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ASYMMETRIC DARK SOLITONS IN NONLINEAR LATTICES

S. Darmanyan*

Institute of Spectroscopy, Russian Academy of Sciences
142190, Troitsk, Russia

A. Kobyakov

School of Optics/CREOL, University of Central Florida
4000 Central Florida Blvd., Orlando, FL 32816-2700, USA

F. Lederer

Institute of Solid State Physics and Theoretical Optics,
Friedrich-Schiller-Universität Jena
07743, Jena, Germany

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New types of stable discrete solitons are discovered. They represent the first example of asymmetric dark solitons and shock waves with a nonzero background. Both types of solutions exhibit a strong intrinsic phase dynamics. Their existence domains and stability criteria are identified. Numerical experiments support the analytical findings.

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Numerous recent studies have evidenced that the inherent discreteness of nonlinear systems can qualitatively alter their dynamical behavior compared to their continuous counterparts. Because many physical systems are discrete by definition, these effects attract a steadily increasing interest in various branches of physics (for a detailed overview, see the review papers [1] and references therein). In these studies, particular emphasis was given to stationary, localized structures that are frequently termed as discrete solitons. One option to categorize them is by their degree of localization. Strongly localized solitons (SLSs), where the excitation is resting and involves only a few lattice sites, exhibit properties that originate from the very discreteness of the system [2]. Thus, their behavior differs in many aspects from solutions of related continuous models. It was shown that SLSs can significantly contribute to the heat transfer and other thermodynamic and magnetic effects in solids. Moreover, certain destabilization scenarios can be used for signal processing and switching applications in discrete optical systems such as the coupled waveguide arrays [1, 2, 3, 4]. Up to now, various types of SLSs have been reported to exist. Bright [3, 4, 5, 6, 7] and dark [8, 9, 10, 11] stable SLSs exhibiting interesting new topologies and shapes were identified in various nonlinear evolution equations.

However, similarly to continuum models, all these solutions are (anti-) symmetric and do not exhibit an intrinsic phase dynamics. The existence and stability of asymmetric bright SLSs that are quasi-periodic in time were studied in [12]. In this paper, we reveal that a discreteness may induce new soliton formation mechanisms resulting in the existence of shock waves with two finite backgrounds as well as asymmetric dark solitons. They exhibit a nontrivial intrinsic phase dynamics, i.e., the backgrounds oscillate at two different frequencies and the transition region is characterized by combinations of these frequencies.

Our model is based on the discrete nonlinear Schrödinger equation (DNLSE), which is among the most prominent model equations in nonlinear physics. Vibron modes in biomolecules, the Heisenberg ferromagnet or Frenkel excitons in a chain with two-level atoms can be mentioned among numerous phenomena

*E-mail: sdarmanyan@yahoo.com

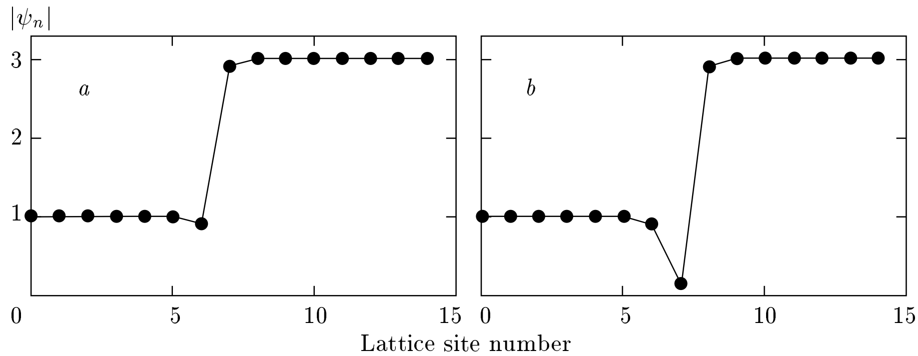

Fig. 1. Discrete shock wave with a finite background (a) and asymmetric discrete dark soliton (b)

described by this equation [1]. Moreover, the DNLSE also describes light propagation in arrays of weakly coupled nonlinear optical waveguides exhibiting the Kerr nonlinearity [13]. Recently, the existence and dynamics of discrete solitons in the latter environment were experimentally verified [14].

