrix $\hat{\Phi}(\epsilon)$ with

$$\hat{\Phi}(\epsilon) \overset{\epsilon\to 0}{\approx} (\epsilon/\hat{w})^\alpha,$$

where $\hat{w}$ is a constant matrix and $\alpha \le 1$ (see below).

Evolution in the $\{x\}$-space is governed by the jump operator

$$\hat{\mathcal{L}} = 1 - \hat{\mathcal{P}},$$

where $\hat{\mathcal{P}}$ is the nondiagonal matrix of jump probabilities. This evolution results in relaxation to the equilibrium state $|e_x\rangle$, satisfying the equation

$$\hat{\mathcal{L}}\hat{w}^\alpha |e_x\rangle = 0$$

and represented as

$$|e_x\rangle = \sum_x P_e(x)|x\rangle,$$

$$\langle e_x| = \sum_x \langle x|$$

(see [9]). We note that (see Eq. (4))

$$\hat{\mathcal{R}}(t) = \langle e_x|\hat{\mathcal{G}}|e_x\rangle \equiv \langle\hat{\mathcal{G}}\rangle. \tag{6}$$

The CTRWA leads to the non-Markovian SLE for $\hat{\mathcal{G}}(x, x_i|t)$ [9]. Solving this SLE yields [9]

$$\hat{\widetilde{\mathcal{G}}} = \hat{\widetilde{P}}_i(\hat{\Omega}) + \hat{\Omega}^{-1}\hat{\Phi}(\hat{\Omega})[\hat{\Phi}(\hat{\Omega}) + \hat{\mathcal{L}}]^{-1}\hat{\mathcal{P}}\hat{\widetilde{W}}_i(\hat{\Omega}), \tag{7}$$

where

$$\hat{\mathcal{L}} = 1 - \hat{\mathcal{P}},$$
$$\hat{\Omega} = \epsilon + i\hat{\mathcal{H}}. \tag{8}$$

In particular, in the case of n-fluctuations ($\hat{W}_i = \hat{W}$),

$$\hat{\tilde{\mathcal{G}}} = \hat{\tilde{\mathcal{G}}}_n = \hat{\Omega}^{-1}\hat{\Phi}(\hat{\Phi} + \hat{\mathcal{L}})^{-1}. \tag{9}$$

For s-fluctuations ($\hat{W}_i = \hat{W}_s$),

$$\hat{\tilde{\mathcal{G}}} = \hat{\Omega}^{-1} - \hat{\tilde{\mathcal{G}}}_n \hat{\mathcal{L}}(\hat{\Omega}\hat{t}_w)^{-1}.$$

Hereafter, for brevity, we omit the argument $\hat{\Omega}$ of all Laplace transforms if this does not result in confusion.

3. USEFUL MODELS AND APPROACHES

3.1. Sudden relaxation model

The sudden relaxation model (SRM) [9] assumes sudden equilibration in the $\{x\}$-space described by the operator

$$\hat{\mathcal{L}} = (1 - |e_0\rangle\langle e_0|)\hat{Q}^{-1}, \quad \hat{Q} = 1 - \sum_x P_0(x)|x\rangle\langle x|, \tag{10}$$

where

$$|e_0\rangle = \sum_x P_0(x)|x\rangle, \quad \langle e_0| = \sum_x \langle x|.$$

For this $\hat{\mathcal{L}}$,

$$|e_x\rangle = \hat{q}|e_0\rangle, \quad \hat{q} = \frac{\hat{Q}\hat{w}^{-\alpha}}{\langle e_0|\hat{Q}\hat{w}^{-\alpha}|e_0\rangle} \tag{11}$$

and

$$\langle e_x| = \langle e_0|.$$

In model (10), one obtains

$$\hat{\tilde{\mathcal{R}}}_i = \langle \hat{\tilde{P}}_{Q_i}\rangle + \langle \hat{q}^{-1}\tilde{P}_Q\rangle[1 - \langle \hat{q}^{-1}\hat{\widetilde{W}}_Q\rangle]^{-1}\langle \hat{\widetilde{W}}_{Q_i}\rangle \tag{12}$$

