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ONE-DIMENSIONAL ANISOTROPIC HEISENBERG MODEL IN THE TRANSVERSE MAGNETIC FIELD

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The one-dimensional spin-1/2 XXZ model in a transverse magnetic field is studied. It is shown that the field induces a gap in the spectrum of the model with the easy-plain anisotropy. Using conformal invariance, the field dependence of the gap is found at small fields. The ground state phase diagram is obtained. It contains four phases with the long-range order of different types and a disordered phase. These phases are separated by critical lines, where the gap and the long-range order vanish. Using scaling estimates, the mean-field approach, and numerical calculations in the vicinity of all critical lines, we find the critical exponents of the gap and the long-range order. It is shown that the transition line between the ordered and disordered phases belongs to the universality class of the transverse Ising model.

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1. INTRODUCTION

The effect of the magnetic field on an antiferromagnetic chain has been attracting much interest from theoretical and experimental standpoints. In particular, a strong dependence of the properties of quasi-one-dimensional anisotropic antiferromagnets on the field orientation was observed experimentally [1]. It is therefore interesting to study the dependence of properties of the one-dimensional antiferromagnet on the direction of the applied field. The simplest model of the one-dimensional anisotropic antiferromagnet is the spin-1/2 XXZ model. This model in a uniform longitudinal magnetic field (along the z axis) was studied in great detail [2]. Because the longitudinal field commutes with the XXZ Hamiltonian, the model can be exactly solved by the Bethe ansatz. This is not the case if the symmetry-breaking transverse magnetic field is applied and the exact integrability is lost. Because of its mathematical complexity, this model has not been studied much. From this standpoint, it is of a particular interest to study the ground state properties of the $1D$ XXZ model in the transverse magnetic field. The Hamiltonian of this model is given by

$$H = \sum_{n=1}^{N}(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z) + h\sum_{n=1}^{N} S_n^x \quad (1)$$

with periodic boundary conditions and even N.

The spectrum of the XXZ model is gapless for $-1 < \Delta \leq 1$. In the longitudinal field, the spectrum remains gapless if the field does not exceed the saturation value $(1+\Delta)$. On the other hand, a gap in the excitation spectrum seems to open up when the transverse magnetic field is applied. It is supposed [3] that this effect can explain the peculiarity of the low-temperature specific heat in Yb_4As_3 [1]. The magnetic properties of this compound are described by the XXZ Hamiltonian with $\Delta \approx 0.98$; it was shown that the magnetic field in the easy plain induces a gap in the spectrum resulting in a dramatic decrease of the linear term in the specific heat [3].

First of all, what do we know about model (1)?

The first part of the Hamiltonian is the well-known XXZ model, whose exact solution is given by the

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Bethe ansatz. In the Ising-like region $\Delta > 1$, the ground state of the XXZ model has a Neel long-range order along the z axis and there is a gap in the excitation spectrum. In the region $-1 < \Delta \leq 1$, the system is in the so-called spin-liquid phase with a power-law decay of correlations and a linear spectrum. Finally, for $\Delta < -1$, the classical ferromagnetic state is the ground state of the XXZ model with a gap over the ferromagnetic state.

In the transverse magnetic field, the total spin projection S^z is not a good quantum number and the model is essentially complicated, because the transverse field breaks rotational symmetry in the xy plane and destroys the integrability of the XXZ model, except at some special points. In particular, the exact diagonalization study of this model is difficult for finite systems because of a nonmonotonic behavior of energy levels.

The first special case of model (1) is the limit as $\Delta \to \pm\infty$. In this case, the model reduces to the $1D$ Ising model in a transverse field (ITF), which can be exactly solved by transforming it to the system of noninteracting fermions. In both limits, the system has the phase transition point $h_c = |\Delta|/2$, where the gap closes and the long-range order in the z direction vanishes.

It is suggested [4] that the phase transition of the ITF type occurs for any $\Delta > 0$ at some critical value $h = h_c(\Delta)$. It can also be expected that such a transition exists for any Δ and the transition line connects two limiting points $h_c = |\Delta|/2$, $\Delta \to \pm\infty$.

Similarly to these limiting cases, for any $|\Delta| > 1$ and $h < h_c(\Delta)$, the system has a long-range order in the z direction (the Neel order for $\Delta > 1$ and the ferromagnetic order for $\Delta < -1$). But for $|\Delta| < 1$ and $h < h_c(\Delta)$, the ground state changes and instead of the long-range order in the z direction, a staggered magnetization along the y axis appears at $h < h_c(\Delta)$.

This assumption is confirmed on the «classical» line $h_{cl} = \sqrt{2(1+\Delta)}$ ($h_{cl} < h_c(\Delta)$), where the quantum fluctuations of the XXZ model are compensated by the transverse field and the exact ground state of (1) at $h = h_{cl}$ is a classical one [5]. The excited states on the classical line are generally unknown, although it is assumed that the spectrum is gapped.

The second case where model (1) remains integrable is the isotropic antiferromagnetic case $\Delta = 1$. In this case, the direction of the magnetic field is not important and the ground state of the system remains the spin-liquid one up to the point $h = 2$, where a phase transition of the Pokrovsky–Talapov type occurs and the ground state becomes a completely ordered ferromagnetic state.

The last special case is $\Delta = -1$. Model (1) then reduces to the isotropic ferromagnetic model in a staggered magnetic field. This model is nonintegrable, but as shown [6], the system remains gapless up to some critical value $h = h_0$, where a phase transition of the Kosterlitz–Thouless type occurs.

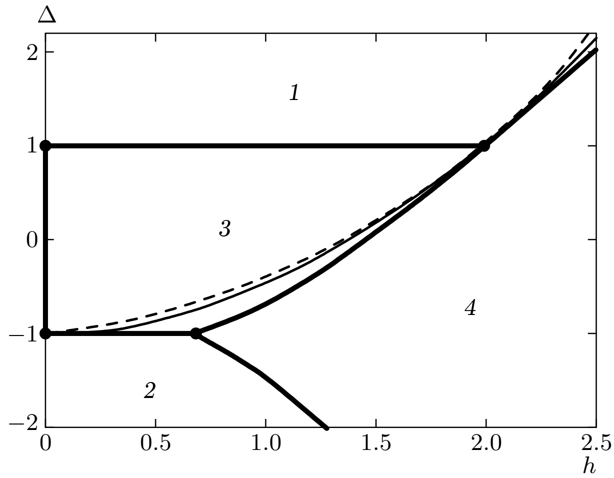

Fig. 1. Phase diagram of model (1). The thick solid lines are the critical lines, the thin solid line is the «classical» line, and the dashed line is the line $h_1(\Delta)$

Summarizing, we expect that the phase diagram of model (1) (in the (Δ, h) plane) has the form shown in Fig. 1. The phase diagram contains four regions that correspond to different phases and are separated by transition lines. Each phase is characterized by a long-range order of its own type: the Neel order along the z axis in region (1); the ferromagnetic order along the z axis in region (2); the Neel order along the y axis in region (3); in the region (4), there is no long-range order except the magnetization along the field direction x (which certainly exists in all the above regions). By the long-range order, we hereafter understand the one of the type corresponding to a given region.

