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OPTICAL BISTABILTY

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We consider the problem of the wave propagation through a nonlinear medium. We derive a dynamical system that governs the behavior of standing (or solitary) waves. The form of this system alone suffices to understand the qualitative dependence of solutions of the original equation on the intensity of the incident wave. We solve this dynamical system in the leading order in the nonlinearity strength. We find multiple solutions of the original problem for a given incoming wave and turning points of these solutions as a function of the intensity of the wave. We briefly investigate stability of different branches. Our results yield analytic description of the optical bistability phenomenon.

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1. INTRODUCTION

The problem of the light propagation through nonlinear media is of a great theoretial and practical interest. At large intensities, the dielectric constant ε is not a constant but varies as the intensity does. Even a tiny dependence of ε on the intensity can produce significant effects over large distances. Controlling and utilizing these effects is one of the main challenges of the theory of wave propagation. In this paper, we consider the propagation of light through a slab of medium whose dielectric constant depends on the intensity of light. A key effect of interest here is the bistability phenomenon — existence of several solutions (with different transmission coefficients) with alternating stability properties for a given intensity of the incoming beam. This phenomenon was predicted about 20 years ago in Ref. [1] and it has been a subject of intensive research since then. The research in this area further intensified about 10 years ago with the theoretical discovery of gap solitons in Ref. [2]. See Refs. [2–10] for some of the important original works and Refs. [11–13] for recent reviews and the background material.

In this paper, we address the problem of the propagation of electromagnetic waves through a nonlinear dialectic slab in a systematic way. To keep the exposition as simple as possible, we consider the simplest possible dependence of the slab dialectic constant on the intensity of light. Our goal is to clarify some conceptual points and to perform concrete computations. Specifically, we

establish a minimum action principle and consequently a Hamiltonian structure for the basic (phenomenological) equation;

find a criterion of bistability in terms of linear resonances, which offers a possibility for multidimensional extensions;

find the location of turning points;

estimate the number of solutions for incoming waves of high intensities;

discuss general features of the stability analysis.

To our knowledge, the results summarized above are new.

2. THE MODEL

In the local and nondissipative approximation, the equation describing the propagation of light through

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a medium with the dielectric constant ε and without charges or currents is given by (see, e.g., Refs. [14–18])

$$\partial_t^2(\varepsilon \mathbf{E}) = \Delta \mathbf{E}, \tag{2.1}$$

where $\mathbf{E}(\mathbf{x}, t)$ is the electric field at a point $\mathbf{x} \in \mathbb{R}^3$ at time t: the speed of light c is set to 1. This equation arises from the principle of minimum (or more precisely, stationary) action. We write the action for the electromagnetic field in a medium whose dialectric constant ε depends on the amplitude of the electric field $\mathbf{E}$ (i.e., $\varepsilon = \varepsilon(|\mathbf{E}|^2)$) as

$$S(\mathbf{A}) = \frac{1}{2} \iint \left(f(|\mathbf{E}|^2, x) - |\mathbf{B}|^2 \right),$$

where $\mathbf{A}$ is the transverse vector potential, $\operatorname{div} \mathbf{A} = 0$ (we work in the Coulomb gauge), $\mathbf{E} = -\partial \mathbf{A}/\partial t$, $\mathbf{B} = \operatorname{rot} \mathbf{A}$ (magnetic field), and

$$f(s, \mathbf{x}) = \int_0^s \varepsilon(u, \mathbf{x}) du.$$

Moreover, we set the magnetic permeability μ to 1. Here, we modified only the part of the action related to the electric field $\mathbf{E}$, leaving the part connected to the magnetic field $\mathbf{B}$ unchanged. The reason for this is that the electric susceptibility $\chi_e = \varepsilon - 1$ can take relatively large values, even much larger than 1, while the magnetic susceptibility $\chi_m = \mu - 1$ is always much smaller than 1 in nonmagnetic materials, namely of the order 10^{-5}–10^{-8}.

The critical points of the above functional are given by the Euler–Lagrange equation

$$-\frac{\partial}{\partial t} \left(\varepsilon(|\mathbf{E}|^2, \mathbf{x}) \frac{\partial \mathbf{A}}{\partial t} \right) + \Delta \mathbf{A} = 0. \tag{2.2}$$

Differentiating this equation with respect to t and using that $\partial \mathbf{A}/\partial t = -\mathbf{E}$, we arrive at (2.1). Conversely, Eq. (2.1) implies Eq. (2.2) if we require that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{A} dt = 0,$$

i.e. the vector potential $\mathbf{A}$ has no zero harmonic. The latter is consistent with Eq. (2.2) because that equation contains only odd powers of $\mathbf{A}$.

A reformulation of Eq. (2.1) (or (2.2)) in terms of the minimum action principle immediately leads to the energy conservation with the energy functional

$$\mathcal{E}(\mathbf{A}) = \int (\mathbf{A} \partial_{\mathbf{A}} L) - L,$$

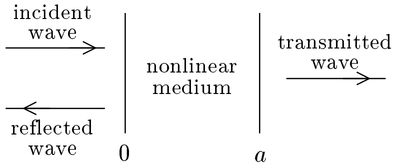

Fig. 1.

where

$$L = \frac{1}{2} \int (f(|\mathbf{A}|^2, x) - |\operatorname{rot} \mathbf{A}|^2).$$

This functional can be explicitly computed as

$$\mathcal{E}(\mathbf{A}) = \frac{1}{2} \int \{ f(|\mathbf{E}|^2, \mathbf{x}) + |\operatorname{rot} \mathbf{A}|^2 \}.$$

Moreover, the variational formulation given above shows that Eq. (2.2) is Hamiltonian, with the standard Poisson brackets and with the Hamiltonian functional found via the Legendre transform as

$$H(\mathbf{A}, \boldsymbol{\pi}) = \frac{1}{2} \int \left(\phi(|\mathbf{E}|^2) + |\operatorname{rot} \mathbf{A}|^2 \right),$$

where the momentum field $\boldsymbol{\pi}(\mathbf{x})$ is related to the electric field $\mathbf{E}(\mathbf{x})$ as $\boldsymbol{\pi} = -f'(|\mathbf{E}|^2)\mathbf{E}$ and we set $\phi(s) = f'(s)s - f(s)/2$.

In what follows, we consider the symplest model of the nonlinear wave propagation. We assume that

(α) the medium in question is uniform in the y and z directions, i.e. ε does not explicitly depend on y and z;

(β) apart from its dependence on x, ε depends on $\mathbf{E}$ only through the amplitude $|\mathbf{E}|^2$, i.e.,

$$\varepsilon = \varepsilon(|\mathbf{E}|^2, x);$$

(γ) the nonlinear part of the medium forms a slab of the thickness a perpendicular to the x axis (see Fig. 1), i.e.,

$$\varepsilon(|\mathbf{E}|^2, x) = \begin{cases} 1 & \text{if } x < 0 \text{ or } x > a, \\ n^2 \bar{\varepsilon}(|\mathbf{E}|^2) & \text{if } 0 \le x \le a, \end{cases} \tag{2.3}$$

where n is the refractive index. The function $\bar{\varepsilon}(|\mathbf{E}|^2)$ is taken in the simplest possible form

$$\bar{\varepsilon}(|\mathbf{E}|^2) = 1 + \bar{\mu} |\mathbf{E}|^2. \tag{2.4}$$

In real materials, $\bar{\mu} \sim |\mathbf{E}_0|^{-2}$, where $\mathbf{E}_0$ is the internal (atomic) electric field. Because the electric breakdown already occurs when $|\mathbf{E}| \ll |\mathbf{E}_0|$, the second term in the right-hand side of (2.4), which is of the order $(|\mathbf{E}|^2/|\mathbf{E}_0|)^2$, is indeed very small,

$$\bar{\mu} |\mathbf{E}|^2 \ll 1. \tag{2.5}$$

We consider only waves of a fixed polarization, i.e., assume that $\mathbf{E}(\mathbf{x}, t)$ in Eq. (2.1) can be written as

$$\mathbf{E}(\mathbf{x}, t) = E(\mathbf{x}, t)\mathbf{e}, \tag{2.6}$$

where $\mathbf{e}$ is a fixed vector (the polarization vector) propagating in the x direction, i.e., $\mathbf{e}$ is perpendicular to the x axis. In this case, $E(\mathbf{x}, t)$ can be assumed to depend on x only, and therefore, Eq. (2.1) reduces to the equation

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2}{\partial t^2}(\varepsilon E), \tag{2.7}$$

where $E = E(x, t)$ and $\varepsilon = \varepsilon(|E|^2, x)$.