for any $\hat{\widetilde{W}}_i$, where

$$\hat{\tilde{P}}_{Q_i} = \frac{1 - \hat{\widetilde{W}}_{Q_i}}{\hat{\Omega}}, \quad \hat{\widetilde{W}}_Q = (1 + \hat{\Phi}\hat{Q})^{-1} \tag{13}$$

and

$$\hat{\widetilde{W}}_{Q_i} = \hat{\widetilde{W}}_i(\hat{\widetilde{W}}_Q/\hat{\widetilde{W}}).$$

3.2. Short correlation time limit

In practical applications, of special importance is the short correlation time limit (SCTL) for $V(t)$-fluctuations, in which Eq. (12) can be markedly simplified. It corresponds to large characteristic rates w_c of the dependence $\hat{\Phi}(\hat{\Omega}) \equiv \hat{\Phi}(\hat{\Omega}/w_c)$:

$$w_c \gg \|V\|.$$

In this limit, the relaxation kinetics is described by the first terms of the expansion of $\hat{\Phi}(\hat{\Omega}/w_c)$ in small $\hat{\Omega}/w_c$, because $\hat{\Phi}(\epsilon)$ is an increasing function of ϵ with

$$\hat{\Phi}(\epsilon) \xrightarrow{\epsilon \to 0} 0.$$

Some important general conclusions, however, can be made independently of the form of $\hat{\Phi}(\Omega)$ (see below).

3.3. Models for quantum evolution and fluctuations

The obtained general results are conveniently illustrated with the quantum two-level model and the stochastic two-state SRM for $V(t)$-fluctuations.

Quantum evolution of the two-level system is governed by the Hamiltonian (assumed to be a real matrix)

$$H_s = \frac{\omega_s}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{V} = \begin{array}{cc} \begin{bmatrix} \mathcal{V}_d & \mathcal{V}_n \\ \mathcal{V}_n & -\mathcal{V}_d \end{bmatrix} & \begin{array}{c} |+\rangle \\ |-\rangle \end{array} \end{array}. \tag{14}$$

The two-state SRM suggests that fluctuations result from jumps between two states (in the $\{x\}$-space), for example, $|x_+\rangle$ and $|x_-\rangle$, whose kinetics is described by

$$\hat{\mathcal{L}} = 2(1 - |e_x\rangle\langle e_x|), \quad |e_x\rangle = \frac{1}{2}\big||x_+\rangle + |x_-\rangle\big\rangle. \tag{15}$$

Below, we consider two examples of these models.

1. Diagonal noise [10]:

$$\omega_s = 0, \quad \mathcal{V}_n = 0,$$

$$\mathcal{V}_d = \omega_0(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|),$$

and

$$H_{\nu=\pm} = \pm\frac{1}{2}\omega_0(|+\rangle\langle +| - |-\rangle\langle -|). \tag{16}$$

2. Nondiagonal noise:

$$\mathcal{V}_d = 0, \quad \mathcal{V}_n = v(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|),$$

and hence

$$H_{\nu=\pm} = H_s \pm v(|+\rangle\langle -| + |-\rangle\langle +|). \tag{17}$$

The first model describes dephasing and the second is useful for studying population relaxation.

In model (14), dephasing and population relaxation are characterized by two functions.