In this paper, we investigate the behavior of the gap and the long-range order near the transition (critical) lines. In Sec. 2, devoted to the classical line, we review the exact ground state and construct three exact excitations. In Sec. 3, we study the transition line $h_c(\Delta)$ using the mean-field approach and the exact diagonalization of finite systems. In Sec. 4, we find the critical exponents in the vicinity of the line $h = 0$. The properties of the model near the critical lines $\Delta = \pm 1$ and in the vicinity of the points ($\Delta = \pm 1$, $h = 0$) in particular are studied in Secs. 5 and 6.

2. THE CLASSICAL LINE

We first we consider the classical line

$$h_{cl} = \sqrt{2(1+\Delta)}, \quad \Delta > -1,$$

because we often refer to it in what follows. It is remarkable in the sense that the ground state is identical to the classical one on this line and quantum flucuations are missing. It was shown in [5] that the ground state of (1) is two-fold degenerate on this line and the ground state wave functions with the momentum $k = 0$ and $k = \pi$ are given by

$$\Psi_{1,2} = \frac{1}{\sqrt{2}}(\Phi_1 \pm \Phi_2),$$

where $\Phi_{1(2)}$ are direct products of single-site functions,

$$|\Phi_1\rangle = |\alpha_1\bar{\alpha}_2\alpha_3\bar{\alpha}_4\ldots\rangle,$$

$$|\Phi_2\rangle = |\bar{\alpha}_1\alpha_2\bar{\alpha}_3\alpha_4\ldots\rangle.$$

Here, $|\alpha_i\rangle$ is the state of the ith spin lying in the xy plane for $|\Delta| < 1$ (or in the xz plane for $\Delta > 1$) at the angle φ with the x axis. These states can be written as

$$|\alpha_i\rangle = (e^{i\varphi}S_i^+ - 1)\,|{\downarrow}\rangle, \quad |\Delta| < 1,$$

$$|\alpha_i\rangle = (e^{\varphi}S_i^+ - 1)\,|{\downarrow}\rangle, \quad \Delta > 1$$

with

$$\cos\varphi = h_{cl}/2, \quad |\Delta| < 1$$

and

$$\operatorname{ch}\varphi = h_{cl}/2, \quad \Delta > 1.$$

The state $|\bar{\alpha}_i\rangle$ is obtained by rotation of the ith spin by π about the magnetic field axis x,

$$|\bar{\alpha}_i\rangle = e^{i\pi S_i^x}\,|\alpha_i\rangle.$$

The ground state has a two-sublattice structure and is characterized by the presence of the long-range order in the y ($|\Delta| < 1$) or in the z ($\Delta > 1$) directions. In particular, for $|\Delta| < 1$, the staggered magnetization $\langle S_n^y\rangle$ is

$$\langle S_n^y\rangle = \frac{(-1)^n}{2}\sqrt{1 - \frac{h_{cl}^2}{4}}.$$

In general, the excited states of (1) on the classical line are nontrivial. Some of them can nevertheless be found exactly. For this, it is convenient to introduce the operator overturning the ith spin,

$$R_i = e^{i\pi S_i^z}, \quad |\Delta| < 1,$$

$$R_i = e^{i\pi S_i^y}, \quad \Delta > 1,$$

such that the states of the «overturned» ith spin $|\beta_i\rangle = R_i\,|\alpha_i\rangle$ and $|\bar{\beta}_i\rangle = R_i\,|\bar{\alpha}_i\rangle$ are orthogonal to $|\alpha_i\rangle$ and $|\bar{\alpha}_i\rangle$,

$$\langle\alpha_i|\beta_i\rangle = \langle\bar{\alpha}_i|\bar{\beta}_i\rangle = 0.$$

The exact excited states are then written as

$$\left|\psi_{1(2)}^1\right\rangle = \sum_m R_m\left|\Phi_{1(2)}\right\rangle,$$

$$\left|\psi_{1(2)}^2\right\rangle = \sum_n (-1)^n R_n R_{n+1}\left|\Phi_{1(2)}\right\rangle,$$

$$\left|\psi_{1(2)}^3\right\rangle = \sum_{n,m} (-1)^n R_n R_{n+1} R_m\left|\Phi_{1(2)}\right\rangle,$$

and therefore, each of the three exact excitations is also two-fold degenerate. This degeneracy is in fact a consequence of the Z_2 symmetry describing the rotation of all spins by π about the magnetic field axis x.

To show that these states are indeed the exact ones, it is convenient to rotate the coordinate system such that in one of the ground states, for example Φ_1, all spins point down. In the case where $|\Delta| < 1$, this transformation is the rotation of the spins at even (odd) sites by an angle φ ($-\varphi$) around the z axis followed by the rotation by $\pi/2$ around the y axis,

$$\begin{aligned} S_n^x &= \sigma_n^z\cos\varphi + (-1)^n\sigma_n^y\sin\varphi, \\ S_n^y &= (-1)^n\sigma_n^z\sin\varphi - \sigma_n^y\cos\varphi, \\ S_n^z &= -\sigma_n^x. \end{aligned} \tag{2}$$

In the case where $\Delta > 1$, the transformation of the spin operators is defined by

$$\begin{aligned} S_n^x &= \sigma_n^z\cos\varphi + (-1)^n\sigma_n^x\sin\varphi, \\ S_n^y &= \sigma_n^y, \\ S_n^z &= -(-1)^n\sigma_n^z\sin\varphi + \sigma_n^x\cos\varphi. \end{aligned} \tag{3}$$

On the classical line, Hamiltonian (1) then becomes

$$H_1 = \Delta\sum_n \sigma_n\sigma_{n+1} + (1+\Delta)\sum_n \sigma_n^z + h_{cl}\sqrt{1 - \frac{h_{cl}^2}{4}}\sum_n (-1)^n\sigma_n^y(\sigma_{n+1}^z + \sigma_{n-1}^z + 1) \tag{4}$$

for $\Delta < 1$ and

$$H_2 = \sum_n \sigma_n\sigma_{n+1} - (\Delta-1)\sum_n \sigma_n^z\sigma_{n+1}^z + 2\sum_n \sigma_n^z + \sqrt{h_{cl}^2 - 4}\sum_n (-1)^n\sigma_n^x(\sigma_{n+1}^z + \sigma_{n-1}^z + 1) \tag{5}$$

for $\Delta > 1$.