This equation is subject to the boundary conditions

$$E(x, t) = \begin{cases} f(x - t) + f_1(x + t) & \text{if } x < 0, \\ f_2(x - t) & \text{if } x > a \end{cases} \tag{2.8}$$

with the function f given and the functions f_1 and f_2 unknown. These boundary conditions say that the field on the left of the sample consists of the incoming wave $f(x - t)$ and some reflected wave $f_1(x + t)$, while the field on the right of the sample consists of only the outgoing wave $f_2(x - t)$. In addition, we specify the incoming wave as

$$f(x - t) = \mathrm{Re}(e^{-ik_0(x-t)}) \tag{2.9}$$

for some $k_0 > 0$.

3. SOLITARY WAVES

We study solitary waves for Eq. (2.7), i.e., waves of the form

$$E(x, t) = \mathrm{Re}\left(e^{-ik_0 t}\psi_0(x)\right) \tag{3.1}$$

where ψ_0 is a complex function. In the leading approximation in the nonlinearity parameter $\overline{\mu}$, it then follows that ψ_0 satisfies the stationary equation (see Appendix 1)

$$\Delta\psi_0 + k_0^2\varepsilon_0\psi_0 = 0, \tag{3.2}$$

with $\varepsilon_0 = \varepsilon((3/4)|\psi_0|^2, x)$. We are interested in the problem of the solitary wave passage through the nonlinear slab, which amounts to taking the solutions ψ_0 such that

$$\psi_0 = \begin{cases} Ae^{ik_0x} + RAe^{-ik_0x} & \text{for} \quad x < 0, \\ TAe^{ik_0x} & \text{for} \quad x > a \end{cases} \tag{3.3}$$

with a given A and for some R and T. Here Ae^{ik_0x}, RAe^{-ik_0x}, and TAe^{ik_0x} are the incident, reflected, and transmitted waves, respectively (see Fig. 2), and R and T are the reflection and transmission coefficients.

Fig. 2. Reflection and transmission coefficients

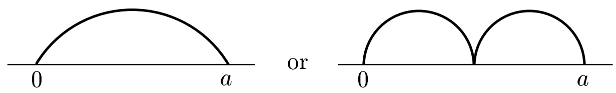

Fig. 3. Two solutions satisfying b.c. $\psi(0) = 0$ and $\psi(a) = 0$

The flux conservation (see below) implies that R and T satisfy

$$|R|^2 + |T|^2 = 1. \tag{3.4}$$

In the linear case, R and T are independent of A, and the amplitude A drops out of the equation. This is not so in the nonlinear case. The goal of this paper is to find the dependence of $|R|$ (or $|T|$) on A. The main point here is that although two initial conditions uniquely define a solution of a second-order ODE, two boundary conditions can be satisfied by several (a finite number of) solutions of a nonlinear secord-order ODE. Figure 3 shows two solutions satisfying the boundary conditions $\psi(0) = 0$ and $\psi(a) = 0$. In constrast, in the linear case, two boundary conditions determine a solution of a second-order ODE uniquely (modulo eigenfunctions).

4. THE BOUNDARY VALUE PROBLEM

Instead of considering Eq. (3.2) and conditions (3.3) on the entire real axis, we study this problem on the interval $[0, a]$,

$$\psi_0'' + k_0^2\overline{\varepsilon}\left(\frac{3}{4}|\psi_0|^2\right)\psi_0 = 0, \quad 0 \le x \le a, \tag{4.1}$$

and use conditions (3.3) to set the boundary conditions at $x = 0$ and $x = a$ as

$$\psi_0(0) = A(1 + R), \quad \psi_0'(0) = ik_0A(1 - R) \tag{4.2}$$

and

$$\psi_0(a) = ATe^{ik_0a}, \quad \psi_0'(a) = ik_0ATe^{ik_0a}. \tag{4.3}$$

We thus arrive at a boundary value problem on $[0, a]$. It is convenient to rescale this problem as

$$\psi_0(x) = A\psi(kx),$$

where $k = k_0 n$, the wave vector in the medium with the refraction coefficient n. The new boundary value problem is given by

$$\psi'' + \varepsilon_1(|\psi|^2)\psi = 0, \quad 0 \le x \le b, \tag{4.4}$$

where $b = ka$ and $\varepsilon_1(|\psi|^2) = \overline{\varepsilon}((3/4)A^2|\psi|^2)$, with the boundary conditions

$$\psi(0) = 1 + R, \quad \psi'(0) = \frac{i}{n}(1 - R) \tag{4.5}$$

and

$$\psi(b) = Te^{ib/n}, \quad \psi'(b) = \frac{i}{n}Te^{ib/n}. \tag{4.6}$$

Because Eq. (4.4) is invariant under the gauge transformation $\psi(x) \to e^{i\beta}\psi(x)$, we can assume $T \ge 0$.

Recalling expression (2.4) for $\overline{\varepsilon}$, we find

$$\varepsilon_1(|\psi|^2) = 1 + 2\mu|\psi|^2, \quad \mu = 3\overline{\mu}|A|^2/8, \tag{4.7}$$

and therefore, the incident beam amplitude A enters the new equation only through the parameter μ and varying A is the same as varying μ.

We note that Eq. (2.5) implies that in real material, $\mu \ll 1$.

Although Eqs. (4.5), (4.6) appear to represent four (complex) constraints, these equations in fact constitute only two conditions because R and T are unknown. Eliminating the unknowns R and T from boundary conditions (4.5), (4.6), we obtain the conditions

$$\psi(0) - in\psi'(0) = 2, \tag{4.8}$$

$$\psi(b) + in\psi'(b) = 0. \tag{4.9}$$

Equation (4.6) shows that a solution of Eq. (4.4) with boundary conditions (4.8), (4.9) determines the transmission coefficient $T = |\psi(b)|$; on the other hand, knowing T determines the solution of (4.4). Our goal in what follows is to find $T = |\psi(b)|$, where ψ solves (4.4), (4.8), and (4.9) as a function of μ.

5. RESONANCES AND THE EFFECTIVE WAVE VECTOR

We now describe the physical mechanism underlying the nonlinear phenomenon under consideration. We begin with the linear component of this mechanism, and therefore set $\mu = 0$. In this case, Eq. (4.4) can be solved explicitly with the result

$$\psi^{lin} = \frac{1}{2}\left(1 + R^{lin} + \frac{1 - R^{lin}}{n}\right)e^{ix} + \frac{1}{2}\left(1 + R^{lin} - \frac{1 - R^{lin}}{n}\right)e^{-ix} \tag{5.1}$$

and

$$R^{lin} = \frac{(n^2 - 1)(e^{ib} - e^{-ib})}{-(n-1)^2 e^{ib} + (n+1)^2 e^{-ib}}. \tag{5.2}$$

The last equation shows that as a function of $b = ka$, R^{lin} has a series of minima and maxima,

$$b = ka = \pi m \Rightarrow |R^{lin}| = 0 \quad (= |R^{lin}|_{\min}),$$

$$b = ka = \pi\left(m + \frac{1}{2}\right) \Rightarrow |R^{lin}| = \frac{n^2 - 1}{n^2 + 1} \quad (= |R^{lin}|_{\max}).$$

For $n \gg 1$, this resonance behavior is rather sharp: if the width a of the slab contains an integer number of the half-wave lengths, $\lambda/2 = 2\pi/2k$, then the transmission is perfect and the slab is therefore transparent. If the width of the slab contains an odd number of quarter-wave lengths, then there is almost no transmission and the slab is opaque.

The resonance structure of the linear case plays a crucial role in the peculiar behavior of the nonlinear solution. This solution can be considered as a linear one with a varying effective wave vector,

$$k_{eff} \equiv k\varepsilon_1^{1/2} = k(1 + 2\mu|\psi|^2)^{1/2}. \tag{5.3}$$

As the intensity (i.e., μ) varies, so does the effective wave-length and the medium goes through a series of resonances in which it is either perfectly transparent, $|T| = 1$, or almost opaque, $|T| \approx 0$.