1. The spectrum $I(\omega)$, which is taken in the form corresponding to Fourier transformed free-induction-decay (FTFID) experiments [11]

$$I(\omega) = \frac{1}{\pi}\,\mathrm{Re}\langle s|\hat{\tilde{\mathcal{R}}}(i\omega)|s\rangle. \tag{18}$$

2. The difference of level populations

$$N(t) = \langle n|\hat{\mathcal{R}}(t)|n\rangle. \tag{19}$$

In these two functions,

$$\begin{aligned} |s\rangle &= \frac{1}{\sqrt{2}}\big||+-\rangle + |-+\rangle\big\rangle, \\ |n\rangle &= \frac{1}{\sqrt{2}}\big||++\rangle - |--\rangle\big\rangle. \end{aligned} \tag{20}$$

4. GENERAL RESULTS IN THE SCTL

Within the SCTL ($\|V\|/w_c \ll 1$), especially simple results are obtained for $\|H_s\|/w_c \ll 1$. In the lowest order in $\|\hat{\Phi}(\hat{\Omega}/w_c)\| \ll 1$,

$$\hat{\tilde{\mathcal{R}}} \approx \hat{\tilde{\mathcal{R}}}_n \approx \frac{\langle \hat{q}^{-1}\hat{Q}\hat{\Omega}^{-1}\hat{\Phi}(\hat{\Omega})\rangle}{\langle \hat{q}^{-1}\hat{Q}\hat{\Phi}(\hat{\Omega})\rangle} = \tag{21}$$

$$= \frac{\langle \hat{w}^{\alpha}\hat{\Omega}^{-1}\hat{\Phi}(\hat{\Omega})\rangle}{\langle \hat{w}^{\alpha}\hat{\Phi}(\hat{\Omega})\rangle}. \tag{22}$$

This formula holds for any initial matrix $\hat{\tilde{W}}_i$ and, in particular, for s-fluctuations, if

$$\|\hat{t}_w\| \sim \frac{1}{w_c} \ll \frac{1}{\|\hat{\Omega}\|}.$$

The more complicated SCTL-case

$$\|H_s\|/w_c \approx 1$$

can be analyzed by expanding $\hat{\tilde{\mathcal{G}}}$ in powers of the parameter

$$\xi = \|V\|/\|H_s\| \ll 1.$$

In particular, within the general two-level model (Eq. (14)) with $V_d = 0$, in the second order in ξ, the diagonal and nondiagonal elements of $\rho(t)$ are decoupled and the corresponding elements of $\hat{\mathcal{R}}(t)$ are expressed in terms of the universal function

$$R_k(t) = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} d\epsilon\, \frac{e^{i\epsilon t}}{\epsilon + k\epsilon/\langle\hat{\Phi}(\epsilon)\rangle}, \tag{23}$$

$$\begin{aligned} &\langle\mu|\hat{\mathcal{R}}(t)|\mu\rangle = \exp(-i\omega_\mu t)R_{k_\mu}(t), \\ &(\mu = n, +-, -+), \end{aligned} \tag{24}$$

where

$$\omega_\mu = \langle\mu|\hat{H}_s|\mu\rangle,\ k_n = 2\,\mathrm{Re}(k_{+-}),$$

$$k_{+-} = k_{-+}^{*} = \frac{1}{2}\omega_s^{-2}\langle\mathcal{V}_n\hat{q}^{-1}[1 - \hat{\tilde{W}}_Q(2i\omega_s)]\mathcal{V}_n\rangle. \tag{25}$$

5. ANOMALOUS FLUCTUATIONS

The simplest model for anomalous fluctuations can be written as [12]

$$\hat{\Phi}(\epsilon) = (\epsilon/\hat{w})^{\alpha}, \quad 0 < \alpha < 1, \tag{26}$$

where $\hat{w}$ is the matrix of fluctuation rates, diagonal in the $|x\rangle$-basis. For simplicity, $\hat{w}$ is assumed to be independent of x, i.e., $\hat{w} \equiv w$ (this parameter can be associated with w_c mentioned above). Model (26) describes the anomalously slow decay of the matrix

$$\hat{W}(t) \propto 1/t^{1+\alpha}$$

(very long memory effects in the system [12]), for which only the case of n-fluctuations is physically sensible.