The ground state of both Hamiltonians and of (1) is two-fold degenerate. Obviously, in one of the ground states, all spins σ_n point down,

$$\Phi_1 = |0\rangle \equiv |\downarrow\downarrow\downarrow\ldots\rangle .$$

The energy of this state is

$$E_0 = -\frac{N}{2} - N\frac{\Delta}{4}. \tag{6}$$

In this representation, the second ground state Φ_2 has a more complicated form,

$$\widetilde{\Phi}_2 = \prod_n (\cos\varphi + (-1)^n \sigma_n^+ \sin\varphi)\,|0\rangle .$$

It is now easy to see that the following three excited states are exact:

$$\begin{gathered} \left|\psi_1^{(1)}\right\rangle = \sum_n \sigma_n^+ |0\rangle , \quad E_1 - E_0 = 1 + \Delta, \\ \left|\psi_1^{(2)}\right\rangle = \sum_n (-1)^n \sigma_n^+ \sigma_{n+1}^+ |0\rangle , \\ E_2 - E_0 = 2 + \Delta, \\ \left|\psi_1^{(3)}\right\rangle = \sum_{n,m} (-1)^n \sigma_n^+ \sigma_{n+1}^+ \sigma_m^+ |0\rangle , \\ E_3 - E_0 = 3 + 2\Delta. \end{gathered} \tag{7}$$

It can be verified that the last terms in (4) and (5) annihilate these three functions and are therefore the exact excited states of (1) for any even N. Similarly to the ground state, excited states (7) are degenerate with the states $|\psi_2^k\rangle$. These states $|\psi_2^k\rangle$ can be represented in the same form (7), but in the coordinate system where the function Φ_2 describes all spins pointing down.

The states $\left|\psi_{1(2)}^1\right\rangle$ are especially interesting because they define the gap of model (1) on the classical line at small values of h_{cl}. Our numerical calculations of finite systems show that as $h_{cl} \to 0$ ($\Delta \to -1$), the lowest branch of the excitations has a minimum at $k = 0$ and the corresponding excitation energy is $(1 + \Delta)$ (of course, because of the Z_2 symmetry, there is another branch with the minimum at $k = \pi$ and the same minimum energy, but we consider one branch only). The excitation energy at $k = \pi$ obtained by the extrapolation of numerical calculations as $N \to \infty$ is $2(1 + \Delta)$. As h_{cl} increases, the excitation energies at $k = 0$ and $k = \pi$ are drawn together and become equal to each other at some $\widetilde{h}_{cl}$. Our numerical results give

$$\widetilde{h}_{cl} \approx 0.76 \quad (\Delta \approx -0.79).$$

On the classical line, the gap is therefore $(1 + \Delta)$ for $-1 < \Delta < -0.79$.

3. THE TRANSITION LINE $h = h_c(\Delta)$

The existence of the transition line $h_c(\Delta)$ passing through the entire phase diagram is quite natural, because all types of the long-range order except the long-range order along the field must vanish at some value of the magnetic field. The transition line connects two obvious limits as $\Delta \to \pm\infty$, where model (1) reduces to the ITF model. The line passes through the exactly solvable point ($\Delta = 1$, $h = 2$) and the point ($\Delta = -1$, $h = h_0$) studied in [6]. We suppose that the entire line $h_c(\Delta)$ is of the ITF type with algebraically decaying correlations with the corresponding critical exponents [7].

The transition line can also be observed from the numerical calculations of finite systems. As an example, the dependences of the excitation energies of three lowest levels on h are shown in Fig. 2 for $\Delta = 0$ and for $N = 10$–18. From this figure, it can be seen that the two lowest states cross each other $N/2$ times and the last crossing occurs at the classical point $h_{cl} = \sqrt{2}$. These two states form a two-fold degenerate ground state in the thermodynamic limit. They have different momenta $k = 0$ and $k = \pi$ and different quantum numbers describing the Z_2 symmetry that remains in the system after applying the field. As for the first excitation above the degenerate ground state, we also see numerous level crossings in Fig. 2. These level crossings lead to incommensurate effects that manifest themselves in the oscillatory behavior of the spin cor-

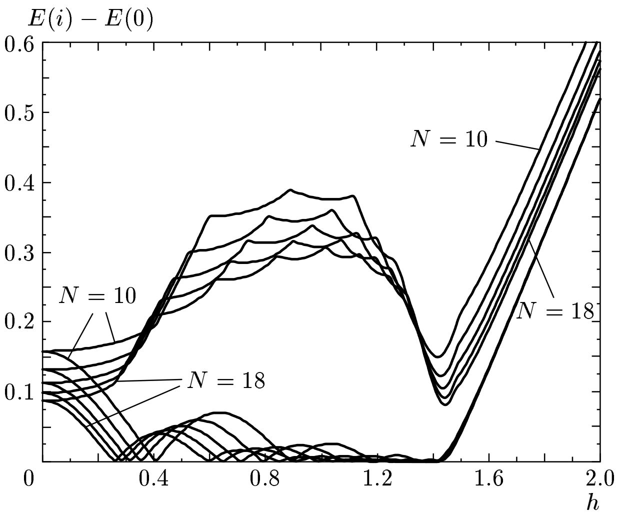

Fig. 2. The dependence of the difference between the energy of two lowest levels $E(1)$, $E(2)$ and the ground state energy $E(0)$ on magnetic field h for finite chains with $N = 10, \ldots, 18$

relation functions. The correlation functions at $n \gg 1$ are given by

$$\langle S_1^\alpha S_n^\alpha \rangle - \langle S^\alpha \rangle^2 = f(n) e^{-\kappa n}, \quad (8)$$

where $\langle S^\alpha \rangle$ ($\alpha = x, y, z$) is the corresponding magnetization (the long-range order) and $f(n)$ is the oscillatory function of n with the oscillation period depending on h and Δ. All crossings disappear at $h > h_{cl}(\Delta)$ and the correlation functions do not contain oscillatory terms in this region of the phase diagram.

The energy of the first excitation near h_{cl} decreases rapidly, and after extrapolation we found that for $\Delta = 0$, the gap vanishes at the magnetic field $h_c \approx 1.456(6) > h_{cl}$. Inside the region $h_{cl} < h < h_c$, the ground state remains two-fold degenerate, although there are no level crossings. At $h > h_c$, the mass gap appears again; for a large field, the gap is proportional to h.

To determine the transition line $h_c(\Delta)$ and to study the model in the vicinity of $h_c(\Delta)$, we use the Fermi representation of (1). This representation gives the exact solution in the limits as $\Delta \to \pm\infty$ and in addition yields the exact ground state on the classical line.

First, it is convenient to perform a rotation of the spins around the y axis by $\pi/2$ in (1) such that the magnetic field is directed along the z axis,

$$H = \sum_n (\Delta S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z) + h \sum_n S_n^z. \quad (9)$$

After the Jordan–Wigner transformation to Fermi operators a_n^+ and a_n,

$$S_n^+ = e^{i\pi \sum a_j^+ a_j} a_n, \qquad S_n^z = a_n^+ a_n - \frac{1}{2}, \quad (10)$$

Hamiltonian (9) becomes

$$H_f = -\frac{hN}{2} + \frac{N}{4} + \sum_k \left(h - 1 - \frac{1+\Delta}{2} \cos k \right) a_k^+ a_k + \frac{1-\Delta}{4} \sum_k \sin k (a_k^+ a_{-k}^+ + a_{-k} a_k) + \sum_n a_n^+ a_n a_{n+1}^+ a_{n+1}. \quad (11)$$