Thus, the presence of sharp minima and maxima of the reflection (or transmission) coefficient offers a simple criterion for the occurrence of the bistability phenomenon. One way to extend this criterion to the multidimensional case is to relate it to the resonance structure of the scattering process considered above. Indeed, we observe that R^{lin} (and therefore, ψ^{lin}) display a resonance structure in the sense that it has complex poles at

$$b(= ka) = \pi m - i\ln\frac{1 + 1/n}{1 - 1/n}, \quad m = 0, \pm 1, \ldots \tag{5.4}$$

The real parts of these poles exactly give the position of maxima of the transmission coefficient. If we recall that $b = ka$, we can rewrite (5.4) as

$$k = \frac{\pi m}{a} - \frac{i}{a}\ln\frac{1 + 1/n}{1 - 1/n}, \quad m = 0, \pm 1, \ldots \tag{5.5}$$

The real part of this expression takes the values $2\pi m/a$ that coincide with the eigenvalues of the operator

$\sqrt{-d^2/dx^2}$ on the interval $[0,a]$ with the periodic boundary conditions.

To obtain resonance solutions, we must solve the original wave equations

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2}{\partial t^2}(\varepsilon E) \tag{5.6}$$

(where $\varepsilon = n^2$) on the interval $[0,a]$ with the boundary conditions representing an outgoing wave. For this, we set $E = e^{-i\sigma t}\xi$ with $\sigma > 0$ and

$$\xi = \begin{cases} A_1 e^{-i\sigma x} & \text{for} \quad x < 0, \\ A_2 e^{i\sigma x} & \text{for} \quad x > a, \end{cases} \tag{5.7}$$

where A_1 and A_2 are arbitrary constants. Eliminating these constants, we obtain

$$\frac{\xi'(0)}{\xi(0)} = -i\sigma \quad \text{and} \quad \frac{\xi'(a)}{\xi(a)} = i\sigma. \tag{5.8}$$

On the other hand, Eq. (5.6) implies the equation for ξ,

$$-\varepsilon\sigma^2\xi = \frac{\partial^2}{\partial x^2}\xi, \tag{5.9}$$

on the interval $[0,a]$. Solving Eqs. (5.8), (5.9) and recalling that $\varepsilon = n^2 (> 1)$, we find

$$\sigma = \frac{\pi m}{an} - \frac{i}{an}\ln\frac{1+1/n}{1-1/n}, \quad m = 0, \pm 1, \ldots \tag{5.10}$$

and the corresponding expression for ξ, which we omit here.

We have thus arrived at the following conclusion: resonances of the transmission coefficient, which are responsible for the bistable behavior of our nonlinear system, coincide with the resonances of the linear wave equation (5.6) (in (5.6), ε is n^2 times the characteristic function of the interval $[0,a]$). This is important because there are well developed techniques for finding resonances in multidimensional linear systems. Thus, we have a possibility of identifying the bistability phenomenon in the multidimensional case.

We indicate the connection between the above resonance solutions and the stability problem for Eqs. (2.7)–(2.9) with $\mu = 0$ (the linear problem). In this case, we seek solutions to Eqs. (2.7)–(2.9) in the form

$$E = \operatorname{Re}\left(e^{-ik_0 t}\psi_0(x) + \eta(x,t)\right), \tag{5.11}$$

where $|\eta| \ll |\psi_0|$. From (2.8), (2.9) and (3.3), we can assume that η satisfies the boundary conditions

$$\eta = \begin{cases} e^{-i\sigma(x+t)} & \text{for} \quad x < 0, \\ e^{i\sigma(x-t)} & \text{for} \quad x > a, \end{cases} \tag{5.12}$$

with σ that is complex but close to k_0. Clearly, η is of the form $\eta = e^{-i\sigma t}\xi$, where ξ satisfies Eqs. (5.8), (5.9), and consequently, σ is given in (5.10). The resonance eigenvalues therefore serve as the stability exponents for solution (3.1) in the linear case (because $\operatorname{Im}\sigma < 0$, the solution $e^{-ik_0 t}\psi_0(x)$ is stable).

6. CONSERVATION LAWS

In this section, we describe conservation laws obeyed by Eq. (4.4). We consider x as a time variable. We first define the «energy» density

$$e(\psi) = |\psi'|^2 + G(|\psi|^2), \tag{6.1}$$

where

$$G(u) = \int_0^u \varepsilon_1(v)\,dv = u + \mu u^2. \tag{6.2}$$

Using Eq. (4.4), we conclude that $\partial e(\psi)/\partial x = 0$, and therefore,

$$e(\psi) \equiv |\psi'|^2 + |\psi|^2 + \mu|\psi|^4 = C > 0. \tag{6.3}$$

In the same way, it follows that the flux density $j = \operatorname{Im}(\overline{\psi}\partial\psi/\partial x)$ is also conserved, $j = C_1$. To combine these two conservation laws, it is convenient to pass to the polar representation

$$\psi = \sqrt{\rho}\, e^{i\alpha}. \tag{6.4}$$

The conservation of the flux then gives

$$\rho\alpha' = C_1. \tag{6.5}$$

In classical mechanics, this equation expresses the angular momentum conservation or the Kepler law: the rate of change of the area swept by the radius vector of a particle in a central potential is constant. Together with the energy conservation equation, this equation gives

$$\rho'^2 = 4(-C_1^2 + C\rho - g(\rho)), \tag{6.6}$$

where $g(\rho) = \rho G(\rho) = \rho^2 + \mu\rho^3$. Starting with this equation and boundary conditions (4.8), (4.9), we derive our main equations in the next section.

7. THE MAIN EQUATIONS

In this section, we derive the equations for ρ and $|T|^2 = \rho(b)$ on which we base our analysis. We observe that the right-hand side of Eq. (6.6) contains two integration constants (or conservation constants). We use boundary conditions (4.8), (4.9) to express these constants in terms of $\rho(0)$ and $\rho(b)$.

The boundary condition at $x = b$ gives

$$\left. \frac{\frac{1}{2}\frac{\rho'}{\sqrt{\rho}} + i\alpha'\sqrt{\rho}}{\sqrt{\rho}} \right|_{x=b} = \frac{i}{n}, \qquad (7.1)$$

which implies

$$\alpha'(b) = \frac{1}{n} \quad \text{and} \quad \rho'(b) = 0. \qquad (7.2)$$

Equation (6.5) and the first equation in (7.2) yield $C_1 = \rho(b)/n$, and therefore,

$$\alpha' = \frac{\rho(b)}{n\rho}. \qquad (7.3)$$

Equations (6.6) and $C_1 = \rho(b)/n$ give

$$\rho'^2 = f(\rho), \qquad (7.4)$$

where

$$f(\rho) = 4\Big(-\frac{\rho(b)^2}{n^2} + C\rho - g(\rho)\Big). \qquad (7.5)$$

Equations (7.2) and (7.4), (7.5) imply that $\rho(b) = |T|^2$ is a root of

$$f_1(\rho(b)) = 0, \qquad (7.6)$$

where

$$f_1(u) = Cu - \frac{1}{n^2}u^2 - g(u). \qquad (7.7)$$

Equations (7.4)–(7.7) constitute all the basic equations of our analysis except one eqaution. We now find the constant C as a function of $\rho(b)$ by solving (7.6), (7.7) for C,

$$C = \left(1 + \frac{1}{n^2}\right)\rho(b) + \mu\rho(b)^2. \qquad (7.8)$$

This is a quadratic relation between $\rho(0)$ and $\rho(b)$. Substituting (7.8) in (7.5), we find

$$f(\rho) = 4\big(-\frac{\rho(b)^2}{n^2} + \rho(b)\big(1 + \frac{1}{n^2}\big)\rho + \mu\rho(b)^2\rho - \rho^2 - \mu\rho^3\big). \qquad (7.9)$$

Finally, we find the remaining basic equation using the boundary condition at $x = 0$. Using (6.4), (7.3), and (7.4), we rewrite Eq. (4.9) as

$$\left(\rho(0) \mp \frac{in}{2}\sqrt{f(\rho(0))} + \rho(b)\right)\frac{e^{i\alpha(0)}}{\sqrt{\rho(0)}} = 2, \qquad (7.10)$$

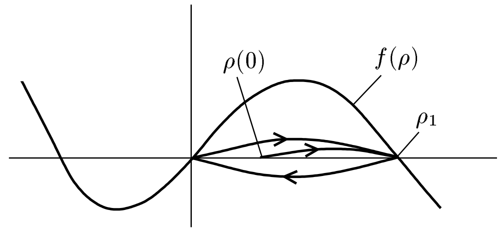

Fig. 4. A trajectory of $\rho' = \pm\sqrt{f(\rho)}$

which implies, after taking the absolute value,

$$\big(\rho(0) + \rho(b)\big)^2 + \frac{1}{4}n^2 f\big(\rho(0)\big) = 4\rho(0). \qquad (7.11)$$

Inserting expression (7.9) for $f(\rho)$ in (7.11), we obtain, after simple transformations,

$$\varphi\big(\rho(b), \rho(0)\big) = 0, \qquad (7.12)$$

where

$$\varphi(u,v) = \mu n^2 v^2 + (n^2 - 1)v - (n^2 + 3)u - n^2\mu u^2 + 4. \qquad (7.13)$$

We now can formulate the problem as follows. We must solve the differential equation $\rho'^2 = f(\rho)$ with the boundary values $\rho(0)$ and $\rho(b)$ satisfying the equation $\varphi\big(\rho(b), \rho(0)\big) = 0$, where f and φ are given by the respective equations (7.9) and (7.13).