In SCTL (22), model (26) yields the expression

$$\begin{aligned} &\hat{\tilde{\mathcal{R}}}_n(\epsilon) = \langle\hat{\Omega}^{\alpha-1}(\epsilon)\rangle\langle\hat{\Omega}^{\alpha}(\epsilon)\rangle^{-1}, \\ &\hat{\Omega}(\epsilon) = \epsilon + i\hat{\mathcal{H}}, \end{aligned} \tag{27}$$

which shows that $\hat{\tilde{\mathcal{R}}}_n(\epsilon)$ (and $\hat{\mathcal{R}}_n(t)$) is independent of the characteristic rate w. For $\alpha = 0$ and $\alpha = 1$, Eq. (27) reproduces the static and fluctuation narrowing limits [1]:

$$\hat{\tilde{\mathcal{R}}}_n(\epsilon) = \langle\hat{\Omega}^{-1}(\epsilon)\rangle$$

and

$$\hat{\tilde{\mathcal{R}}}_n(\epsilon) = \frac{1}{\langle\hat{\Omega}(\epsilon)\rangle},$$

respectively.

Of certain interest is the limit as $\alpha \to 1$, in which formula (27) predicts the Bloch-type exponential relaxation

$$\hat{\tilde{\mathcal{R}}}_n(\epsilon) \approx \Big[\epsilon + i\hat{H}_s + (\alpha - 1)\langle\hat{\Omega}\ln(\hat{\Omega})\rangle_{\epsilon\to 0}\Big]^{-1}, \tag{28}$$

controlled by the relaxation rate matrix

$$\hat{W}_r = (\alpha - 1)\,\mathrm{Re}\langle\hat{\Omega}\ln(\hat{\Omega})\rangle_{\epsilon\to 0}$$

and accompanied by frequency shifts represented by

$$\hat{h} = i(\alpha - 1)\,\mathrm{Im}\langle\hat{\Omega}\ln(\hat{\Omega})\rangle_{\epsilon\to 0}.$$

However, the matrices $\hat{W}_r$ and $\hat{h}$ (unlike those in the conventional Bloch equation) are independent of the characteristic rate w of $V(t)$-fluctuations.

5.1. Dephasing for diagonal noise

In model (16), the spectrum $I(\omega)$ can be obtained in the general SRM (10),

$$I(\omega) = n_\alpha \frac{\psi_-^\alpha \psi_+^{\alpha-1} + \psi_-^{\alpha-1}\psi_+^\alpha}{(\psi_-^\alpha)^2 + (\psi_+^\alpha)^2 + 2\psi_-^\alpha \psi_+^\alpha \cos(\pi\alpha)}, \quad (29)$$

where

$$\psi_\pm^\beta(\omega) = \langle |\omega - 2V_d|^\beta \theta[\pm(\omega - 2V_d)]\rangle$$

with $\theta(z)$ being the Heaviside step-function and

$$n_\alpha = \sin(\pi\alpha)/\pi.$$

In the two-state SRM (16),

$$I(\omega) = \frac{n_\alpha}{2\omega_0}\theta(y)\,\frac{y + y^{-1} + 2}{y^\alpha + y^{-\alpha} + 2\cos(\pi\alpha)}, \quad (30)$$

where

$$y = \frac{\omega_0 + \omega}{\omega_0 - \omega}$$

(see also Ref. [6]). According to this formula, the anomalous dephasing (unlike the conventional one [1]) leads to broadening of $I(\omega)$ only in the region $|\omega| < \omega_0$ and singular behavior of $I(\omega)$ at $\omega \to \pm\omega_0$:

$$I(\omega) \sim \frac{1}{(\omega \pm \omega_0)^{1-\alpha}}.$$

For $\alpha > \alpha_c \approx 0.59$ (α_c satisfies the relation $\alpha_c = \cos(\pi\alpha_c/2)$), the two-state-SRM formula also predicts the occurrence of the central peak (at $\omega = 0$) [6] of the Lorenzian shape and width

$$w_L \approx \frac{\omega_0 \cos(\pi\alpha/2)}{\sqrt{\alpha^2 - \cos^2(\pi\alpha/2)}},$$