Treating the Hamiltonian H_f in the mean-field approximation, we find the ground state energy E_0 and the one-particle excitation spectrum $\varepsilon(k)$,

$$\frac{E_0}{N} = (h-1)\left(\gamma_1 - \frac{1}{2}\right) + \frac{1}{4} - \left(1 - \frac{g}{2}\right)\gamma_2 + \frac{g}{2}\gamma_3 + \gamma_1^2 - \gamma_2^2 + \gamma_3^2, \quad (12)$$

$$\varepsilon(k) = \sqrt{a^2(k) + b^2(k)}, \quad (13)$$

where $g = 1 - \Delta$ and

$$a(k) = (h-1) - \left(1 - \frac{g}{2}\right)\cos k + 2\gamma_1 - 2\gamma_2 \cos k, \qquad b(k) = \left(\frac{g}{2} + 2\gamma_3\right)\sin k. \quad (14)$$

The quantities γ_1, γ_2, and γ_3 are the ground state averages determined by the self-consistent equations:

$$\gamma_1 = \langle a_n^+ a_n \rangle = \sum_{k>0} \left(1 - \frac{a(k)}{\varepsilon(k)}\right), \qquad \gamma_2 = \langle a_n^+ a_{n+1} \rangle = -\sum_{k>0} \frac{a(k)}{\varepsilon(k)} \cos k, \qquad \gamma_3 = \langle a_n^+ a_{n+1}^+ \rangle = -\sum_{k>0} \frac{b(k)}{2\varepsilon(k)} \sin k. \quad (15)$$

The magnetization $S = \langle S_n^x \rangle$ of model (1) is given by

$$S = \frac{1}{2} - \gamma_1. \quad (16)$$

The numerical solution of Eqs. (15) shows that the function $\varepsilon(k)$ has a minimum at k_{min}, which changes from $\pi/2$ at $h = 0$ to zero at $h = h_1(\Delta)$ and $k_{min} = 0$ for $h > h_1(\Delta)$. The gap in the spectrum $\varepsilon(k)$ vanishes at $h_c(\Delta)$ ($h_c > h_1$) and is given by $m = |h - h_c|$ for $h > h_1$. The functions $h_1(\Delta)$ and $h_c(\Delta)$ are shown in Fig. 1. We note that the Hamiltonian H_f differs from the domain-wall fermionic Hamiltonian that is mapped from (1) in [4]. The transition line obtained in [4] is a linear function of Δ in contrast to $h_c(\Delta)$ in Fig. 1.

It is interesting to note that the mean-field approximation gives the exact ground state on the classical line $h_{cl} = \sqrt{2(1+\Delta)}$. On this line, the solution of Eqs. (15) has the simple form

$$\gamma_1 = \frac{1}{2} - \frac{h_{cl}}{4}, \quad \gamma_2 = -\gamma_3 = \frac{4 - h_{cl}^2}{16}, \quad |\Delta| < 1, \qquad \gamma_1 = \frac{1}{2} - \frac{1}{h_{cl}}, \quad \gamma_2 = \gamma_3 = \frac{h_{cl}^2 - 4}{4h_{cl}^2}, \quad \Delta > 1, \quad (17)$$

and the energy is given by

$$\frac{E_0}{N} = -\frac{1}{2} - \frac{\Delta}{4}.$$

On the classical line in the mean-field approximation, the gap is

$$m = \frac{1}{4}(2 - h_{cl})^2, \quad |\Delta| < 1, \qquad m = \frac{h_{cl}^2 - 2}{2h_{cl}^2}(h_{cl} - 2)^2, \quad \Delta > 1. \quad (18)$$

We compared (18) with the results of the extrapolation of finite systems on the classical line. The coincidence is sufficiently good for $\Delta > 0.5$. Equation (18) gives a satisfactory estimate for the gap up to $\Delta \approx -0.5$. For example, at $\Delta = 0$ ($h_{cl} = \sqrt{2}$), it follows that $m = 0.086$ from Eq. (18), while the extrapolated gap is $m \approx 0.076(4)$.

The smaller the fermion density, the better the mean-field approximation works. It becomes worse as the magnetization $S \to 0$. This is the reason of incorrect behavior of the gap as $h_{cl} \to 0$ ($\Delta \to -1$). It follows from (18) that $m = 1$, while m vanishes in this limit as $m = (1 + \Delta)$ (7).

In the mean-field approximation, the Hamiltonian H_f is similar to the well-known bilinear Fermi Hamiltonian describing the anisotropic XY model or the ITF model. Using results in [7], the following facts related to the model under consideration can be established.

1. There is a staggered magnetization $\langle S_n^y \rangle$ along the y axis for $|\Delta| < 1$ or $\langle S_n^z \rangle$ along the z axis for $|\Delta| > 1$, and they vanish as $(h_c - h)^{1/8}$ for $h \to h_c$.

2. The magnetization S has a logarithmic singularity as $h \to h_c$.

3. The spin correlation function decays exponentially (excluding the transition line) as $n \to \infty$,

$$G^\alpha(n) = \langle S_1^\alpha S_n^\alpha \rangle - \langle S^\alpha \rangle^2 = f(n) e^{-\kappa n}. \qquad (19)$$

The function $f(n)$ has an oscillatory behavior for $0 < h < h_{cl}$ and is monotonic for $h > h_{cl}$; $f(n) = 0$ at $h = h_{cl}$ and

$$f(n) \approx \frac{\cos \omega n}{n^2}, \quad \omega = \sqrt{2 \frac{h_{cl} - h}{h_{cl}}}$$

for $h_{cl} - h \ll 1$. The classical line therefore determines the boundary on the phase diagram where the spin correlation functions show the incommensurate behavior.

On the transition line $h = h_c(\Delta)$, the spin correlation functions have a power-law decay,

$$\begin{aligned} &G^x(n) \propto 1/n^2, \quad G^y(n) \propto 1/n^{1/4}, \\ &G^z(n) \propto 1/n^{9/4}, \qquad\qquad |\Delta| < 1, \\ &G^x(n) \propto 1/n^2, \quad G^y(n) \propto 1/n^{9/4}, \\ &G^z(n) \propto 1/n^{1/4}, \qquad\qquad |\Delta| > 1. \end{aligned} \qquad (20)$$

These results show that the transition at $h = h_c(\Delta)$ belongs to the universality class of the ITF model.

In the vicinity of the point $h = 2$, $\Delta = 1$, the fermion density is small ($S \approx 1/2$) and the mean-field approximation of the four-fermion term gives the accuracy up to g^3 or $(2 - h)^4$ at least. In this case, we give the corresponding expressions (for $g \ll 1$):

$$\begin{aligned} &h_c = 2 - \frac{g}{2} - \frac{g^2}{32}, \\ &h_1 = h_c - \frac{g^2}{16}, \\ &m = \begin{cases} |h - h_c|, & h > h_1, \\ \dfrac{g}{2\sqrt{2}} \sqrt{h_c - h - \dfrac{g^2}{32}}, & h < h_1. \end{cases} \end{aligned} \qquad (21)$$

The magnetization S is

$$S = \begin{cases} \dfrac{1}{2} - \dfrac{\sqrt{2}}{\pi} \sqrt{h_c - h} - \dfrac{g}{8\pi}, \\ \qquad\qquad g \ll \sqrt{h_c - h}, \\ \dfrac{1}{2} - \dfrac{g}{4\pi} - \dfrac{2(h_c - h)}{\pi g} \ln\left(\dfrac{g^2}{h_c - h}\right), \\ \qquad\qquad g \ll \sqrt{h_c - h}. \end{cases} \qquad (22)$$

The susceptibility $\chi(h) = dS/dh$ is

$$\chi(h) = \begin{cases} \dfrac{2}{\pi g} \ln\left(\dfrac{g^2}{h_c - h}\right), & g \gg \sqrt{h_c - h}, \\ \dfrac{1}{\sqrt{2}\pi} \dfrac{1}{\sqrt{h_c - h}}, & g \ll \sqrt{h_c - h}. \end{cases} \qquad (23)$$

It follows from (23) that there is a crossover from the square root to the logarithmic divergence of χ as the parameter $g^2/(h_c - h)$ varies from 0 to ∞.