We split our task as follows:

(i) Using Eqs. (7.4) and (7.9), we first determine $\rho(0)$ as a function of $\rho(b)$. Here, we consider $\rho(b)$ as an initial condition for the dynamical system

$$\rho' = \pm\sqrt{f(\rho)_+}, \qquad (7.14)$$

where $x_+ = \max(x, 0)$. We then solve (7.14) backwards, from $x = b$ to $x = 0$ (with the change of signs at turning points, $f(\rho) = 0$!) and find $\rho(0)$ as a function of $\rho(b)$, n, and μ. Here, the cases where $\rho'(0) > 0$ and $\rho'(0) < 0$ can be considered separately.

(ii) We then insert $\rho(0)$ found at step (i) in Eq. (7.12). The result is an algebraic equation for $\rho(b) = |T|^2$. In general, this algebraic equation has several solutions depending on μ and n.

The essence of this analysis can be inferred from the form of Eqs. (7.9) and (7.14), without solving them. Indeed, let ρ_1, ρ_2 and ρ_3 be the roots of the equation $f(\rho) = 0$. As we already know, one of these roots is $\rho(b)$, for example $\rho_1 = \rho(b)$. We let $\rho_3 < \rho_2$. It is not difficult to show (see below) that $\rho_2 < \rho_1$ (in fact, $\rho_2 < (1 + \mu\rho_1)^{-1}n^{-2}\rho_1$) and $\rho_3 < 0$. The roots of $f(\rho)$

are equilibria and turning points of the dynamical system in Eq. (7.14). At these roots, $\rho' = 0$ and $\sqrt{f(\rho)_+}$ changes its sign. The behavior of this dynamical system is shown in Fig. 4.

Open Problem. Dynamical system (7.14) is parameterized by ρ_1. Its phase portrait should qualitatively change as ρ_1 goes through a turning point. How?

8. EQUATION (7.6)

We now show that we can infer much information from Eq. (7.6) without solving differential equation (7.4). We consider the two simplest cases:

(a) linear case: $G(u) = u$, and hence, $g(u) = u^2$. In this case,

$$f_1^{(lin)}(u) = Cu - \left(1 + \frac{1}{n^2}\right) u^2,$$

and therefore, the equation $f_1^{(lin)}(\rho(b)) = 0$ has, in addition to the trivial solution $\rho(b) = 0$, one nontrivial solution

$$\rho(b) = \frac{C}{1 + n^{-2}}.$$

(b) Cubic nonlinearity: $G(u) = u + \mu u^2$, and therefore, $g(u) = u^2 + \mu u^3$. In this case,

$$f_1(u) = Cu - (1 + \frac{1}{n^2})u^2 - \mu u^3.$$

Thus, depending on the coefficients, the equation $f_1(\rho(b)) = 0$ has, in addition to the trivial solution $\rho(b) = 0$, either none or one or two nontrivial solutions. All the possibilities are listed in Fig. 5.

Conclusion. In the linear case, we always have one nontrivial solution (after the division by u, the function $f_1^{(lin)}(u)$ becomes linear). In the simplest, cubic nonlinear case, depending on C and μ, there can be none, one or two nontrivial solutions for $\rho(b) = |T|^2$.

9. SOLUTION OF THE NONLINEAR PROBLEM

We first solve differential equation (7.4), $\rho' = \pm\sqrt{f(\rho)_+}$. Recalling that $\rho(b)$ solves $f(\rho) = 0$, we express $f(\rho)$ as

$$f(\rho) = 4\big(\rho(b) - \rho\big) \left(\mu\rho^2 + (1 + \mu\rho(b))\rho - \frac{\rho(b)}{n^2}\right). \quad (9.1)$$

Integrating the equation $\rho' = \pm\sqrt{f(\rho)_+}$, we find

$$\pm\int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{f(\rho)}} = b \quad (9.2)$$

(we recall that $\rho_1 = \rho(b)$ and $\rho_0 = \rho(0)$). It is shown in Appendix 2 that Eq. (9.2) is equivalent, modulo $O(\mu)$ (but keeping terms of the order $O(\mu b)$), to the equation

$$\frac{4(1-\rho_1)}{\rho_1} \frac{n^2}{(n^2-1)^2} = \sin^2 \frac{b}{1 - \dfrac{3}{4}\mu\rho_1 \dfrac{n^2+1}{n^2}}. \quad (9.3)$$

This equation defines ρ_1 $(= \rho(b) = |T|^2)$ as a multivalued function of μ for given values of the parameters n and b.

Of central interest are turning points of this function. To find them, we differentiate Eq. (9.3) with respect to ρ_1,

$$-\frac{4}{\rho_1^2} \frac{n^2}{(n^2-1)^2} = \frac{3\mu b}{4} \frac{n^2+1}{n^2} \times \\ \times \sin \frac{2b}{1 - \dfrac{3}{4}\mu\rho_1 \dfrac{n^2+1}{n^2}}, \quad (9.4)$$

and then solve the resulting two equations for ρ_1 and μ. Thus, the turning points of ρ_1 as a function of μ are given by solutions of Eqs. (9.3) and (9.4).

10. TURNING POINT IN THE LARGE-n CASE (SEMICLASSICAL LIMIT)

We investigate Eqs. (9.3) and (9.4) in the case where $n \gg 1$. In this case, the factor $\frac{n^2}{(n^2-1)^2}$ in both equations can be replaced by $1/n^2$. We consider two cases:

(a) $|\mu| b \ll 1$. Because $n \gg 1$, Eq. (9.4) shows that either $\rho_1 \sim 1/n^2$ or b is close to πm, where m is an integer. We consider the latter case and set

$$b = \pi m + \delta \quad \text{with} \quad |\delta| \ll 1. \quad (10.1)$$

Equation (9.3) can then be reduced to the equation

$$4(1-\rho_1) = n^2\rho_1 \left(\delta + \frac{3\mu b}{4}\rho_1\right)^2. \quad (10.2)$$

Differentiating this equation with respect to ρ_1, we obtain the equation for the turning points,

$$-4 = \frac{3\mu b n^2 \rho_1}{2} \left(\delta + \frac{3}{4} b\mu\rho_1\right) + \\ + n^2 \left(\delta + \frac{3\mu b}{n}\rho_1\right)^2. \quad (10.3)$$

We now pass from ρ_1 to the variable z defined by

$$n\left(\delta + \frac{3\mu b}{4}\rho_1\right) = -\frac{8}{3z}.$$

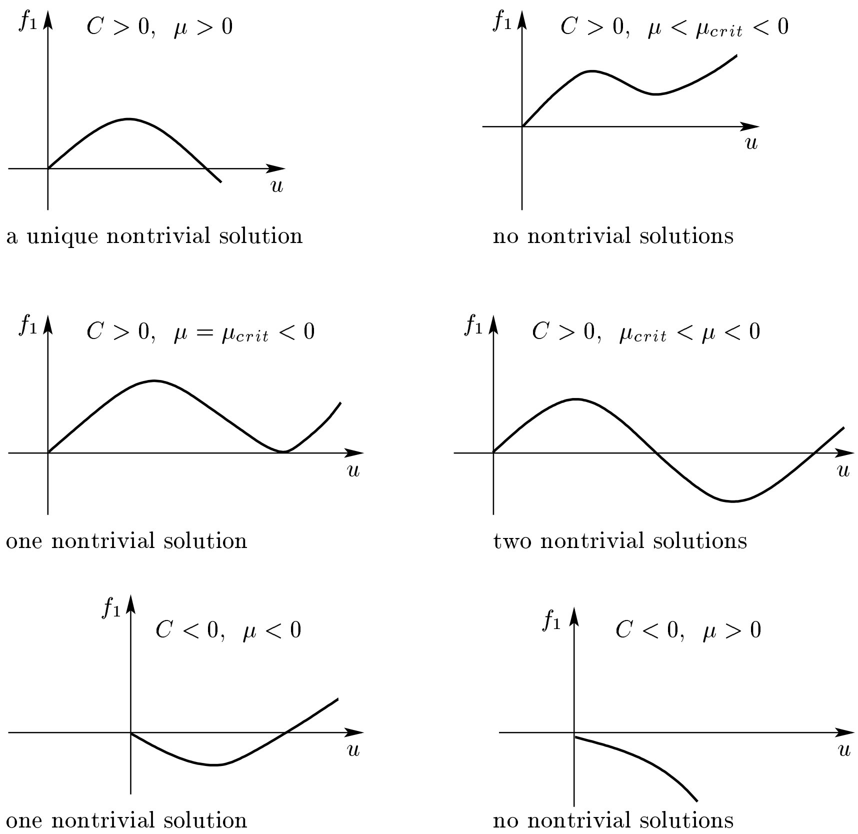

Fig. 5.