We consider the DNLSE in the generic form

$$i\dot{\psi}_n + c(\psi_{n-1} + \psi_{n+1}) + \lambda|\psi_n|^2\psi_n = 0, \tag{1}$$

where ψ_n denotes the amplitude excitation at the n-th site, c and $\lambda = \pm 1$ are the linear and nonlinear coupling coefficients, respectively, and the dot denotes the derivative with respect to the evolution variable t.

Traditionally, one seeks a resting solution to (1) with the common frequency ω in the form $\psi_n = f_n \exp(i\omega t)$, where the localization involves only several lattice sites n. In contrast to this conventional ansatz, we search for solutions that are characterized by a combinational frequency. This combinational frequency is determined by interaction of the lattice sites of the localization region with both backgrounds. We show that typical stable SLSs of this type are, e.g., shock waves with two finite backgrounds (Fig. 1a) or asymmetric dark solitons (Fig. 1b). A new family of symmetric dark solitons without the phase jump π in the soliton center is also identified in what follows.

The asymmetric SLSs displayed in Fig. 1 have the form

$$\psi = \{\psi_n\} = \{(\dots, 1, 1, 1)\exp(i\omega_1 t), \psi_{-N}, \dots \\ \dots \psi_N, (A, A, A, \dots)\exp(i\omega_2 t)\},$$

where the amplitude of the left background is scaled to unity. Strong localization implies $c \ll 1$ [4, 6–11] and a small number N of constituents of the transition region. It is evident from Eq. (1) that the two background frequencies, $\omega_1 = 2c + \lambda$ for $n < -N$ and $\omega_2 = 2c + \lambda A^2$ for $n > N$, do not coincide. For these solutions to exist, the localization between both backgrounds ($n = -N, \dots, N$) must have the form

$$\psi_n(t) = f_n \exp(i\omega_1 t) + g_n \exp(i\omega_2 t) + m_n(t), \tag{2}$$

where $m_n(t)$ contains an infinite sum of terms with various combinational frequencies of both backgrounds. We follow the conventional terminology [3], assuming that odd (even) SLSs have an odd (even) number of transition sites, and we omit the site $n = 0$ for even modes. As can be seen in the ansatz, we assume unstaggered backgrounds, which requires $\lambda = -1$ for modulationally stable solutions [15]. Because (1) is invariant under the transformation $\lambda \to -\lambda$, $t \to -t$, $\psi_n \to (-1)^n\psi_n$, the results also hold for staggered backgrounds with $\lambda = 1$.

In what follows, we assume that A is real-valued, thus dealing with either the in-phase ($A > 0$) or out-of-phase ($A < 0$) background at $t = 0$. Substitution of (2) into (1) results in a system of equations, where in the strong localization limit [2–7], we only keep the terms in the lowest order in the small parameter c.

SHOCK WAVES WITH A FINITE BACKGROUND

We begin with SLSs of the narrowest possible width, namely with finite background shock waves (Fig. 1a). It is an even SLS with $N = 1$, and therefore, only two sites $n = -1, 1$ constitute the transition region. Within the first-order approximation in c, the solution to Eq. (1) is given by

$$\begin{aligned} \psi_{-1} &\approx \left(1 - \frac{c}{2}\right) e^{i\omega_1 t} - \alpha A^3 e^{i\omega_2 t} + \alpha A e^{i(2\omega_1 - \omega_2)t}, \\ \psi_1 &\approx \left(A - \frac{c}{2A}\right) e^{i\omega_2 t} - \alpha e^{i\omega_1 t} + \alpha A^2 e^{i(2\omega_2 - \omega_1)t}, \end{aligned} \tag{3}$$

Fig. 2. Temporal evolution of the amplitude of a stable shock wave; $A = 3$, $c = 0.08$; step-like excitation

where

$$\alpha = \frac{c}{(A^2-1)^2 - 2c(1+A^2)}$$

and the oscillation of each site in the transition region is determined by a combination of three frequencies. Other combinations of the background frequencies ω_1 and ω_2 appear only as higher-order terms in c and do not significantly contribute to the dynamics of this SLS. Two constraints must be satisfied for solution (3) being valid, namely, $|\alpha A^3| \ll 1$ for $|A| > 1$ and $|\alpha| \ll |A|$ for $|A| < 1$.