4. THE LINE $h = 0$, $|\Delta| < 1$

4.1. Scaling estimates

The XXZ model is integrable and its low-energy properties are described by a free massless boson field theory with the Hamiltonian

$$H_0 = \frac{v}{2} \int dx \left[\Pi^2 + (\partial_x \Phi)^2 \right], \qquad (24)$$

where $\Pi(x)$ is the momentum conjugate to the boson field $\Phi(x)$, which can be separated into the left and right moving terms,

$$\Phi = \Phi_L + \Phi_R.$$

The dual field $\tilde{\Phi}$ is defined as the difference

$$\tilde{\Phi} = \Phi_L - \Phi_R.$$

The spin-density operators are represented as

$$S_n^z \approx \frac{1}{2\pi R}\partial_x\Phi + \mathrm{const}(-1)^n \cos\frac{\Phi}{R},$$
$$S_n^x \approx \cos\left(2\pi R\tilde{\Phi}\right)\left[C(-1)^n + \mathrm{const}\cdot\cos\frac{\Phi}{R}\right] \qquad (25)$$

with the constant C found in [8]. The compactification radius R is known from the exact solution

$$2\pi R^2 = \theta = 1 - \frac{\arccos\Delta}{\pi}.$$

The nonoscillating part of the operator S^x in Eq. (25) has the scaling dimension

$$d = \frac{\theta}{2} + \frac{1}{2\theta}$$

and conformal spin $S = 1$. A nonzero conformal spin of the perturbation operator S^x can lead to incommensurability in the system [9], which agrees with Eq. (19). As shown in [10], the general formula for the mass gap

$$m \sim h^\nu, \quad \nu = \frac{1}{2-d} = \frac{2}{4-\theta-1/\theta}, \qquad (26)$$

is not applicable in the entire region $|\Delta| < 1$. Because of a nonzero conformal spin of the nonoscillating part of the operator S^x, higher-order effects in h must be considered. The analysis shows [10] that the original perturbation with a nonzero conformal spin generates another perturbation with zero conformal spin,

$$V = h^2 \cos\left(4\pi R\tilde{\Phi}\right). \qquad (27)$$

This perturbation gives the critical exponent for the mass gap

$$m \sim h^\gamma, \quad \gamma = \frac{1}{1-\theta}. \qquad (28)$$

Comparing Eqs. (26) and (28), we see that perturbation (27) becomes more relevant in the region

$$\Delta < \cos(\pi\sqrt{2}) \approx -0.266.$$

It turns out that the oscillating part of the operator S^x gives another, more relevant index for the gap at $\Delta < 0$. We now reproduce the standard «conformal» chain of arguments for this oscillating part. The perturbed action of the model is given by

$$S = S_0 + h\int dt\,dx\,S^x(x,t), \qquad (29)$$

where S_0 is the Gaussian action of the XXZ model. The time-dependent correlation functions of the XXZ chain show the power-law decay at $|\Delta| < 1$ and have the asymptotic form [11]

$$\langle S^x(x,\tau)S^x(0,0)\rangle \sim$$
$$\sim \frac{(-1)^x A_1}{(x^2+v^2\tau^2)^{\theta/2}} - \frac{A_2}{(x^2+v^2\tau^2)^{\theta/2+1/2\theta}}, \qquad (30)$$

where A_1 and A_2 are known constants [8] and $\tau = it$ is the imaginary time. We can therefore estimate the large-distance contribution to the action of the oscillating part of the operator $S^x(x,\tau)$ as

$$h\int d\tau\,dx\,S^x(x,\tau) \sim h\int d\tau\sum_n \frac{(-1)^n}{(n^2+v^2\tau^2)^{\theta/4}} \sim$$
$$\sim h\int d\tau \sum_{even\,n} \frac{\theta n}{(n^2+v^2\tau^2)^{\theta/4}} \sim$$
$$\sim h\int d\tau\,dx \frac{\theta x}{(x^2+v^2\tau^2)^{\theta/4+1}}.$$

The relevant field $S^x(x,\tau)$ leads to a finite correlation length ξ. This correlation length is such that the contribution of the field $S^x(x,\tau)$ to the action is of the order of unity. That is,

$$h\int_0^{\xi/v} d\tau \int_0^{\xi} dx \frac{\theta x}{(x^2+v^2\tau^2)^{\theta/4+1}} \sim \frac{\theta h \xi^{1-\theta/2}}{v} \sim 1$$

which gives the mass gap

$$m \sim \frac{v}{\xi} \sim h^\mu, \quad \mu = \frac{1}{1-\theta/2}. \qquad (31)$$

In fact, the oscillating factor $(-1)^n$ in the correlator in some sense eliminates one singular integration over x, and the general conformal formula

$$m \propto h^{1/(D-d)},$$

where D is the dimension of space and d is the scaling dimension of the perturbation operator, must be taken with $D = 1$ instead of conventional $D = 2$.

The comparison of Eqs. (26), (28), and (31) shows that for $0 < \Delta < 1$, the leading term is given by Eq. (26) and for $-1 < \Delta < 0$, by Eq. (31). We therefore have

$$m \sim h^\nu, \quad 0 < \Delta < 1,$$
$$m \sim h^\mu, \quad -1 < \Delta < 0. \qquad (32)$$

The functions $\nu(\Delta)$, $\mu(\Delta)$, and $\gamma(\Delta)$ are shown in Fig. 3. In this respect, model (1) is different from the XXZ model in the staggered transverse field, for which

$$m \propto h^{2/(4-\theta)}$$

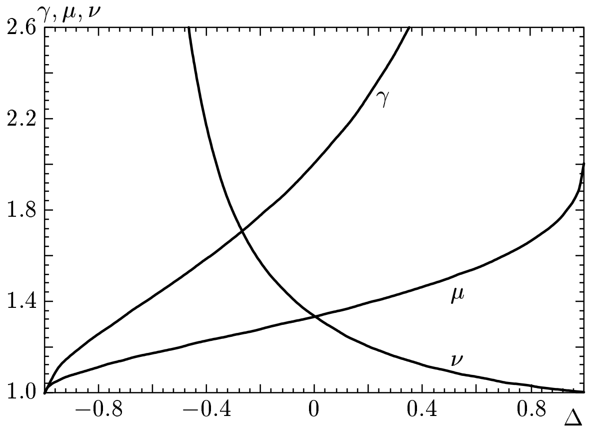

Fig. 3. The dependence of the critical exponents ν, μ, and γ on Δ. The smallest exponent gives the perturbation of the most relevant type and defines the index for the mass gap

for all $|\Delta| < 1$ [12].