The resulting equation for z has the solutions

$$z_{1,2} = \frac{2}{3}\Big(-\delta n \pm \sqrt{\delta^2 n^2 - 12}\Big). \tag{10.4}$$

Substituting in Eqs. (10.2) and (10.3) $-8/3z$ for $n(\delta + 3\mu b\rho_1/4)$ and then solving Eq. (10.2) for ρ_1 and Eq. (10.3), for μnb we find

$$\rho_1^{(k)} = \frac{9z_k^2}{16 + 9z_k^2} \quad \text{and} \quad \mu nb = \frac{z_k}{(\rho_1^{(k)})^2} \tag{10.5}$$

for $k = 1, 2$. In the region between these turning points, all the three solutions of Eq. (10.2) can be represented as

$$\rho_1 = -\frac{4\delta(1+z)}{3\pi\mu m} \tag{10.6}$$

and

$$z = -\frac{1}{3} + \sqrt{\frac{4}{9} - \frac{16}{3\delta^2 n^2}}\cos\varphi, \tag{10.7}$$

where $\varphi = \varphi_j$, $j = 1, 2, 3$, is given by

$$\varphi_j = \frac{1}{3}\left(\frac{\pi}{2} + 2\pi j + \arcsin\left\{\frac{1 + \dfrac{36}{\delta^2 n^2} + \dfrac{81 b\mu}{2n^2\delta^3}}{\left(1 - \dfrac{12}{\delta^2 n^2}\right)^{3/2}}\right\}\right). \tag{10.8}$$

Because $\rho_1 > 0$, solutions (10.6) exist only in the region $\delta\mu < 0$.

(b) $n|\mu|b \gg 1$. In this case, n can take an arbitrary value larger than one. Eliminating trigonometric functions from Eqs. (9.3) and (9.4), we arrive at the equation

$$\left(\frac{4n^3}{3\mu b(n^2+1)^2}\right)^2 = \rho_1^2(1-\rho_1)\left(\rho_1 - \frac{4n^2}{(n^2+1)^2}\right). \tag{10.9}$$

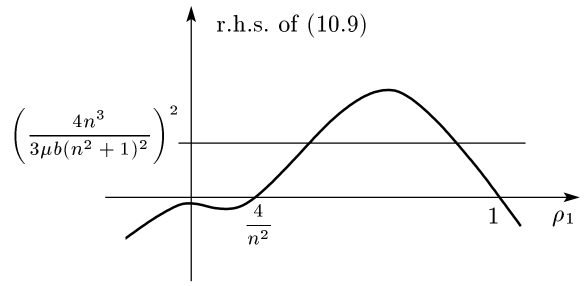

Fig. 6. Graphical solution to Eq. (10.9)

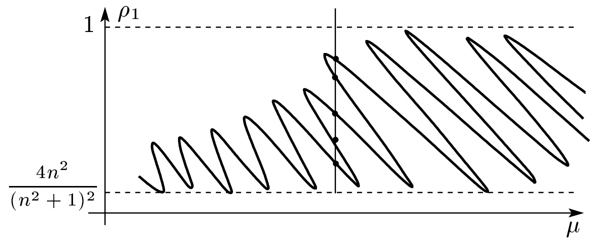

Fig. 7. Dependence of ρ_1 on μ; the number of solutions for a given μ

This equation can have either two positive solutions for ρ_1 or none. In the first case, one of the solutions is close to 1,

$$\rho_1^{(1)} = 1 - \frac{1}{2}\left(\frac{n^2-1}{n^2+1}\right)^2 + \\ + \sqrt{\frac{1}{4}\left(\frac{n^2-1}{n^2+1}\right)^4 - \left(\frac{4n^3}{3\mu b(n^2+1)^2}\right)^2}, \quad (10.10)$$

while the other one, $\rho_1^{(2)}$, is a solution of a cubic equation. For $\mu bn \gg 1$ and $n \gg 1$, the cubic equation is given by

$$\rho_1^3 - \frac{4\rho_1^2}{n^2} - \left(\frac{4}{3\mu bn}\right)^2 = 0.$$

A positive solution to the last equation is

$$\rho_1^{(2)} = \frac{8}{3n^2}\left(\frac{1}{2} + \operatorname{ch}\beta\right), \\ \text{where } \operatorname{ch} 3\beta = 1 + \frac{3}{8}\left(\frac{n^2}{\mu b}\right)^2. \quad (10.11)$$

A graphical solution to Eq. (10.9) is shown in Fig. 6.

Using values (10.10) and (10.11) for ρ_1, we obtain the values for μ from Eq. (9.3) as

$$\mu^{(1)} = \frac{4}{3}\,\frac{\pi m - b}{b}\,\frac{n^2}{n^2+1} \quad (10.12)$$

and

$$\mu^{(2)} = \frac{n^2+1}{3}\,\frac{\pi(m+1/2)-b}{b}, \quad (10.13)$$

provided $|\pi m - b| \ll b$.

Equations (10.10)–(10.13) give the top (Eqs. (10.10) and (10.12)) and bottom (Eqs. (10.11) and (10.13)) turning points. The distance between the neighboring turning points in the first and the second sets is

$$\delta\mu_1 = \frac{4}{3}\,\frac{\pi}{b}\,\frac{n^2}{n^2+1} \quad \text{and} \quad \delta\mu_2 = \frac{n^2+1}{3}\,\frac{\pi}{b}. \quad (10.14)$$

The dependence of ρ_1 on μ is shown graphically in Fig. 7.

We finally compute the number of solutions for a given μ in the region $|\mu| b \gg 1$. It is given by

$$N(\mu) = 2(N_{top}(\mu) - N_{bottom}(\mu)) + 1, \quad (10.15)$$

where $N_{top}(\mu)$ and $N_{bottom}(\mu)$ is the number of the top and bottom turning points in the interval $[0, \mu]$ and the coefficient 2 accounts for the fact that there are two solutions corresponding to each turning point (see Fig. 7). We have roughly $N_{top}(\mu) \approx \mu/\delta\mu_1$ and $N_{bottom} \approx \mu/\delta\mu_2$, and therefore,

$$N(\mu) \approx 2\mu\left(\frac{1}{\delta\mu_1} - \frac{1}{\delta\mu_2}\right) \approx \frac{3}{2\pi}\mu b. \quad (10.15)$$

11. STABILITY

In this section, we study the general stability properties of solutions to boundary value problem (4.4)–(4.6). A detailed analysis will be given elsewhere. Clearly, given $\rho_1 \equiv \rho(b) \equiv |\psi(b)|^2 = T^2$, the problem in Eqs. (4.4)–(4.6) has a unique solution. In other words, solutions of the latter problem can be parameterized by ρ_1. (This can be done explicitly by expressing μ in terms of ρ_1.) In what follows, we tacitly assume that the curve (=multivalued function) $\psi = \psi(\mu)$ is parameterized by ρ_1. With this parameterization in mind, we sometimes speak about stability of a point (ρ_1, μ) understanding by it the stability of the corresponding point (ψ, μ).