We also mention that the limit $A \to 1$ requires taking the second-order terms in c into account. The transformation $\alpha \to \alpha(1 \pm c)$, where the respective signs «$+$» and «$-$» correspond to the second and third terms in Eq. (3), provides a more accurate solution in this case. Without loss of generality, we consider the case where $|A| \geq 1$, thereby normalizing with respect to the lower background amplitude.

We performed numerical experiments to prove the existence and to probe the robustness of this new SLS. We directly integrated Eq. (1) using solution (3) as the initial condition. The results have shown that the soliton can be easily excited. Moreover, the solution is very robust against rather strong perturbations of the initial conditions. We used a step-like profile

$$f_n = (\ldots, 1, 1, 1, A, A, A, \ldots)$$

for excitation and obtained the robust propagation displayed in Fig. 2. A zoomed picture of the amplitude and phase evolution of the two sites in the transition region is shown in Fig. 3, where an excellent agreement between analytical (Eq. (3)) and numerical results can be recognized. However, this SLS exists only in a restricted domain in the parameter space because for A approaching $A_\pm \to 1 \pm \sqrt{c} + c^{3/2}/8$, the approximate solution diverges, see Eq. (3). For example, if $c = 0.08$, then $A_+ \approx 1.29$. Indeed, the numerical integration of Eq. (1) with the step-like initial condition reveals a rapid decay of the initial excitation even for $A = 1.45$ (Fig. 4). This behavior can be easily explained by realizing that, e.g., for $n = -1$, the ratio of the amplitudes oscillating at ω_2 and ω_1 amounts to approximately 0.35. Thus higher-order terms become essential and evoke the SLS decay. If we require that this ratio should be of the order of c, we can estimate the SLS robustness domain. The condition $\alpha A^3 \sim c \ll 1$ gives the approximate threshold value of the amplitude A as $A_{th} \approx 1.9 + c$. For $A \gtrsim 1.9$, one can therefore expect a robust SLS behavior that has been confirmed by our numerical simulations.

ASYMMETRIC DARK SOLITONS

Following the same approach, we can find an odd solution that takes form

$$\psi_{-1} \approx \left(1 - \frac{c}{2}\right) e^{i\omega_1 t},$$
$$\psi_0 \approx -ce^{i\omega_1 t} - \frac{c}{A} e^{i\omega_2 t}, \tag{4}$$
$$\psi_1 \approx \left(A - \frac{c}{2A}\right) e^{i\omega_2 t}.$$

To our best knowledge, the solution represents the first example of an asymmetric dark soliton (Fig. 1b) exhibiting a strong intrinsic phase dynamics. Numerical solution of Eq. (1) with initial condition (4) proves the robustness of the solution. Although both amplitudes with $n = 0$ are small, the presence of two frequency components is essential, because the backgrounds interact via the excitation at $n = 0$. Precisely this interaction affects the stability of the dark soliton. The existence domain of this mode depends on the coupling constant c and the ratio of the background amplitudes A. If the backgrounds are separated by more than two lattice constants, wide solitons form. In fact, such solitons can be viewed as two noninteracting discrete front waves reported recently [7].