The staggered magnetization (long-range order) along the y axis behaves as

$$\langle S_n^y \rangle \sim \frac{(-1)^n}{\xi^{\theta/2}} \sim (-1)^n m^{\theta/2}. \tag{33}$$

Hence, the long-range order also has two different critical exponents,

$$\begin{aligned} \langle |S^y| \rangle &\sim h^{\theta/(4-\theta-1/\theta)}, \quad 0 < \Delta < 1, \\ \langle |S^y| \rangle &\sim h^{\theta/(2-\theta)}, \quad -1 < \Delta < 0. \end{aligned} \tag{34}$$

4.2. Perturbation series

The critical exponents ν and μ can also be derived from the analysis of infrared divergences of the perturbation theory in h. Obviously, only even orders in h give contributions. We now estimate the large-distance behavior of the operator

$$U = \frac{1}{E_0 - H_0} V \frac{1}{E_0 - H_0} V \tag{35}$$

determining the perturbation theory order, where $V = h \sum S_i^x$ and H_0 is the Hamiltonian of the XXZ model. The perturbation series for the ground state energy is given by

$$\delta E \sim V \frac{1}{E_0 - H_0} V (1 + U + U^2 + \ldots). \tag{36}$$

We consider a large but finite system of the length N. We keep the powers of N and h only, omitting all other factors. We first consider the nonoscillating part of correlator (30). Taking only low-lying excitations of the spectrum of the XXZ model into account (these excitations give the most divergent part) and estimating the large-distance behavior of the nonoscillating part of correlator (30), we arrive at

$$U \sim h^2 \frac{\sum\limits_{i,j} \langle S_i^x S_j^x \rangle}{(1/N)^2} \sim h^2 N^2 \frac{N^2}{N^{\theta+1/\theta}} = \\ = h^2 N^{4-\theta-1/\theta}. \tag{37}$$

It follows that if $4 - \theta - 1/\theta > 0$, then each next order in perturbation theory (36) diverges more and more strongly. To absorb these infrared divergences, we must introduce the scaling parameter $y = Nh^\nu$ and assume that the series $\left(1 + U + U^2 + \ldots\right)$ in (36) forms some function of the scaling parameter y. In our case,

$$\nu = \frac{2}{4 - \theta - 1/\theta}$$

(see Eq. (26)) and

$$U \propto y^{2/\nu}.$$

The leading second-order divergence of the ground state energy can be found similarly to (37),

$$\delta E^{(2)} = V \frac{1}{E_0 - H_0} V \sim h^2 N^{3-\theta-1/\theta}. \tag{38}$$

Combining Eqs. (37) and (38), we can write

$$\delta E \sim N h^{2\nu} f(y)$$

with some unknown function $f(y)$ whose small-y expansion is given by

$$f(y) = \frac{1}{y^2} \sum_{n=1}^{\infty} c_n y^{2n/\nu}.$$

In the thermodynamic limit as $N \to \infty$, the scaling parameter $y = Nh^\nu$ also tends to infinity, $y \to \infty$. Because the energy is proportional to N, the function $f(y)$ has a finite limit $f(\infty) = a$. In the thermodynamic limit for the correction to the ground state energy, we therefore have

$$\delta E \sim a N h^{2\nu}. \tag{39}$$

For the first excited state, the perturbation theory divergences have the same form as in (37) and (38). For the gap, we therefore find the same scaling parameter $y = Nh^\nu$ and

$$m \sim N h^{2\nu} g(y).$$

In the thermodynamic limit, the mass gap is of the order of unity (in terms of N), and therefore, the function $g(y) \propto 1/y$ as $y \to \infty$. Thus, finally we arrive at the Eq. (26).

We now consider the more subtle, oscillating part of correlator (30). For the oscillating part at large distances, we can write

$$\sum_{i,j} \langle S_i^x S_j^x \rangle \sim N \sum_r \frac{(-1)^r}{r^\theta} \sim N \sum_r \frac{1}{r^{\theta+1}} \sim N \frac{1}{N^\theta}.$$

The oscillating part of the perturbation operator V connects the low-lying gapless states with finite-energy states. That is, each second level in all orders of the perturbation series is separated from the ground state by a finite gap. For the operator U, we therefore have

$$U \sim h^2 \frac{\sum_{i,j} \langle S_i^x S_j^x \rangle}{(1/N)} \sim h^2 N^{2-\theta}.$$

Because θ is always less than 2, the divergences grow with the order of the perturbation theory. To eliminate these divergences, we introduce the scaling parameter $y = Nh^\mu$, with μ defined in Eq. (31), such that $U \sim y^{2/\mu}$.

The second-order correction to the ground state energy is then given by

$$\delta E^{(2)} \sim h^2 \frac{\sum_{i,j} \langle S_i^x S_j^x \rangle}{1} \sim h^2 N^{1-\theta}$$

and the total correction to the ground state energy is

$$\delta E \sim N h^{2\mu} f(y),$$

where $f(y)$ is an unknown function with a finite limit $f(\infty) = b$.

In the thermodynamic limit, the ground state energy therefore behaves as

$$\delta E \sim bNh^{2\mu}.$$

The mass gap is found similarly,

$$m \sim Nh^{2\mu} g(y)$$

with the function $g(y) \propto 1/y$ as $y \to \infty$. We thus reproduce Eq. (31) in the thermodynamic limit.

We note that we have estimated only the long wavelength divergent part of the perturbation theory. In addition, the regular part of the perturbation theory gives the leading term of the order h^2. Combining all the above facts, we thus arrive at

$$\frac{\delta E}{N} = -\frac{\chi}{2} h^2 + ah^{2\nu} + bh^{2\mu}. \tag{40}$$

As can be seen from Eq. (40), δE consists of a regular term h^2 and two singular terms. Because $\nu > 1$ and $\mu > 1$, the susceptibility χ is finite for any Δ in contrast to the model with the staggered transverse field [12], where the singular term is h^η with $\eta = 4/(4-\theta) < 2$.

It follows from Eqs. (26) and (31) that $\nu \to 1$ as $\Delta \to 1$ and $\mu \to 1$ as $\Delta \to -1$. In both limits, one of the singular terms therefore becomes proportional to h^2, and hence, contributes to the susceptibility. This implies that the susceptibility has a jump at the symmetric points $\Delta = \pm 1$.

5. THE LINE $\Delta = 1$

In the vicinity of the line $\Delta = 1$, it is convenient to rewrite Hamiltonian (1) as

$$\begin{aligned} H &= H_0 + V, \\ H_0 &= \sum_n (\mathbf{S}_n \cdot \mathbf{S}_{n+1}) + h \sum_n S_n^x, \\ V &= -g \sum_n S_n^z S_{n+1}^z, \end{aligned} \tag{41}$$

where the parameter $g = 1 - \Delta \ll 1$ is small. On the isotropic line $\Delta = 1$, model (1) is exactly solvable by the Bethe ansatz. The properties of the system remain critical up to the transition point $h_c = 2$, where the ground state becomes ferromagnetic. Therefore, for $h < 2$ and small perturbation V, we can use conformal estimates.