Our task is to find stability of solutions of Eqs. (2.7)–(2.9) of form (3.1)–(3.3). To fit this problem into the standard framework, one would have to rewrite (2.7) as a system of the first order Hamiltonian equations and apply to it a rather subtle stability theory for solitary waves (see e.g. [19] and references therein). We adopt a direct approach instead. We seek solutions of Eqs. (2.7)–(2.9) in the form

$$E = \begin{cases} \mathrm{Re}[e^{-i\omega t}(\psi_0 + e^{\lambda t}\xi)] \text{ if } \lambda \text{ is real} & (11.2) \\ \mathrm{Re}[e^{-i\omega t}(\psi_0 + e^{\lambda t}\xi_1 + e^{\bar\lambda t}\xi_2)] & \\ \quad \text{if } \lambda \text{ is complex,} & (11.3) \end{cases}$$

where $\omega = k_0$, ψ_0 satisfies Eqs. (3.2), (3.3), and ξ are small and such that

$$e^{-i(\omega+i\lambda)t}\xi \text{ is an outgoing wave for } x<0 \text{ and } x>a \quad (11.4)$$

and similarly for λ complex. This implies

$$\xi = \begin{cases} A_1 e^{-i(\omega+i\lambda)x} & \text{for } x<0, \\ A_2 e^{i(\omega+i\lambda)x} & \text{for } x>a \end{cases} \quad (11.5)$$

for some constants A_1 and A_2, which gives

$$\frac{\xi'(0)}{\xi(0)} = -i(\omega+i\lambda) \text{ and } \frac{\xi'(a)}{\xi(a)} = i(\omega+i\lambda) \quad (11.6)$$

for λ real. For λ complex, the boundary conditions are

$$\frac{\xi_1'(0)}{\xi_1(0)} = -i(\omega+i\lambda), \quad \frac{\xi_1'(a)}{\xi_1(a)} = i(\omega+i\lambda) \quad (11.7)$$

and

$$\frac{\xi_2'(0)}{\xi_2(0)} = -i(\omega+i\bar\lambda), \quad \frac{\xi_2'(a)}{\xi_2(a)} = i(\omega+i\bar\lambda). \quad (11.8)$$

For simplicity, we deal only with the case Eq. (11.2), the case (11.3) is treated in a similar way. Substituting (11.2) in (2.7), we derive the linearized equation for ξ (see Appendix 1 for a similar derivation),

$$L_\lambda(\xi) = 0, \quad (11.9)$$

where, with $\sigma = \omega + i\lambda$ and $\varepsilon'(s,x) = \partial\varepsilon(s,x)/\partial s$,

$$L_\lambda(\xi) = \partial_x^2\xi + \sigma^2\varepsilon(|\psi_0|^2, x)\xi + \\ + \sigma^2\varepsilon'(|\psi_0|^2, x)\psi_0\,\mathrm{Re}(\overline{\psi}_0\xi). \quad (11.10)$$

Equation (11.9) is a nonlinear eigenvalue problem. We observe that the operator family L_λ satisfies, $L_\lambda^* = L_{-\lambda}$, with respect to the inner product

$$(\xi,\eta) = \mathrm{Re}\int \overline{\xi}\eta. \quad (11.11)$$

A crucial role in our analysis is played by the following result which is stated directly for the rescaled function $\psi(x) = A^{-1}\psi_0(x/k)$, $k = k_0 n$.

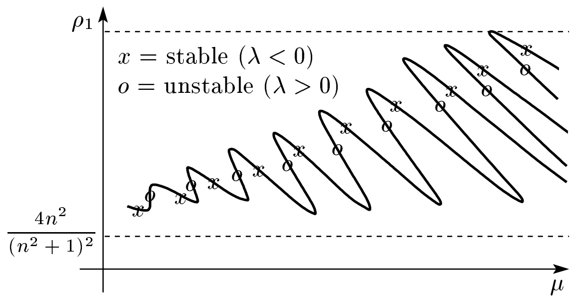

Fig. 8.

Theorem. $(\overline{\psi}, \overline{\mu})$ is a turning point iff

$$\frac{\partial\psi}{\partial\rho_1} \quad \text{solves} \quad L_0(\xi) = 0 \quad (11.12)$$

at that point

Proof. We write Eq. (4.4) as $F(\psi,\mu) = 0$ and let $\psi = \psi(\rho_1)$ and $\mu = \mu(\rho_1)$. Differentiating the last equation with respect to ρ_1, we obtain

$$F_\psi(\psi,\mu)\frac{\partial\psi}{\partial\rho_1} + \frac{\partial F}{\partial\mu}(\psi,\mu)\frac{\partial\mu}{\partial\rho_1} = 0, \quad (11.13)$$

where $F_\psi(\psi,\mu)$ is the variational derivative of $F(\psi,\mu)$ with respect to ψ. We note that $F_\psi(\psi,\mu)$ is equal to L_0 up to a rescaling. Now, $(\overline{\psi}, \overline{\mu})$ is a turning point iff $\partial\mu/\partial\rho_1 = 0$ at that point, and therefore, iff $F_\psi(\overline{\psi}, \overline{\mu})\partial\psi/\partial\rho_1 = 0$.

This theorem implies that

$$\lambda = 0 \text{ is an eigenvalue of (rescaled) Eq. (11.9)} \leftrightarrow (\rho_1,\mu) \text{ is a turning point.} \quad (11.14)$$

We claim that λ changes its sign as ρ_1 passes a turning point,

$$\frac{\partial\lambda}{\partial\rho_1} \begin{array}{ll} <0 & \text{at a top turning point} \\ >0 & \text{at a bottom turning point,} \end{array} \quad (11.15)$$

where top and bottom refers to the turning points of $\rho_1 = \rho_1(\mu)$.

Equations (11.14) and (11.15) suggest that the real eigenvalue λ, is negative on the top branches of $\rho_1 = \rho_1(\mu)$ (see Fig. 8), while $\lambda > 0$ on the bottom ones. Hence, the bottom branches are unstable. In order to understand the stability properties of the top branches, one has to envoke the remaining, complex eigenvalues. We expect that they are stable near the top turning points and unstable elsewhere.

Equation (11.15) is proved by a perturbation theory, which requires the information about solutions to

(11.9) only at the turning point. The details will be presented elsewhere.

Comparing Eqs. (11.6) and (11.9) with Eqs. (5.8) and (5.9), i.e. with the equations for the resonance solution in the linear case, we conclude that the former equations describe the resonance solution in the nonlinear case (the nonlinear resonance). It is remarkable that while the corresponding problem is always stable in the linear case (see Eq. (5.10)), the stable and unstable branches alternate in the nonlinear case.

12. EXPRESSION FOR ψ

In this section, we find an approximate form of the solution ψ to Eq. (4.4). This information is needed, in particular, for a more detailed study of the stability of various branches.

We first find the function $\rho(x)$ for $0 < x < b$. For this, we replace Eq. (9.2) with the equation

$$\pm \int\limits_{\rho_0}^{\rho(x)} \frac{d\rho}{\sqrt{f(\rho)}} = x,$$

which is integrated in the same way as (9.2) to yield (see Appendix 2)

$$\rho(x) = \frac{1}{2}\rho_1 \left\{ 1 + \frac{1}{n^2} + \left(1 - \frac{1}{n^2}\right)\cos 2\gamma(x) \right\}, \quad (12.1)$$

where

$$\gamma(x) = \frac{b - x}{1 - \frac{3\mu\rho_1}{4}(1 + \frac{1}{n^2})}. \quad (12.2)$$

We next find the expression for $\alpha = \arg(\psi)$ that matches (12.1). Observing that

$$1 + \frac{1}{n^2} + \left(1 - \frac{1}{n^2}\right)\cos 2\gamma = \frac{(n+1)^2}{2n^2}\left| e^{i\gamma} + \frac{n-1}{n+1}e^{-i\gamma} \right|^2,$$

we seek α in the form

$$\alpha(x) = \overline{\alpha}(x) + \beta(x), \quad (12.3)$$

where $\overline{\alpha}(x) = -\operatorname{arctg}\left[n^{-1}\operatorname{tg}\gamma(x)\right]$ and where the function β is to be found using Eq. (7.3) for α. The latter gives, modulo $O(\mu)$,

$$\beta(x) = -\frac{3\mu\rho_1}{4}\left(1 + \frac{1}{n^2}\right)\left(\overline{\alpha}(x) - \overline{\alpha}(0)\right) + \beta(0). \quad (12.4)$$

The initial condition $\beta(0)$ is found from

$$\beta(0) = \alpha(0) - \operatorname{arctg}\left(\frac{1}{n}\operatorname{tg}\gamma(0)\right) \quad (12.5)$$

and

$$\alpha(0) = \operatorname{arcctg}\left(\frac{n\rho'}{2(\rho+\rho_1)}\right)\bigg|_{x=0}. \quad (12.6)$$

