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THE INSTABILITY POINT H_s OF THE MEISSNER STATE FOR LARGE-κ SUPERCONDUCTORS

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The critical field H_s corresponding to the emergence of vortices in a superconductor without a threshold is found near the transition temperature and in limit as $T \to 0$ for an arbitrary value of the depairing factor Γ. In superconductors of the second kind, this field value coincides with the absolute instability point of the Meissner state. In large-κ superconductors, the order parameter tends to zero on the surface of the superconductor if the external magnetic field reaches the value H_s. We obtain that $H_s = H_{cm}$ (where H_{cm} is the thermodynamic critical field) for an arbitrary value of the depairing factor Γ in the temperature region near T_c and at $T = 0$.

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1. INTRODUCTION

In superconductors with a large value of the Ginzburg–Landau parameter κ, the critical magnetic fields H_{c1}, H_{cm}, and H_{c2} are widely separated by their κ-values [1, 2]. Here, H_{c1} is the magnetic field value for the transition to the Shubnikov phase (vortex state), H_{cm} is the thermodynamic critical field, and H_{c2} is the bifurcation point corresponding to the formation of a vortex state in the volume of the superconductor. If $\kappa = 1$, all the three magnetic field values coincide. An isolated vortex is attracted to the boundary of a superconductor if the external magnetic field is weaker than a certan critical value H_s, the field of a barrierless penetration of vortices into the superconductor [3]. Near the transition temperature T_c, the problem of entering a vortex into the superconductor was considered by de Gennes [3]. He estimated that H_s is of the same order as the thermodynamic critical field H_{cm}. The exact value has not been found, because this requires considering small distances of the order of the correlation length. The value of the critical field H_s can be found as the linear instability point of the Meissner state. This means that there exists a stage of the transformation of the linear instability to the formation of a single vortex.

From this standpoint, the problem of calculating the critical field H_s is closely related to the problem of determining the superheating field H_{sh}. The last problem was considered by Ginzburg [4]. In what follows, we show that both problems (the calculation of the critical field H_s in the $\kappa \gg 1$ limit and the calculation of the critical field H_{sh} in the $\kappa \ll 1$ limit) can be solved using a single method near the transition temperature T_c, where the Ginzburg–Landau equations are applicable. The linear instability problem is simpler than the calculation of the vortex energy and some results for the H_s value can be found outside the framework of the Ginzburg–Landau free energy. We also find H_s in the zero-temperature limit. We show that near T_c, we have $H_s = H_{cm}$ and $H_{sh} = H_{cm}/\sqrt{\kappa}$, with the κ value related to the original definition of κ_{GL} as $\kappa_{GL} = \kappa/\sqrt{2}$.

2. THE CRITICAL FIELD H_s NEAR THE TRANSITION TEMPERATURE

The Ginzburg–Landau equations can be used near the transition temperature. We write them as

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$$\left\{-\tau-\frac{\pi D}{8T}\left(\frac{\partial}{\partial r}-2ieA\right)^2+\frac{7\zeta(3)|\Delta|^2}{8\pi^2T^2}\right\}\Delta=0,$$
$$j=-ie\frac{\pi\nu D}{4T}(\Delta^*\partial_-\Delta-\Delta\partial_+\Delta^*), \qquad (1)$$
$$\text{rot rot } A=4\pi j,$$

where

$$\tau=1-T/T_c,\quad \partial_\pm=\frac{\partial}{\partial r}\pm 2ieA,$$

D is the effective diffusion coefficient, A is the vector potential, $\zeta(x)$ is the Riemann zeta function, and $\nu=mp/2\pi^2$ is the density of states on the Fermi surface. The value of D was found by Gor'kov [5] and is equal to

$$D=\frac{vl_{tr}}{3}\times \times\left\{1+\frac{8T\tau_{tr}}{\pi}\left[\psi(1/2)-\psi\left(\frac{1}{2}+\frac{1}{4\pi T\tau_{tr}}\right)\right]\right\}, \qquad (2)$$

where v is the Fermi velocity, τ_{tr} is the transport collision time, and $\psi(x)$ is the Euler psi-function.