The asymptotic form of the correlation function on this line is given by

$$\langle S_i^z S_{i+n}^z \rangle \sim \frac{(-1)^n}{n^{\alpha(h)}}, \tag{42}$$

where $\alpha(h)$ is a known function obtained from the Bethe ansatz [13]. It has the asymptotic forms

$$\alpha(h) \sim \begin{cases} 1 - \dfrac{1}{2\ln(1/h)}, & h \to 0, \\ \dfrac{1}{2}, & h \to 2. \end{cases} \tag{43}$$

The scaling dimension of the operator S^z is $d_z = \alpha(h)/2$ and the scaling dimension of $S_i^z S_{i+1}^z$ is four times greater, $d_{zz} = 4d_z = 2\alpha(h)$. Because $\alpha(h) < 1$, the perturbation V is relevant and leads to the mass gap and the staggered magnetization given by

$$\begin{aligned} m &\sim |g|^{1/(2-d_{zz})} = |g|^{1/(2-2\alpha)}, \\ \langle |S^y| \rangle &\sim |g|^{\alpha/(4-4\alpha)}, \quad \Delta < 1, \\ \langle |S^z| \rangle &\sim |g|^{\alpha/(4-4\alpha)}, \quad \Delta > 1. \end{aligned} \tag{44}$$

From the general expressions for the mass gap in Eq. (44), we obtain that $m \sim g$ in the limit as $h \to 2$, which agrees with the result of the mean-field approximation in Eq. (21).

In the vicinity of the point $\Delta = 1$, $h = 2$, the long-range order vanishes on both lines: at $\Delta = 1$ as $g^{1/4}$ (see (44)) and at $h = h_c$ as $|h_c - h|^{1/8}$. We also have the exact expression for the long-range order on the classical line,

$$\langle |S^y| \rangle_{cl} = \frac{\sqrt{g}}{2\sqrt{2}}. \tag{45}$$

Combining all these facts, we arrive at the formula

$$\langle |S^y| \rangle = 2^{-7/8} g^{1/4} \left| h_c - h \right|^{1/8}. \tag{46}$$

The behavior of the system near the point $\Delta = 1$, $h = 0$ is more subtle. As follows from Eq. (32), the mass gap is $m \sim h$ for very small h; on the other hand, Eq. (44) implies a different scaling $m \sim g^{\ln(1/h)}$. Therefore, there are two regions near this point with different behaviors of the mass gap. The boundary between these two regions can be found as follows. We rewrite the perturbation in Hamiltonian (41) as

$$V = V_1 + V_2,$$

$$V_1 = -\frac{g}{2} \sum_n \left(S_n^y S_{n+1}^y + S_n^z S_{n+1}^z \right),$$

$$V_2 = \frac{g}{2} \sum_n \left(S_n^y S_{n+1}^y - S_n^z S_{n+1}^z \right).$$

The part $H_0 + V_1$ of the Hamiltonian corresponds to the XXZ model in the longitudinal magnetic field, which is gapless for the magnetic field

$$h > \exp\left(-\frac{\pi^2}{2\sqrt{g}} \right).$$

Therefore, in the region of very small magnetic field

$$h < \exp\left(-\frac{\pi^2}{2\sqrt{g}} \right),$$

the perturbation V_1 is relevant, leading to the mass gap $m \sim h$. The two-cutoff scaling procedure [9, 10] leads to the mass gap

$$m \approx h \exp\left(-\frac{\pi^2}{2\sqrt{g}} \right)$$

for

$$h > \exp\left(-\frac{\pi^2}{2\sqrt{g}} \right).$$

Finally, when g is much less than h, the scaling dimension of the operator V_2 defines the exponent for the gap in Eq. (44). Summarizing, the mass gap in the vicinity of the isotropic point $\Delta = 1$, $h = 0$ is given by

$$\begin{aligned} &m \sim h, \quad \ln h \ll -\frac{1}{\sqrt{g}}, \\ &m \sim h e^{-\pi^2/2\sqrt{g}}, \quad \frac{1}{\sqrt{g}\ln g} \gg \ln h \gg -\frac{1}{\sqrt{g}}, \\ &m \sim g^{-\ln h}, \quad \ln h \gg \frac{1}{\sqrt{g}\ln g}. \end{aligned} \tag{47}$$

6. THE LINE $\Delta = -1$

In this section, we consider model (1) in the vicinity of the line $\Delta = -1$, where

$$1 + \Delta = \delta \ll 1$$

is a small parameter. It is convenient to rotate spins on each odd site by π around the z axis, such that model (1) becomes

$$H = -\sum_n (\mathbf{S}_n \cdot \mathbf{S}_{n+1}) + \delta \sum_n S_n^z S_{n+1}^z - - h \sum_n (-1)^n S_n^x. \tag{48}$$

At $\delta = 0$ and $h = 0$, the ground state of (48) is the ferromagnetic state with zero momentum degenerate with respect to total S^z. The states that can be reached from the ground state by means of the transition operator

$$\sum_n (-1)^n S_n^x$$

are the states with $q = \pi$ and a finite gap over the ground state. For $\delta \ll 1$, the transition operator connects the low-energy states and the states with the energies $\varepsilon_s \approx 2$. The second-order correction to low-energy states is given by

$$\delta E_l^{(2)} = h^2 \sum_{s,n,m} \frac{\langle l | (-1)^n S_n^x | s \rangle \langle s | (-1)^m S_m^x | l \rangle}{E_l - E_s}, \tag{49}$$

where $|l\rangle$ is a low-energy state and $|s\rangle$ is a state with the high energy $E_s - E_l \approx 2$. For $\delta \ll 1$, Eq. (49) can therefore be rewritten as

$$\delta E_l^{(2)} = -\frac{h^2}{2} \sum_{n,m} \langle l | (-1)^{n-m} S_n^x S_m^x | l \rangle = = -\frac{h^2 N}{8} - h^2 \sum_{n<m} \langle l | (-1)^{n-m} S_n^x S_m^x | l \rangle. \tag{50}$$

The spin correlation function $\langle l| S_n^x S_m^x |l\rangle$ is a slowly varying function of $|m-n|$ for $\delta \ll 1$. Therefore,

$$\sum_{n<m} \langle l| (-1)^{n-m} S_n^x S_m^x |l\rangle \approx \approx -\frac{1}{2}\sum_n \langle l| S_n^x S_{n+1}^x |l\rangle . \quad (51)$$

In accordance with Eqs. (49)–(51), the low-lying states of (48) are therefore described for $|\delta| \ll 1$ and $h \ll 1$ by the XYZ Hamiltonian

$$H = -\frac{h^2 N}{8} - \sum_n \left[\left(1-\frac{h^2}{2}\right) S_n^x S_{n+1}^x + S_n^y S_{n+1}^y - \Delta S_n^z S_{n+1}^z\right]. \quad (52)$$

The coincidence of the low-energy spectra of (48) and (52) in the vicinity of the ferromagnetic point $\Delta = -1$, $h = 0$ has been checked numerically for finite systems. The spectrum of low-lying excitations of the $s = 1/2$ XYZ model in Eq. (52) and of the original model in Eq. (1) near the ferromagnetic point $\Delta = -1$, $h = 0$ can be asymptotically exactly described by the spin-wave theory, which gives

$$\begin{aligned} m &= h\sqrt{(1+\Delta)/2}, \quad \Delta > -1, \\ m &= \sqrt{(1+\Delta)(1+\Delta+h^2/2)}, \quad \Delta < -1. \end{aligned} \quad (53)$$

It can be verified that Eq. (53) yields the exact gap of the XYZ model [14] for $|\delta|$, $h \ll 1$. The validity of the spin-wave approximation is quite natural because the number of magnons forming the ground state is small in the vicinity of the ferromagnetic point $\Delta = -1$, $h = 0$.