Whereas the canonical case where $A = -1$ has been investigated previously and both even and odd dark solutions have been found [8, 9], the case where $A = 1$

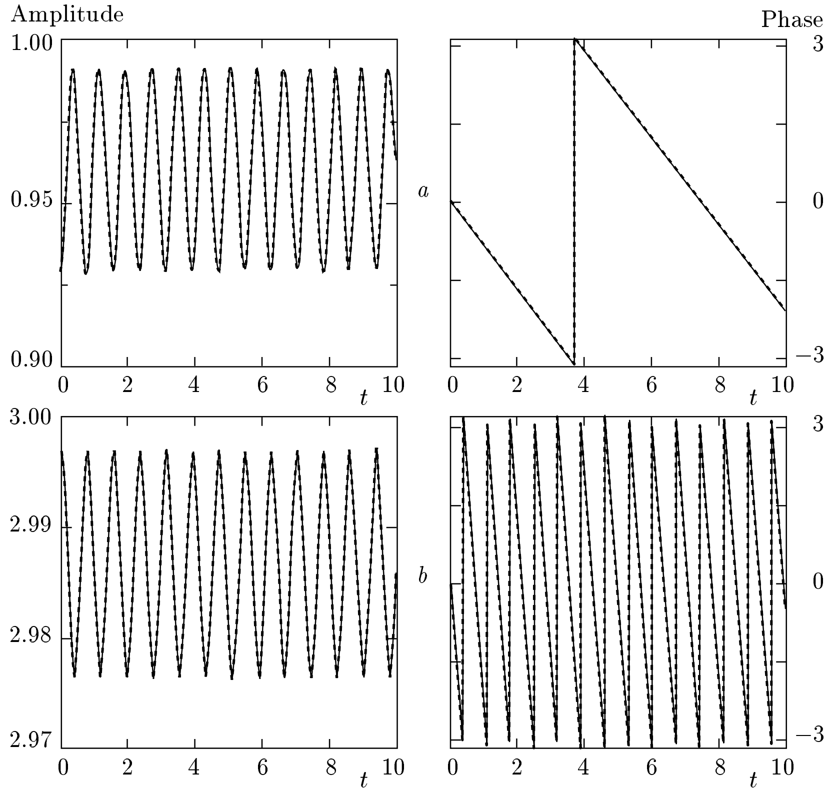

Fig. 3. Amplitude oscillations and phase evolution of the excitations in the shock wave transition region: *a*) $n=-1$; *b*) $n=1$; the parameters are as in Fig. 2. The solid lines show analytical results (3) and the dashed lines correspond to the numerical integration of Eq. (1)

provides a new type of solutions, namely symmetric dark solitons without a phase jump in the center representing a genuine dark soliton with regard to the amplitude. This particular solution has no intrinsic phase dynamics, i.e., all excitations oscillate with frequency $\omega = 2c+\lambda$. There are odd,

$$\psi_{-1}=\psi_1\approx\left(1-\frac{c}{2}\right)e^{i\omega t},\quad \psi_0\approx -2ce^{i\omega t}, \tag{5}$$

and even,

$$\psi_{-2}=\psi_2\approx\left(1-\frac{c}{2}\right)e^{i\omega t},\quad \psi_{-1}=\psi_1\approx -ce^{i\omega t}, \tag{6}$$

solutions.

Because asymmetric dark soliton (4) is a fairly exotic object, it is worthwhile to probe its stability by the linear stability analysis. Introducing a complex perturbation at each site via $\psi_n\to\psi_n+\epsilon_n$ and linearizing Eq. (1) with respect to perturbations ϵ_n, we obtain the set of equations

$$\begin{aligned}
&i\dot\epsilon_{-2}-2\epsilon_{-2}+c\epsilon_{-1}-\epsilon^*_{-2}e^{2i\omega_1 t}=0,\\
&i\dot\epsilon_{-1}-2(1-c)\epsilon_{-1}+c(\epsilon_0+\epsilon_{-2})-\\
&-(1-c)\epsilon^*_{-1}e^{2i\omega_1 t}=0,\\
&i\dot\epsilon_0+c(\epsilon_{-1}+\epsilon_1)=0,\\
&i\dot\epsilon_1-2(A^2-c)\epsilon_1+\\
&+c(\epsilon_0+\epsilon_2)-(A^2-c)\epsilon^*_1e^{2i\omega_2 t}=0,\\
&i\dot\epsilon_2-2A^2\epsilon_2+c\epsilon_1-A^2\epsilon^*_2e^{2i\omega_2 t}=0,
\end{aligned} \tag{7}$$

where only the sites that belong to the transition region and one site from each background were taken into account. Nevertheless, this set of equations can be easily extended to any number of background sites.