We use the gauge where

$$A=(A(y),0,0),\quad H=(0,0,-\partial A/\partial y). \qquad (3)$$

In the extreme case of large κ, we can use the local relation between A and Δ,

$$\Delta^2=\frac{8\pi^2T^2}{7\zeta(3)}\left[\tau-\frac{\pi De^2}{2T}A^2\right]. \qquad (4)$$

As the result, we obtain only one equation for the vector potential A instead of system (1),

$$-\frac{\partial^2\tilde{A}}{\partial Y^2}+\frac{1}{\kappa^2}\tilde{A}\left[1-\frac{1}{2}\tilde{A}^2\right]=0,$$
$$\tilde{H}=-\kappa\frac{\partial\tilde{A}}{\partial Y}. \qquad (5)$$

In Eq. (5), we use the dimensionless variables

$$y=\xi Y,\quad A=H_{cm}\kappa\xi\tilde{A}(Y),\quad \xi=\sqrt{\frac{\pi D}{16T\tau}},$$
$$\lambda_L^2=\frac{7\zeta(3)}{32\pi^4e^2\nu DT\tau},\quad \kappa=\frac{\lambda_L}{\xi}=\sqrt{\frac{7\zeta(3)}{2\pi^5e^2\nu D^2}}, \qquad (6)$$
$$\frac{H_{cm}^2}{8\pi}=\frac{4\pi^2T^2\nu\tau^2}{7\zeta(3)},\quad H=H_{cm}\tilde{H}.$$

The definition of κ in Eq. (6) differs from the original definition of κ_{GL} by the factor $\sqrt{2}$:

$$\kappa_{GL}=\kappa/\sqrt{2}.$$

With this definition of κ, the boundary between superconductors of the first and the second kind is at $\kappa=1$. For $\kappa=1$, all the critical fields H_{c1}, H_{cm}, and H_{c2} coincide. The definitions of ξ and κ in Eq. (6) can be continued in a natural way to the entire temperature region [6, 7].

Equation (5) can be reduced to the first-order equation

$$\frac{\partial\tilde{A}}{\partial Y}=-\frac{1}{\kappa}\tilde{A}\sqrt{1-\frac{1}{4}\tilde{A}^2}. \qquad (7)$$

The solution of (7) is

$$\tilde{A}=\frac{4C\exp(-Y/\kappa)}{1+C^2\exp(-2Y/\kappa)},$$
$$\left.\frac{\partial\tilde{A}}{\partial Y}\right|_{Y=0}=-\frac{4C}{\kappa}\frac{1-C^2}{(1+C^2)^2}, \qquad (8)$$

where C is an arbitrary positive constant. The function

$$\tilde{A}_1=\frac{\partial}{\partial C}\tilde{A}=\frac{4\exp(-Y/\kappa)[1-C^2\exp(-2Y/\kappa)]}{[1+C^2\exp(-2Y/\kappa)]^2} \qquad (9)$$

is a solution of system (1) linearized near $\tilde{A}$ with Δ given by Eqs. (4) and (8). The function $\tilde{A}$ is the eigenfunction of this system of equations with zero eigenvalue if the boundary condition

$$\left.\frac{\partial\tilde{A}_1}{\partial Y}\right|_{Y=0}=0 \qquad (10)$$

is satisfied. Using Eqs. (8) and (10), we obtain the critical value of the coefficient C,

$$C^2=3-2\sqrt{2}. \qquad (11)$$

Therefore, the Meissner state becomes absolutely unstable at the magnetic field value

$$H_s=H_{cm}. \qquad (12)$$

This linear instability leads to the formation of vortices. Hence, there exists some stage of the transformation from the linear instability to the formation of vortices. This stage cannot be studied in the framework of the Ginzburg–Landau equations. The energy δE of the vortex–antivortex pair at distances $2a$ such that $2a\ll\xi$ decreases very slowly with a, only as $1/\ln(\xi/2a)$ [8]. This energy was found as the minimum of the Ginzburg–Landau free energy for fixed positions of zeros of Δ and fixed vorticities (±1) [8]. From this point of view, a single vortex enters the superconductor without a threshold only if the order parameter Δ