We also note that the gap in Eq. (53) for $\Delta \geq -1$ agrees with the conformal theory result (32) and gives the preexponential factor for the gap. On the classical line

$$h_{cl} = \sqrt{2(1+\Delta)},$$

Eq. (53) yields the gap $m = 1+\Delta$, which confirms that the function $\psi_1^{(1)}$ in Eq. (7) gives the exact gap.

A similar mapping of model (1) with an arbitrary spin s to the XYZ model can be performed for $\Delta \approx -1$, $h \ll 1$. Taking into account that $\varepsilon_s = 4s$, the corresponding XYZ Hamiltonian is

$$H = -\sum_n \left[\left(1-\frac{h^2}{2}\right) S_n^x S_{n+1}^x + S_n^y S_{n+1}^y - \Delta S_n^z S_{n+1}^z\right] - \frac{h^2}{4s}\sum_n (S_n^x)^2, \quad (54)$$

where S_n^α are spin-s operators.

The leading term of the gap of model (1) with an arbitrary spin s in the vicinity of the point $\Delta = -1$, $h = 0$ is exactly given by the spin-wave theory,

$$\begin{aligned} m &= h\sqrt{(1+\Delta)/2}, \quad \Delta > -1, \\ m &= 2s\sqrt{(1+\Delta)(1+\Delta+h^2/8s^2)}, \quad \Delta < -1. \end{aligned} \quad (55)$$

On the classical line h_{cl}, Eq. (55) gives the correct result $m = 2s\delta$.

Strictly on the line $\Delta = -1$, model (1) reduces to the isotropic ferromagnet in the staggered magnetic field. This model is nonintegrable, but it was suggested in [6] that the system is governed by a $c = 1$ conformal field theory up to some critical value $h = h_0$, where the phase transition of the Kosterlitz–Thouless type occurs.

For $h \ll 1$, where the mapping of (48) to XYZ model (52) is valid, the line $\Delta = -1$ is described by the XXZ model and the correlation functions have a power-law decay,

$$\begin{aligned} \langle S_i^z S_{i+n}^z\rangle &= \langle S_i^y S_{i+n}^y\rangle \sim \frac{(-1)^n}{n^{1/\beta(h)}}, \\ \langle S_i^x S_{i+n}^x\rangle &\sim \frac{(-1)^n}{n^{\beta(h)}}. \end{aligned} \quad (56)$$

We believe that the relation between the indices of x and y, z correlators on the line $\Delta = -1$ is given by (56) for $0 < h < h_0$. The scaling dimensions of the operators S_i^x and S_i^y, S_i^z on this line are therefore given by $d_x = \beta/2$ and $d_y = d_z = 1/2\beta$.

On the line $\Delta = -1$, model (1) is gapless for $h < h_0$. This implies that the magnetic field term is irrelevant for $h < h_0$ ($\beta(h) > 4$) and becomes marginal at $h = h_0$, where $d_x = 2$ and $\beta(h_0) = 4$. Therefore, at the point $h = h_0$, the transition is of the Kosterlitz–Thouless type, and for $h > h_0$, the mass gap is exponentially small.

In the vicinity of the line $\Delta = -1$, the term

$$\delta \sum_n S_n^z S_{n+1}^z$$

in (48) can be considered as a perturbation and the scaling dimension of the perturbation operator $S_n^z S_{n+1}^z$ is

$$d_{zz} = 4d_z = 2/\beta(h).$$

Because $\beta(h) \geq 4$ for $h < h_0$, the perturbation is relevant and leads to the mass gap and the long-range order,

$$\begin{aligned} m &\sim |\delta|^{1/(2-2/\beta)}, \\ \langle|S^y|\rangle &\sim \delta^{1/4(\beta-1)}, \quad \delta > 0, \\ \langle|S^z|\rangle &\sim |\delta|^{1/4(\beta-1)}, \quad \delta < 0. \end{aligned} \quad (57)$$

In particular, $m \propto |\delta|^{2/3}$ and $\langle|S^y|\rangle \sim |\delta|^{1/12}$ as $h \to h_0$.

The function $\beta(h)$ is generally unknown, except in the case where $h \ll 1$, the mapping to the XXZ model is valid, and

$$\beta(h) = \left[1 - \frac{1}{\pi} \arccos\left(\frac{h^2}{2} - 1\right)\right]^{-1} \approx \frac{\pi}{h}.$$

But because model (1) is conformally invariant at $\Delta = -1$ and $h < h_0$, we can use a finite-size scaling analysis to determine the exponent $\beta(h)$ and the value of h_0. According to the standard scaling approach [15],

$$\beta(h) = \frac{2\pi v}{A},$$

where v is the speed of sound and A/N is the difference between the two lowest energies of the system. We calculated $\beta(h)$ for finite systems. The extrapolated function $\beta(h)$ agrees well with the dependence π/h at $h \ll 1$ and $\beta = 4$ at $h_0 \approx 0.52$. This estimate is close to our direct numerical estimates $h_0 \approx 0.549$. On the other hand, the mean-field approach gives a rather crude value

$$h_0 = h_c(-1) = 0.69.$$

7. CONCLUSIONS

In summary, we have studied the effect of the symmetry-breaking transverse magnetic field on the $s = 1/2$ XXZ chain. Unlike the longitudinal field, the transverse field generates the staggered magnetization in the y direction and the gap in the spectrum of the model with the easy-plain anisotropy. Using conformal invariance, we have found the critical exponents of the field dependence of the gap and the long-range order. We have shown that the spectrum of the model is gapped on the entire $h\Delta$ plain except at several critical lines, where the gap and the long-range order vanish. The behavior of the gap and the long-range order in the vicinity of the critical lines $\Delta = \pm 1$ is considered on the base of the conformal field theory. We note that in the vicinity of the points ($\Delta = 1$, $h = 0$) and ($\Delta = 1$, $h = 2$), there is a crossover between different regimes of the behavior of the system. We have shown that near the point ($\Delta = -1$, $h = 0$), the original model can be mapped to the effective exactly solvable $1D$ XYZ model and has the spin-wave spectrum. The transition line $h_c(\Delta)$ between the ordered phases and the disordered one is studied in the mean-field approximation. This study shows that this transition is similar to that in the Ising model in the transverse field. But the behavior of the model on the transition line near the Kosterlitz–Thouless point ($\Delta = -1$, $h = h_0$) is not so clear. The mean-field approximation worsens as $\Delta \to -1$ and a more sophisticated theory is needed.

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