The approach successfully used in studying the stability of bright SLSs [4] cannot be applied here, because the coefficients in Eqs. (7) depend explicitly on the evolution variable. We therefore follow a different procedure to tackle the stability issue of multifrequency localized structures. In doing this, we introduce the Fourier transform of the perturbations,

Fig. 4. Temporal evolution of the amplitude of an unstable shock wave; $A = 1.45$, $c = 0.08$; step-like excitation

$$\epsilon_n = \int\limits_{-\infty}^{\infty} F_n(\Omega) e^{i\Omega t} d\Omega, \quad \epsilon_n^* = \int\limits_{-\infty}^{\infty} \Phi_n(\Omega) e^{i\Omega t} d\Omega,$$

where $\Phi_n(\Omega) = F_n^*(-\Omega)$, and rewrite Eqs. (7) in the frequency domain. We eliminate functions Φ_0 and F_0 and reduce the total number of equations to eight,

$$(2+\Omega)F_{-2}(\Omega) + \Phi_{-2}(\Omega - 2\omega_1) - cF_{-1}(\Omega) = 0, \quad (8)$$

$$F_{-2}(\Omega) - (\Omega - 4c)\Phi_{-2}(\Omega - 2\omega_1) - \\ - c\Phi_{-1}(\Omega - 2\omega_1) = 0, \quad (9)$$

$$- cF_{-2}(\Omega) + \left(2 - 2c - \frac{c^2}{\Omega} + \Omega\right) F_{-1}(\Omega) + \\ + (1-c)\Phi_{-1}(\Omega - 2\omega_1) - \frac{c^2}{\Omega} F_1(\Omega) = 0, \quad (10)$$

$$-c\Phi_{-2}(\Omega - 2\omega_1) + (1-c)F_{-1}(\Omega) + \\ + \left(2c - \Omega + \frac{c^2}{\Omega + 2 - 4c}\right) \Phi_{-1}(\Omega - 2\omega_1) + \\ + \frac{c^2}{\Omega + 2 - 4c} \Phi_1(\Omega - 2\omega_1) = 0, \quad (11)$$

$$- \frac{c^2}{\Omega} F_{-1}(\Omega) + \left(\Omega + 2A^2 - 2c - \frac{c^2}{\Omega}\right) F_1(\Omega) + \\ + (A^2 - c)\Phi_1(\Omega - 2\omega_2) - cF_2(\Omega) = 0, \quad (12)$$

$$\frac{c^2}{\Omega + 2A^2 - 4c} \Phi_{-1}(\Omega - 2\omega_2) + (A^2 - c)F_1(\Omega) - \\ - \left(\Omega - 2c - \frac{c^2}{\Omega + 2A^2 - 4c}\right) \Phi_1(\Omega - 2\omega_2) - \\ - c\Phi_2(\Omega - 2\omega_2) = 0, \quad (13)$$

$$-cF_1(\Omega) + (\Omega + 2A^2)F_2(\Omega) + A^2\Phi_2(\Omega - 2\omega_2) = 0, \quad (14)$$

$$- c\Phi_1(\Omega - 2\omega_2) + A^2 F_2(\Omega) - \\ - (\Omega - 4c)\Phi_2(\Omega - 2\omega_2) = 0, \quad (15)$$