The function $\beta(x)$ can also be represented as (again modulo $O(\mu)$)

$$\beta(x) = \beta(0) - \frac{3\mu\rho_1}{2n}\left(1 + \frac{1}{n^2}\right) \times \int\limits_0^x \frac{dy}{1 + \frac{1}{n^2} + \left(1 - \frac{1}{n^2}\right)\cos 2\gamma(y)}. \quad (12.7)$$

Putting Eqs. (12.1) and (12.3) together, we write $\psi(x) = \sqrt{\rho(x)}e^{i\alpha(x)}$ as

$$\psi(x) = \frac{n+1}{2n}\sqrt{\rho_1}e^{i\delta(x)} \times \left[e^{-i\gamma(x)} + \frac{n-1}{n+1}e^{i\gamma(x)} \right], \quad (12.8)$$

where $\gamma(x)$ is given by Eq. (12.2) and

$$\delta(x) = \text{const} + \frac{\mu\rho_1}{2n}(b - x).$$

The explicit form of ψ reflects the picture of a nonlinear wave propagation: it is a superposition of two waves travelling in opposite directions with slightly different speed. The nonlinearity leads to a renormalization of the wave vector,

$$k_0 \to k_0\left(1 - \frac{3\mu\rho_1}{4}\left(1 + \frac{1}{n^2}\right)\right)^{-1},$$

and to the appearance of a slowly varying phase $\beta(x)$.

Expression (12.8) for ψ will be used in the study of the stability problem which will appear elsewhere.

13. CONCLUSION

As we see from Fig. 8, a small change of the light intensity (i.e., of μ) near a turning point is capable of switching the system from one state (solution ψ) to another. Namely, moving around a turning point changes a stable solution into an unstable one; under the action of a random perturbation, the system then jumps to a stable solution as shown in Fig. 9. This either turns on or turns off the light passing through the slab.

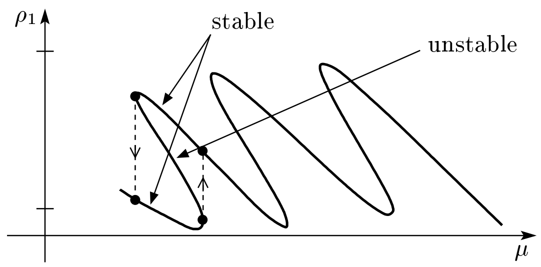

Fig. 9. Switching of solutions

At the next step, we would like to address the nonlinear stability problem, in other words, to study solutions of Eqs. (2.7)–(2.9) with the initial conditions close to the solitary wave $e^{-ik_0t}\psi_0(x)$ in the cases where ψ_0 is on a stable branch and, more interestingly, in the case where ψ_0 is on the unstable branch (see Eq. (5.11)). In the latter case, it is desirable to find a mathematical description to the processes described above (see Fig. 9).

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APPENDIX 1

One-mode approximation

In this appendix, we derive Eq. (3.2), which is a one-mode approximation to Eq. (2.7). We seek a solution to Eq. (2.7) in the form

$$E(x,t) = \sum_{n=0}^{\infty} \mathrm{Re}\left(e^{-i(2n+1)k_0t}\psi_n(x)\right). \qquad \text{(A1.1)}$$

Substituting (A1.1) in (2.7), we obtain an infinite system of coupled equations for the coefficients $\psi_n(x)$, $n = 0, 1, \ldots$, the $\pm n$-th equation read off from the coefficient in front of $e^{\mp i(2n+1)k_0t}$. Because of boundary conditions (2.8), (2.9), it is easy to show that $\psi_n = O(\overline{\mu}^n)$. Hence, the contribution of ψ_n, $n \ge 1$, to the $n = 0$ equation is of the order $O(\overline{\mu}^2)$, and we drop this contribution in the leading-order approximation. Finally, to derive the $n = 0$ equation, we use the relation

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left(|\,\mathrm{Re}(e^{-ik_0t}\psi_0)|^2\,\mathrm{Re}(e^{-ik_0t}\psi_0)\right) = \\
&= \frac{\partial^2}{\partial t^2}\left(\frac{1}{4}2|\psi_0|^2\left(\frac{1}{2}e^{-ik_0t}\psi_0\right)\right) + \\
&+ \frac{\partial^2}{\partial t^2}\left(\frac{1}{4}e^{-2ik_0t}\psi_0^2\left(\frac{1}{2}e^{ik_0t}\bar{\psi}_0\right)\right) + \\
&+ \text{ terms proportional to } e^{ik_0t} \text{ and } e^{\pm 3ik_0t} = \\
&= \frac{\partial^2}{\partial t^2}\left(\frac{3}{4}|\psi_0|^2\left(\frac{1}{2}e^{-ik_0t}\psi_0\right)\right) + \\
&+ \text{ terms proportional to } e^{ik_0t} \text{ and } e^{\pm 3ik_0t}. \qquad \text{(A1.2)}
\end{aligned}$$

In the above approximation, this expression immedicately implies Eq. (3.2).

APPENDIX 2

Computation of $\int \frac{d\rho}{\sqrt{f(\rho)}}$

The derivative of (9.3). We set $\rho_1 = \rho(b)$ and recall expression (9.1) for $f(\rho)$,

$$f(\rho) = 4(\rho_1 - \rho)\left[\mu\rho^2 + (1+\mu\rho_1)\rho - \frac{\rho_1}{n^2}\right]. \qquad \text{(A2.1)}$$

We observe that ρ_1 is a root of the equation $f(\rho) = 0$. We find the other two roots,

$$\begin{aligned}
\rho_{2,3} &= (2\mu)^{-1}\left[-(1+\mu\rho_1) \pm \sqrt{(1+\mu\rho_1)^2 + \frac{4\mu\rho_1}{n^2}}\right] = \\
&= \frac{1+\mu\rho_1}{2\mu}\left[-1 \pm \sqrt{1 + \frac{4\mu\rho_1}{n^2(1+\mu\rho_1)^2}}\right]. \qquad \text{(A2.2)}
\end{aligned}$$

The expansion in powers of μ shows that

$$\begin{aligned}
\rho_2 &= \frac{\rho_1}{n^2(1+\mu\rho_1)}\left(1 - \frac{\mu\rho_1}{n^2}\right) + O(\mu^2) = \\
&= \frac{\rho_1}{n^2}\left(1 - \mu\rho_1\left(1 + \frac{1}{n^2}\right)\right) + O(\mu^2) \qquad \text{(A2.3)}
\end{aligned}$$

and

$$\rho_3 = -\frac{1}{\mu}\left[1 + \mu\rho_1\left(1 + \frac{1}{n^2}\right)\right] + O(\mu). \qquad \text{(A2.4)}$$

Clearly, for μ small and $n \ge 1$, a safe way to expand $f^{-1/2}(\rho)$ is by expanding

$$\begin{aligned}
&(\mu\rho - \mu\rho_3)^{-1/2} = \\
&= \left[1 + \mu\rho_1\left(1 + \frac{1}{n^2}\right) + \mu\rho + O(\mu^2)\right]^{-1/2} = \\
&= 1 - \frac{1}{2}\mu\left(\rho_1 + \frac{\rho_1}{n^2} + \rho\right) + O(\mu^2).
\end{aligned}$$

Hence, for

$$f(\rho) = -4(\rho-\rho_1)(\rho-\rho_2)(\mu\rho-\mu\rho_3) \qquad \text{(A2.5)}$$

we obtain, modulo $O(\mu^2)$,

$$2f^{-1/2}(\rho) = \frac{1-\frac{1}{2}\mu\left(\rho_1+\frac{\rho_1}{n^2}+\rho\right)}{[-(\rho-\rho_1)(\rho-\rho_2)]^{1/2}}. \qquad \text{(A2.6)}$$

Our aim is to integrate this expression. For this, we present it as

$$2f^{-1/2}(\rho) = \frac{1-\frac{1}{2}\mu\left(\frac{3}{2}\rho_1+\frac{1}{2}\rho_2+\frac{\rho_1}{n^2}\right)}{\sqrt{-(\rho-\rho_1)(\rho-\rho_2)}} + \frac{1}{2}\mu\frac{-\rho+\frac{1}{2}(\rho_1+\rho_2)}{\sqrt{-(\rho-\rho_1)(\rho-\rho_2)}}. \qquad \text{(A2.7)}$$