$$I(\omega) \approx \frac{1}{2\pi}\,\frac{\mathrm{tg}(\pi\alpha/2)\omega_0^{-1}}{1 + (\omega/w_L)^2},$$

whose intensity increases with the increase of $\alpha - \alpha_c$ (Fig. 1). At $\alpha \sim 1$, the parameters of this peak are reproduced by Eq. (28) in which

$$\langle \hat\Omega \ln \hat\Omega\rangle_\epsilon = -\frac{\pi}{2}\omega_0.$$

The origin of the peak indicates the transition from static broadening at $\alpha \ll 1$ to narrowing at $\alpha \sim 1$ (see Eq. (27)). For systems with complex spectra, this transition can, of course, be strongly smoothed. The behavior of $I(\omega)$ is illustrated in Fig. 1 for different values of the parameters of the model.

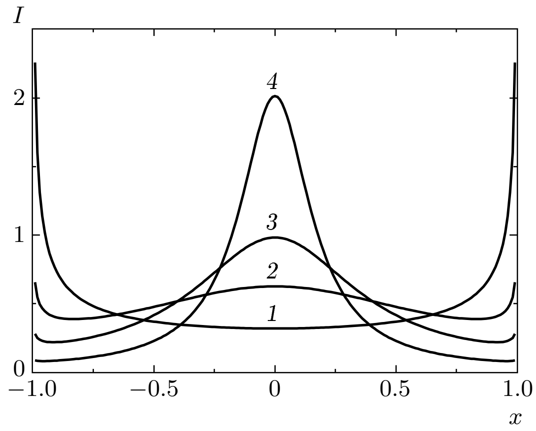

Fig. 1. The spectrum $I(x) = I(\omega)\omega_0$, where $x = \omega/\omega_0$, calculated in model (16) (using Eq. (29)) for different values of α = 5 (*1*), 7 (*2*), 8 (*3*), and 9 (*4*)

5.2. Dephasing for nondiagonal noise

The model in Eq. (17) allows revealing some additional specific features of dephasing. We restrict ourselves to the analysis of the case where $\|H_s\| \sim \omega_s \gtrsim w$ and the most interesting part of the spectrum at $|\omega| \sim \omega_s$. Equations (23) and (24) show that the elements $\langle\mu|\mathcal{R}(t)|\mu\rangle$, ($\mu = +-, -+$), which describe phase relaxation, are then given by

$$\langle\mu|\mathcal{R}(t)|\mu\rangle = \exp(-i\omega_\mu t)E_\alpha[-k_\mu(wt)^\alpha], \quad (31)$$

where

$$E_\alpha(-z) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} \frac{dy\, e^y}{y + zy^{1-\alpha}}$$

is the Mittag–Leffler function [12]. Therefore, for $|\omega| \sim \omega_s$,

$$I(\omega) = I_0(\omega_s + \omega) + I_0(\omega_s - \omega), \quad (32)$$

where

$$I_0(\omega) = n_0 \sin\phi_x (|x|^{1+\alpha} + |x|^{1-\alpha} + 2|x|\cos\phi_x)^{-1} \quad (33)$$

with

$$x = \frac{\omega}{|k_{+-}|^{1/\alpha}w}, \quad n_0 = (\pi|k_{+-}|^{1/\alpha}w)^{-1},$$

$$\phi_x = \frac{\pi\alpha}{2} + \mathrm{sign}\, x\, \mathrm{arctg}\left[\frac{\sin(\pi\alpha/2)}{\cos(\pi\alpha/2) + 2^{-\alpha^{-1}}\omega_s/w}\right]. \quad (34)$$