is equal to zero at the boundary. Using Eqs. (4), (8), and (11), it is easy to prove that the condition

$$\Delta|_{y=0} = 0$$

is satisfied in the case under consideration. Our conjecture is that this condition is the boundary condition for the problem of calculating the critical field H_s for $\kappa \gg 1$. It is satisfied in all the cases considered in what follows. We note that the critical field H_s is separated from the critical fields H_{c1} and H_{c2} by the large parameter κ. The critical field H_{c2} was introduced in [1] as

$$H_{c2} = \frac{4}{\pi e D}(T_c - T). \tag{13}$$

Therefore,

$$\frac{H_{c2}}{H_s} = \frac{H_{c2}}{H_{cm}} = \kappa. \tag{14}$$

The critical field H_{c1} was found in [2] as

$$\frac{H_{c1}}{H_{cm}} = \frac{\ln\kappa + 0.146}{\kappa}. \tag{15}$$

3. THE SUPERHEATING FIELD H_{sh} FOR $\kappa \ll 1$

The superheating field H_{sh} was found by Ginzburg [4]. In the range $\kappa \ll 1$, this problem can be solved analytically. In the superconductor with a small value of κ, the magnetic field is screened near the surface at the distances much smaller than the correlation length ξ. In the leading approximation, we then obtain

$$A(y) = A_0 \exp\left(-\sqrt{\frac{4\pi^2 e^2 \nu D}{T}\Delta^2(0)}\; y\right), \tag{16}$$

where $\Delta(0)$ is the order parameter on the surface of the superconductor. With the help of Eq. (16), we reduce the system of equations (1) to one equation for the order parameter Δ and to the effective boundary condition for this equation,

$$\left(-1 - 2\frac{\partial^2}{\partial \tilde{Y}^2} + \tilde{\Delta}^2\right)\tilde{\Delta} = 0, \qquad \tilde{\Delta}^2 \frac{\partial\tilde{\Delta}}{\partial Y}\bigg|_{Y=0} = \frac{\tilde{H}^2\kappa}{8}, \tag{17}$$

where

$$\Delta = \left(\frac{8\pi^2 T^2 \tau}{7\zeta(3)}\right)^{1/2}\tilde{\Delta}$$

and the quantities ξ, κ, Y, and $\tilde{H}$ are defined in Eqs. (6). Equation (17) has the solution

$$\tilde{\Delta} = \frac{C\exp Y - 1}{1 + C\exp Y}, \quad \frac{\partial\tilde{\Delta}}{\partial Y}\bigg|_{Y=0} = \frac{2C}{(1+C)^2}, \tag{18}$$

where $C > 1$ is an arbitrary parameter related to the external magnetic field by Eq. (17). The function

$$\tilde{\Delta}_1 = \frac{\partial\tilde{\Delta}}{\partial C} = \frac{2\exp Y}{(1 + C\exp Y)^2} \tag{19}$$

is the solution of system (1) linearized near the function $\tilde{\Delta}$ given by Eq. (18).

This function becomes the eigenfunction of this system of equations with zero eigenvalue if the boundary condition

$$\left(\frac{\partial\tilde{\Delta}_1}{\partial Y} + 2\frac{\tilde{\Delta}_1}{\tilde{\Delta}}\frac{\partial\tilde{\Delta}}{\partial Y}\right)\bigg|_{Y=0} = 0 \tag{20}$$

is satisfied. With the help of Eqs. (18), (19), and (20), we obtain the critical value of the coefficient C as

$$C = 3 + 2\sqrt{2}. \tag{21}$$

The superheating field H_{sh} is therefore related to the field H_{cm} by the simple equation

$$\frac{H_{sh}}{H_{cm}} = \frac{1}{\sqrt{\kappa}} = \frac{2^{-1/4}}{\sqrt{\kappa_{GL}}}. \tag{22}$$