We let $\rho_0 = \rho(0)$ and recall that $\rho_1 = \rho(b)$. We then have

$$\int_{\rho_0}^{\rho_1} d\rho \frac{-\rho+\frac{1}{2}(\rho_1+\rho_2)}{\sqrt{-(\rho-\rho_1)(\rho-\rho_2)}} = \sqrt{-(\rho-\rho_1)(\rho-\rho_2)}\,\Big|_{\rho_0}^{\rho_1} = -\sqrt{(\rho_1-\rho_0)(\rho_0-\rho_2)}. \qquad \text{(A2.8)}$$

Using that

$$\left(\arcsin\frac{x-p}{q}\right)' = \frac{1}{\sqrt{1-\dfrac{(x-p)^2}{q^2}}}\frac{1}{q} = \frac{1}{\sqrt{q^2-(x-p)^2}} = \frac{1}{\sqrt{(p+q-x)(x-p+q)}}$$

and setting $p = (\rho_1+\rho_2)/2$ and $q = (\rho_1-\rho_2)/2$, we obtain

$$\int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{(\rho_1-\rho)(\rho-\rho_2)}} = \arcsin 2\left(\frac{\rho-\dfrac{\rho_1+\rho_2}{2}}{\rho_1-\rho_2}\right)\Bigg|_{\rho_0}^{\rho_1} = \frac{\pi}{2} - \arcsin\left[\frac{2\rho_0-(\rho_1+\rho_2)}{\rho_1-\rho_2}\right]. \qquad \text{(A2.9)}$$

Combining Eqs. (A2.7)–(A2.9), we obtain

$$2\int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{f(\rho)}} = -\frac{1}{2}\mu\sqrt{(\rho_1-\rho_0)(\rho_0-\rho_2)} + \left[1-\frac{1}{4}\mu\left(3\rho_1+\rho_2+2\frac{\rho_1}{n^2}\right)\right]\times$$
$$\times\left[\frac{\pi}{2}-\arcsin\left[\frac{2\rho_0-(\rho_1+\rho_2)}{\rho_1-\rho_2}\right]\right]. \qquad \text{(A2.10)}$$

Together with Eqs. (9.2) and (A2.3), this gives, modulo $O(\mu^2)$,

$$\left[1-\frac{3}{4}\mu\left(1+\frac{1}{n^2}\right)\rho_1\right]\left[\frac{\pi}{2}-\arcsin\left[\frac{2\rho_0-(\rho_1+\rho_2)}{\rho_1-\rho_2}\right]\right] - \frac{1}{2}\mu\sqrt{(\rho_1-\rho_0)(\rho_0-\rho_2)} = \pm 2b. \qquad \text{(A2.11)}$$

This can be rewritten as

$$\frac{2\rho_0-(\rho_1+\rho_2)}{\rho_1-\rho_2} = \cos\frac{\pm 2b+\mu\sqrt{(\rho_1-\rho_0)(\rho_0-\rho_2)}/2}{1-\frac{3}{4}\mu\left(1+\frac{1}{n^2}\right)\rho_1}. \qquad \text{(A2.12)}$$

Equations (7.11)–(7.12) now imply

$$\rho_0 = \frac{(n^2+3)\rho_1-4}{n^2-1} + \frac{n^2\mu}{n^2-1}\left[\rho_1^2 - \frac{\left((n^2+3)\rho_1-4\right)^2}{(n^2-1)^2}\right] + O(\mu^2). \qquad \text{(A2.13)}$$

Inserting this expression for ρ_0 and Eq. (A2.3) for ρ_2 in Eq. (A2.12), we finally obtain an equation for ρ_1 only,

$$\frac{4(1-\rho_1)}{\rho_1}\frac{n^2}{(n^2-1)^2}\left[1-\frac{2n^2\mu}{(n^2-1)^2}\big((n^2+1)\rho_1-2\big)-\frac{\mu\rho_1(n^2+1)}{n^2(n^2-1)}\right]= \\ =\sin^2\left[\frac{b+\dfrac{\mu}{2(n^2-1)}\sqrt{(1-\rho_1)n^{-2}\big((n^2+1)^2\rho_1-4n^2\big)}}{1-\dfrac{3\mu}{4}\left(1+\dfrac{1}{n^2}\right)\rho_1}\right], \tag{A2.14}$$

modulo $O(\mu^2)$ terms. We simplify this equation by dropping terms of the order $O(\mu)$ (but keeping terms of the order $O(\mu b)$!) to obtain Eq. (9.3).

The derivation of (12.1), (12.2). Proceeding as above, we obtain

$$\int\limits_{\rho_0}^{\rho(x)}\frac{d\rho}{\sqrt{f(\rho)}}=\left[1-\frac{3\mu}{4}(1+\frac{1}{n^2})\rho_1\right]\left[\arcsin\left[\frac{2\rho(x)-(\rho_1+\rho_2)}{\rho_1-\rho_2}\right]-\arcsin\left[\frac{2\rho_0-(\rho_1+\rho_2)}{\rho_1-\rho_2}\right]\right]+ \\ +\frac{1}{2}\mu\left[\sqrt{(\rho_1-\rho(x))(\rho(x)-\rho_2)}-\sqrt{(\rho_1-\rho_0)(\rho_0-\rho_2)}\right] \tag{A2.15}$$

(Eq. (A2.15) with $x = b$ yields, as it should, Eq. (A2.10)). Using this expression, we find an approximate solution of the equation

$$\int\limits_{\rho_0}^{\rho(x)}\frac{d\rho}{\sqrt{f(\rho)}}$$

for $\rho(x)$ by dropping terms of the order $O(\mu)$ but keeping terms of the order $O(\mu b)$. This yields

$$\rho(x)=\frac{1}{2}(\rho_1+\rho_2)+\frac{1}{2}(\rho_1-\rho_2)\cos 2\gamma(x), \tag{A2.16}$$

where

$$\gamma(x)=\frac{b-x}{1-\dfrac{3\mu\rho_1}{4}\left(1+\dfrac{1}{n^2}\right)}.$$

Inserting expression (A2.3) for ρ_2 in the right-hand side, we arrive at (12.1), (12.2).

APPENDIX 3

In this appendix, we outline another derivation of the expression for the solution ψ of Eq. (4.4). In this derivation, we consider (4.4) as a linear equation for ψ by assuming that $|\psi|^2=\rho$ in this equation is given by (12.1), (12.2). We seek two linearly independent solutions of the resulting equation in the Bloch function the form

$$[Ae^{i\gamma(x)}+Be^{-i\gamma(x)}]e^{i\nu(b-x)}+O(\mu) \tag{A3.1}$$

for some ν, A, and B. Inserting this in the equation in question and using the solvability condition for the constants A and B, we obtain, after a simple calculation,

$$\nu=\pm\lambda, \quad \text{where} \quad \lambda=\frac{\mu\rho_1}{2n} \tag{A3.2}$$

With these values for ν, we solve for A as

$$A=\frac{n\pm 1}{n\pm 1}B.$$

As a result, the general solution of the above linear equation is given by

$$\psi=C\left\{\left(1+\frac{1}{n}\right)e^{i\gamma(x)}+\left(1-\frac{1}{n}\right)e^{-i\gamma(x)}\right\}\times \\ \times e^{-i\lambda(b-x)}+ \\ +D\left\{\left(1+\frac{1}{n}\right)e^{-i\gamma(x)}+\left(1-\frac{1}{n}\right)e^{i\gamma(x)}\right\}\times \\ \times e^{i\lambda(b-x)}. \tag{A3.3}$$

From boundary condition (4.9), we find

$$D=\frac{\mu\rho_1}{4}\left(\frac{1}{2}+\frac{3}{2n^2}\right)C. \tag{A3.4}$$

Hence, the last term in (A3.3) is $O(\mu)$ and can therefore be omitted. Finally, we use that $|\psi(b)|=\sqrt{\rho_1}$, to find that $|D|=\sqrt{\rho_1}/2$, and therefore,

$$\psi=\frac{n+1}{2n}\sqrt{\rho_1}\left[e^{-i\gamma(x)}+\frac{n-1}{n+1}e^{i\gamma(x)}\right]\times \\ \times e^{i\beta+i\lambda(b-x)} \tag{A3.5}$$

where λ is given by Eq. (A3.2) and β is some constant related to $\alpha(0)$ in a simple way.

The value of $|\psi|$ that follows from (A3.5) is the same as that given by Eq. (12.1).

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