Formula (32) predicts singular behavior of $I(\omega)$ at $\omega \sim \pm\omega_s$:

$$I(\omega) \sim \frac{1}{|\omega \pm \omega_s|^{1-\alpha}},$$

and slow decrease of $I(\omega)$ with the increase of $|\omega \pm \omega_s|$:

$$I(\omega) \sim \frac{1}{|\omega \pm \omega_s|^{1+\alpha}}.$$

In the limit $\omega_s/w \ll 1$, we have

$$\phi_x \approx \pi\alpha\theta(x)$$

and hence

$$I_0(\omega) \sim \theta(\omega).$$

This implies that for $\omega_s/w \ll 1$, the spectrum $I(\omega)$ is localized in the region $|\omega| < \omega_s$ and looks similar to $I(\omega)$ for diagonal dephasing at $\alpha < \alpha_c$. For $\omega_s/w \gtrsim 1$, however, $I(\omega)$ is nonzero outside this region; moreover, in the limit $\omega_s/w \gg 1$, the spectrum $I_0(\omega)$ becomes symmetric, $I_0(\omega) = I_0(-\omega)$, similarly to the conventional spectra.

It is worth noting that for $\omega_s/w \ll 1$, the functions $\langle\mu|\mathcal{R}(t)|\mu\rangle$ and $I(\omega)$ are independent of w (in agreement with Eq. (22)) because $k_\mu \propto (\omega_s/w)^\alpha$ and $k_\mu(wt)^\alpha \propto (\omega_s t)^\alpha$. In the opposite limit, however, $k_\mu \sim w^0$, and therefore the characteristic relaxation time behaves as w^{-1}.

5.3. Population relaxation

Specific features of the anomalous population relaxation can be analyzed with the model of nondiagonal noise (17).

In particular, in the respective limits $\|H_s\| \sim \omega_s \gtrsim w$ and $1-\alpha \ll 1$, Eqs. (23), (24), and (28) imply that

$$N(t) = E_\alpha[-k_n(wt)^\alpha], \quad N(t) = \exp(-w_\alpha t), \qquad (35)$$

where $E_\alpha(-x)$ is the Mittag–Leffler function defined above and

$$w_\alpha \approx k_n(\alpha \to 1)w \sim 1-\alpha.$$

The first of these formulas predicts a very slow population relaxation at

$$t > \tau_r = w^{-1}(k_n/w)^{1/\alpha},$$

namely,

$$N(t) \propto 1/t^\alpha.$$

Similarly to $I(\omega)$, the function $N(t)$ is in fact independent of w in the limit $\omega_s/w \ll 1$ because $k_n \propto (\omega_s/w)^\alpha$

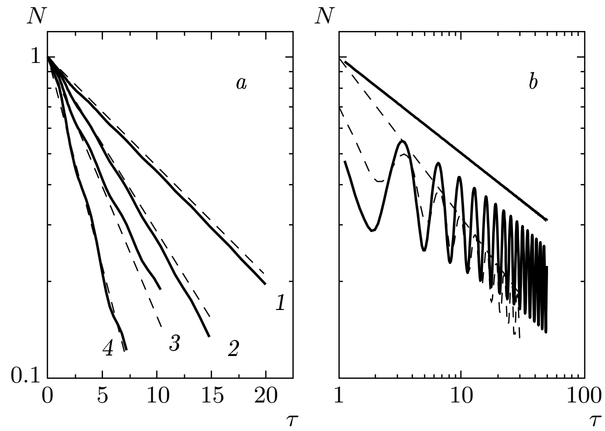

Fig. 2. Population relaxation kinetics $N(\tau)$, where $\tau = E_0 t$, calculated with Eq. (36) (a) for large α and different $r = 2v/\omega_s$: $\alpha = 0.95$, $r = 1.0$ (1); $\alpha = 0.95$, $r = 2.0$ (2); $\alpha = 0.88$, $r = 1.0$ (3); $\alpha = 0.88$, $r = 2.0$ (4); and (b) for small $\alpha = 0.3$ (solid line) and $\alpha = 0.5$ (dashed line) ($r = 0.7$). Straight lines in figures a and b represent exponential (Eq. (35)) and $t^{-\alpha}$ dependences, respectively (in a, they are shown by dashed lines)

in this case. In the opposite limit $\omega_s/w > 1$, the characteristic time population relaxation behaves as w^{-1} because k_n is independent of w (as in the case of phase relaxation).