4. CRITICAL FIELD H_s: THE GENERAL ANALYSIS

The general strategy of calculating H_s is as follows. First, the expression for the current density j must be found as a function of the vector potential A and the order parameter Δ,

$$j = Q(A, \Delta)A. \tag{23}$$

Equation (23) is formal and the relation can be nonlocal. In Eq. (23), the order parameter Δ must be considered as a functional of the vector potential A. Next, we solve the Maxwell equation for A,

$$\frac{\partial^2 A}{\partial y^2} + 4\pi Q(A,\Delta)A = 0, \qquad -\frac{\partial A}{\partial y} = H, \quad H(0) = H_{ext}, \tag{24}$$

where H_{ext} is the external magnetic field. Solutions of Eq. (24) form a one-parameter family. The value of $A(y)$ at the point $y = 0$ can be considered as this parameter, and hence, the function

$$A_1 = \frac{\partial A}{\partial A(0)} \tag{25}$$

is a solution of linearized equation (24). It is an eigenfunction of this equation if the boundary condition

$$\left.\frac{\partial A_1}{\partial y}\right|_{y=0} = \left.\frac{\partial^2 A}{\partial A(0)\partial y}\right|_{y=0} = 0 \tag{26}$$

is satisfied. Equation (26) determines the critical field H_s (if $\kappa > 1$). We apply this strategy in the extreme case where $\kappa \gg 1$ and in the case where the superconductor can be considered as a «dirty» material. The system of equations for the Green's functions α and β can then be taken in the form [9]

$$\begin{gathered}\alpha\Delta - \beta\omega + \frac{D}{2}\left(\alpha\partial_-^2\beta - \beta\frac{\partial^2\alpha}{\partial r^2}\right) = \alpha\beta\Gamma,\\ \Delta = 2\pi T|\lambda|\sum_{\omega>0}\beta(\omega), \quad \alpha^2 + |\beta|^2 = 1,\\ j = -ie\nu D2\pi T\sum_{\omega>0}(\beta^*\partial_-\beta - \beta\partial_+\beta^*).\end{gathered} \tag{27}$$

For $\kappa \gg 1$, the relation of the vector potential to the order parameter Δ is local in the leading approximation and α and β can be taken as real functions. In this case, the system of equations (27) can be essentially simplified to

$$\begin{gathered}\Delta\,\mathrm{tg}\,\varphi - \omega = \sin\varphi(\Gamma + 2e^2DA^2),\\ \Delta = 2\pi T|\lambda|\sum_{\omega>0}\cos\varphi,\\ j = -e^2\nu D8\pi T\sum_{\omega>0}A\cos^2\varphi, \quad -\frac{\partial^2 A}{\partial y^2} = 4\pi j,\\ \alpha = \sin\varphi, \quad \beta = \cos\varphi,\end{gathered} \tag{28}$$

where $\Gamma = \tau_s^{-1}$, with τ_s being the electron spin flip scattering time.

In what follows, we restrict ourselves to the zero-temperature limit. There are two regions

$$\begin{gathered}\Delta > \Gamma + 2e^2DA,\\ \Delta < \Gamma + 2e^2DA.\end{gathered} \tag{29}$$

In the first region, we have

$$\begin{gathered}A = \sqrt{\frac{2}{\pi e^2 D}}\left(\Delta\ln\left(\frac{\pi T_c^0}{\gamma\Delta}\right) - \frac{\pi\Gamma}{4}\right)^{1/2},\\ \frac{\partial}{\partial y}\left(\Delta\ln\left(\frac{\pi T_c^0}{\gamma\Delta}\right) - \frac{\pi\Gamma}{4}\right)^{1/2} = -\Phi(\Delta),\end{gathered} \tag{30}$$

where γ is the Euler constant, $\ln\gamma = 0.577216\ldots$, and T_c^0 is the transition temperature for the superconductor without paramagnetic impurities,