where all functions with shifted arguments must be considered as independent. A complete set of these equations contains an infinite number of equations for the functions $F_n(\Omega - 2l\omega_1)$, $F_n(\Omega - 2l\omega_2)$, $\Phi_n(\Omega - 2l\omega_1)$, and $\Phi_n(\Omega - 2l\omega_2)$ with $n = \pm 1, \pm 2$ and $l = 0, 1, 2, 3 \ldots$ This fact is not surprising because Eqs. (7) explicitly depend on time, and therefore, their solutions contain all harmonics of the background frequencies ω_1 and ω_2. The terms with denominators in Eqs. (8)–(15) are responsible for higher harmonics. They are of the second order in c and could therefore be omitted. The reason to keep them is to account for possible resonances that appear as any denominators approaches zero, i.e., as $\Omega \to 0$, $\Omega - 2\omega_1 \to 0$, $\Omega - 2\omega_2 \to 0$. Outside the resonance regions, these terms can be omitted and Eqs. (8)–(15) reduce to two sets of four closed equations allowing the solution of the respective eigenvalue problem. The solution reveals that all eigenvalues are real, i.e., the SLS is stable. Thus, only the resonance regions are potentially responsible for the onset of instability. To treat the set of equations (8)–(15), one needs to close it by truncating to a finite number of equations. To proceed in this way, we note that only the terms $\Phi_1(\Omega - 2l\omega_1)$ in (11) and $\Phi_{-1}(\Omega - 2l\omega_2)$ in (13) introduce new frequencies into the system. A more thorough analysis of Eqs. (8)–(15) shows that it is not necessary to consider these harmonics in the first-order approximation in c because the amplitudes of these oscillations are of a higher order in c. In seeking the instability gain $\operatorname{Im}\Omega_j \sim c$, we can therefore drop these terms. We then obtain eight closed equations with the coefficients that depend nonlinearly on the eigenvalue Ω. The corresponding eigenvalue problem represents a polynomial

Fig. 5. Amplitude evolution of discrete dark solitons: *a*) stable asymmetric dark soliton, $A = 1.4$, $c = 0.065$; *b*) unstable asymmetric dark soliton, $A = 1.4$, $c = 0.1$; *c*) stable symmetric dark soliton, $A = 1$, $c = 0.07$; *d*) unstable symmetric dark soliton, $A = 1$, $c = 0.1$

of the 11th order possessing complex solutions in some domains of the parameter space (c, A). We found that complex eigenvalues appear for $c > c_{cr1}(A)$. Our analysis also revealed the existence of stability windows for $c_{cr2n}(A) < c < c_{cr2n+1}(A)$, where $n = 1, 2, \ldots$. As was shown recently [10], the existence of such windows is due to the finite size of the system used for modelling. The windows tend to disappear with an increasing number of lattice sites. With additional sites taken into account, we indeed observed this phenomenon. We note that the results obtained also hold for symmetric dark soliton (5).

Thus, we conclude that both asymmetric and symmetric dark solitons destabilize provided the linear coupling exceeds the threshold $c = c_{cr1}(A)$. It is important to note that the value $c_{cr1}(A)$ slightly depends on both the number N_s of sites regarded for the stability analysis, provided $N_s \geq 5$, and the ratio of the background amplitudes A. This value can be calculated with a good accuracy by taking five sites into account. To improve the accuracy, we also considered the case of seven sites involved. The result obtained was $c_{cr1} \approx 0.085$. A direct numerical integration of Eq. (1) confirms this prediction. Representative examples are displayed in Fig. 5 for $A = 1.4$ (asymmetric dark soliton) and $A = 1$ (symmetric dark soliton without a phase jump). Figures 5*a* and 5*c* exhibit stable propagation below the critical coupling ($c < c_{cr1}$), whereas the solitons decay beyond that threshold ($c = 0.1 > c_{cr1}$), which is in agreement with the linear stability analysis (Figs. 5*b* and 5*d*).

In conclusion, we have shown that new types of solitons, not reported before in the literature, may exist in nonlinear lattices described by the discrete nonlinear Schrödinger equation. These solitons are shock waves with a finite background and asymmetric dark solitons. They are peculiar in that they exhibit a nontrivial intrinsic phase dynamics. Additionally, we found a symmetric dark soliton with the conventional phase dynamics but without a phase jump in the center. A linear stability analysis and numerical experiments revealed the domains of their robust behavior.

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