In the limit $\|H_s\|, \|V\| \ll w$, we obtain

$$N(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\epsilon\, e^{\epsilon t}\, \frac{\omega_s^2\epsilon^{\alpha-1} + 4v^2\Omega_{\alpha-1}(\epsilon)}{\omega_s^2\epsilon^\alpha + 4v^2\Omega_\alpha(\epsilon)}, \qquad (36)$$

where

$$\Omega_\beta(\epsilon) = [(\epsilon + 2iE_0)^\beta + (\epsilon - 2iE_0)^\beta]/2 \qquad (37)$$

and

$$E_0 = \sqrt{v^2 + \omega_s^2/4}.$$

Naturally, in the corresponding limits, expression (36) reproduces formulas (35) with

$$k_n \approx 2^{\alpha-1}\cos(\pi\alpha/2)(E_0/w)^\alpha$$

and

$$w_\alpha \approx \pi(1-\alpha)v^2/E_0$$

(see Fig. 2). Outside these limits, $N(t)$ can be evaluated numerically (some results are shown in Fig. 2). In general, $N(t)$ is the oscillating function (of frequency $\sim E_0$) with slowly decreasing average value and oscillation amplitude: for $E_0 t \gg 1$

$$N(t) \sim 1/t^\alpha$$

(except in the limit as $\alpha \to 1$).

6. CONCLUDING REMARKS

The presented analysis of relaxation kinetics in quantum systems induced by anomalous noise demonstrates a number of peculiarities of this kinetics. The peculiarities are analyzed with the use of the two-level quantum model, as an example, although the observed anomalous effects can manifest themselves in more complicated multi-level quantum systems. The proposed theoretical method is quite suitable for the analysis of these systems. This work is currently in progress.

Noteworthy is that in some limits, the developed theory predicts relaxation kinetics described by the Mittag – Leffler function $E_\alpha[-(wt)^\alpha]$. Following a number of recent works (for review, see Ref. [12]) this kinetics can be considered as a result of the anomalous Bloch equation with a fractional time derivative. For brevity, we have not discussed the corresponding representations.

It is also interesting to note that with the increase of α, the effects of anomaly of fluctuations decrease but still persist. To clarify them, we briefly consider the model

$$\Phi(\epsilon) = (\epsilon/w) + \zeta(\epsilon/w)^{1+\alpha},$$

in which $0 < \alpha < 1$, and w and ζ are constants with $\zeta \ll 1$ (a small value of ζ ensures that $W(t) > 0$). Possible effects can be analyzed within the SCTL with the use of Eqs. (22)–(24). For example, in the limit $\|H\|/w \ll 1$, we obtain the formula

$$\widetilde{\mathcal{R}} \approx [\epsilon + i\hat{H}_s + \zeta w^{-\alpha}\langle (i\hat{\mathcal{H}})^{1+\alpha} - (i\hat{H}_s)^{1+\alpha}\rangle]^{-1},$$

predicting the Bloch-type relaxation of both phase and population, but with the rate

$$\hat{W}_r = \zeta w^{-\alpha}\,\mathrm{Re}\langle (i\hat{\mathcal{H}})^{1+\alpha} - (i\hat{H}_s)^{1+\alpha}\rangle$$

that depends on w as $w^{-\alpha}$ and is therefore slower than in the conventional Bloch equation ($\hat{W}_r \sim 1/w$, [1]). Analysis also shows that in the expression for $\widetilde{\mathcal{R}}$, the terms proportional to $w(\epsilon/w)^{1+\alpha}$ occur as well. They lead to the inverse power- type asymptotic behavior of

$$\langle \mu|\hat{R}(t)|\mu\rangle \propto 1/t^{2+\alpha},$$

observed, however, only at very long times $t \gg w^{-1}$.

In our brief analysis, we neglected the effect of a possible natural width of lines corresponding to the additional slow exponential relaxation in the system. It is clear that the developed method allows taking these effects into account straightforwardly whenever needed.

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