$$\begin{aligned}\Phi(\Delta) = (4\pi^2e^2\nu D)^{1/2}\times\\ \times\Bigg\{\frac{\Delta_\infty}{2}\left(\Delta_\infty - \frac{\pi\Gamma}{2}\right) + \Delta^2\left(\ln\left(\frac{\pi T_c^0}{\gamma\Delta}\right) - \frac{1}{2}\right) -\\ - \frac{8}{3\pi}\left[\Gamma\left(\Delta\ln\left(\frac{\pi T_c^0}{\gamma\Delta}\right) - \frac{\pi\Gamma}{4}\right) +\right.\\ \left. + \frac{2}{\pi}\left(\Delta\ln\left(\frac{\pi T_c^0}{\gamma\Delta}\right) - \frac{\pi\Gamma}{\Delta}\right)^2\right]\Bigg\}^{1/2},\end{aligned} \tag{31}$$

where Δ_∞ is the value of the order parameter as $y \to \infty$.

We define the quantity Y as the point where

$$\Delta(Y) = \Gamma + 2e^2DA^2(Y). \tag{32}$$

At this point, we have

$$\Delta(Y) = \frac{\pi T_c^0}{\gamma}\exp\left(-\frac{\pi}{4}\right). \tag{33}$$

In the region $y > Y$, the order parameter Δ is the solution of the equation

$$\int_{\Delta(Y)}^{\Delta}\frac{d\left(\Delta\ln\frac{\pi T_c^0}{\gamma\Delta} - \frac{\pi\Gamma}{4}\right)^{1/2}}{\Phi(\Delta)} = -(y - Y). \tag{34}$$

We now consider the region $y < Y$. In this region, the system of equations (30) for the order parameter Δ and the vector potential A is more complicated,

$$\begin{aligned}\Delta\ln\frac{\pi T_c^0}{\gamma\left((\Gamma+2e^2DA^2)+\left((\Gamma+2e^2DA^2)^2-\Delta^2\right)^{1/2}\right)} =\\ = \frac{\Gamma + 2e^2DA^2}{2}\Bigg\{\arcsin\left(\frac{\Delta}{\Gamma + 2e^2DA^2}\right) -\\ - \frac{\Delta}{\Gamma + 2e^2DA^2}\left(1 - \left(\frac{\Delta}{\Gamma + 2e^2DA^2}\right)^2\right)^{1/2}\Bigg\},\\ -\frac{\partial^2 A}{\partial y^2} + 16\pi e^2\nu DA\Bigg\{\Delta\arcsin\left(\frac{\Delta}{\Gamma+2e^2DA^2}\right) -\\ -(\Gamma+2e^2DA^2)\left[\frac{2}{3} - \left(\left(1 - \left(\frac{\Delta}{\Gamma+2e^2DA^2}\right)^2\right)^{1/2} -\right.\right.\\ \left.\left. - \frac{1}{3}\left(1 - \left(\frac{\Delta}{\Gamma + 2e^2DA^2}\right)^2\right)^{3/2}\right)\right]\Bigg\} = 0.\end{aligned} \tag{35}$$

For $y > Y$, we have obtained the first integral of the equation for A (Eq. (34)). We now show how to obtain the first integral in the region $y < Y$. To simplify the calculations, we set $\Gamma = 0$ in what follows. We then obtain

$$\frac{H_{cm}^2}{8\pi} = \frac{\nu\Delta_\infty^2}{2}, \quad \Delta_\infty = \frac{\pi T_c^0}{\gamma}, \qquad \lambda_L^{-2} = 8\pi^2 e^2 \nu D \Delta_\infty. \tag{36}$$

Equations (36) allow us to pass to the dimensionless variables

$$\Delta = \Delta_\infty \tilde{\Delta}, \quad y = \lambda_L \tilde{y}, \quad Y = \lambda_L \tilde{Y}, \qquad H = H_{cm}\tilde{H}, \quad A = H_{cm}\lambda_L \tilde{A}. \tag{37}$$

Equations (35) and (37) then imply

$$\ln\left(\frac{\pi}{\tilde{A}^2 + (\tilde{A}^4 - \pi^2\tilde{\Delta}^2)^{1/2}}\right) = \frac{\tilde{A}^2}{2\pi\tilde{\Delta}} \times \left\{\arcsin\left(\frac{\pi\tilde{\Delta}}{\tilde{A}^2}\right) - \frac{\pi\tilde{\Delta}}{\tilde{A}^2}\left(1 - \left(\frac{\pi\tilde{\Delta}}{\tilde{A}^2}\right)^2\right)^{1/2}\right\},$$
$$-\frac{\partial^2\tilde{A}}{\partial\tilde{y}^2} + \frac{2}{\pi}\tilde{A}\tilde{\Delta}\left\{\arcsin\left(\frac{\pi\tilde{\Delta}}{\tilde{A}^2}\right) - \frac{\tilde{A}^2}{\pi\tilde{\Delta}}\left[\frac{2}{3} - \left(\left(1 - \left(\frac{\pi\tilde{\Delta}}{\tilde{A}^2}\right)^2\right)^{1/2} - \frac{1}{3}\left(1 - \left(\frac{\pi\tilde{\Delta}}{\tilde{A}^2}\right)^2\right)^{3/2}\right)\right]\right\} = 0. \tag{38}$$

The problem of calculating the critical field H_s is thus reduced to solving system (38) on the interval $\{0, \tilde{Y}\}$ with the boundary conditions

$$\tilde{\Delta}(\tilde{Y}) = \exp(-\pi/4), \quad \tilde{A}(\tilde{Y}) = \sqrt{\pi}\exp(-\pi/8), \qquad \left.\frac{\partial\tilde{A}}{\partial\tilde{y}}\right|_{\tilde{y}=\tilde{Y}} = -\left\{1 + \left(\frac{\pi}{2} - \frac{5}{3}\right)\exp\left(-\frac{\pi}{2}\right)\right\}^{1/2}. \tag{39}$$

We must find the point $\tilde{Y}_{cr}$ such that

$$\left.\frac{\partial^2\tilde{A}}{\partial\tilde{Y}\partial\tilde{y}}\right|_{\tilde{y}=0,\tilde{Y}=\tilde{Y}_{cr}} = 0. \tag{40}$$

The value of the derivative $\partial\tilde{A}/\partial\tilde{y}$ at the point $\tilde{y} = 0$ gives the field H_s via

$$\frac{H_s}{H_{cm}} = -\left.\frac{\partial\tilde{A}}{\partial\tilde{y}}\right|_{\tilde{y}=0,\tilde{Y}=\tilde{Y}_{cr}}. \tag{41}$$

To solve this problem, we set

$$Z = \frac{\pi\tilde{\Delta}}{\tilde{A}^2}. \tag{42}$$

The system of equations (38) then implies

$$\tilde{A}^2 = \frac{\pi}{1 + \sqrt{1 - Z^2}} \times \exp\left[-\frac{1}{2Z}\left(\arcsin Z - Z\sqrt{1 - Z^2}\right)\right],$$
$$\frac{\partial\tilde{A}}{\partial\tilde{y}} = -\left[1 + \left(\frac{\pi}{2} - \frac{5}{3}\right)\exp\left(-\frac{\pi}{2}\right) + \Phi_1(Z)\right]^{1/2}, \tag{43}$$

where

$$\Phi_1(Z) = 2\int_Z^1 dZ R(Z), \tag{44}$$

$$R(Z) = \frac{Z}{(1 + \sqrt{1 - Z^2})^2} \times \exp\left[-\frac{1}{Z}\left(\arcsin Z - Z\sqrt{1 - Z^2}\right)\right] \times \left[\frac{Z}{1 + \sqrt{1 - Z^2}} - \frac{1}{2Z^2}\left(\arcsin Z - Z\sqrt{1 - Z^2}\right)\right] \times \left\{\arcsin Z - \frac{1}{Z}\left[\frac{2}{3} - \left((1 - Z^2)^{1/2} - \frac{1}{3}(1 - Z^2)^{3/2}\right)\right]\right\}.$$

The second equation in (43) is in fact an equation for Z,

$$\frac{\partial Z}{\partial\tilde{y}} = \frac{2}{\sqrt{\pi}}(1 + \sqrt{1 - Z^2})^{1/2} \times \exp\left[\frac{1}{4Z}\left(\arcsin Z - Z\sqrt{1 - Z^2}\right)\right] \times \left[1 + \left(\frac{\pi}{2} - \frac{5}{3}\right)\exp\left(-\frac{\pi}{2}\right) + \Phi_1(Z)\right]^{1/2} \times \left\{\frac{Z}{1 + \sqrt{1 - Z^2}} - \frac{1}{2Z^2}\left(\arcsin Z - Z\sqrt{1 - Z^2}\right)\right\}^{-1}. \tag{45}$$

Condition (40) implies that

$$\left.\frac{\partial Z}{\partial\tilde{Y}}\right|_{\tilde{y}=0} = 0 \quad \text{or} \quad \left.\frac{\partial\Phi_1}{\partial Z}\right|_{\tilde{y}=0} = 0. \tag{46}$$

The function Z is a monotonic function of $\tilde{y}$, and hence, the first equation in (46) cannot be satisfied. It follows from Eq. (44) and the second equation in (46) that

$$Z = 0 \tag{47}$$

at the critical point, and therefore,

$$\frac{H_s}{H_{cm}} = \left\{1 + \left(\frac{\pi}{2} - \frac{5}{3}\right)\exp\left(-\frac{\pi}{2}\right) + \Phi_1(0)\right\}^{1/2} = 1. \tag{48}$$

We note that at $T = 0$, the order parameter behaves as

$$\Delta(y) \propto y^{1/2} \tag{49}$$

for small values of y.

5. THE CRITICAL FIELD H_s IN A SUPERCONDUCTOR WITH MAGNETIC IMPURITIES

The system of equations (35) can be solved for an arbitrary value of the spin flip scattering time $\Gamma = \tau_s^{-1}$. To do this, we put

$$Z = \frac{\Delta}{\Gamma + 2e^2DA^2}. \tag{50}$$

From the first equation in (35), we then obtain

$$\Gamma + 2e^2DA^2 = \frac{\pi T_c^0}{\gamma}\frac{1}{1+\sqrt{1-Z^2}} \times \exp\left[-\frac{1}{2Z}\left(\arcsin Z - Z\sqrt{1-Z^2}\right)\right]. \tag{51}$$

With the help of Eq. (51), we obtain the first integral of the second equation in (35),

$$\frac{\partial A}{\partial y} = -\left[B + \tilde{\Phi}_1(Z)\right]^{1/2}, \tag{52}$$

where

$$\tilde{\Phi}_1(Z) = 8\pi\nu\left(\frac{\pi T_c^0}{\gamma}\right)^2 \int\limits_Z^b dZ\, R(Z), \tag{53}$$

$$B = \begin{cases} 8\pi\nu\left\{\Delta_\infty\left(\dfrac{\Delta_\infty}{2} - \dfrac{\pi\Gamma}{4}\right) + \left(\dfrac{\pi T_c^0}{\gamma}\right)^2 \times \left(\dfrac{\pi}{4} - \dfrac{5}{6}\right)\exp\left(-\dfrac{\pi}{2}\right) + \dfrac{\Gamma^2}{3}\right\} & \text{if } \Delta_\infty > \Gamma, \\ 0 & \text{if } \Delta_\infty < \Gamma. \end{cases}$$

The upper integration limit in Eq. (53) is defined as

$$b = \begin{cases} 1 & \text{if} \quad \Delta_\infty > \Gamma, \\ \dfrac{\Delta_\infty}{\Gamma} & \text{if} \quad \Delta_\infty < \Gamma. \end{cases} \tag{54}$$

In Eqs. (53) and (54), the quantity Δ_∞ is the value of the order parameter in the Meissner state at large distances from the surface. The value of Δ_∞ can be found from Eqs. (30) and (35) as

$$\begin{aligned} &\Delta_\infty \ln\left(\frac{\pi T_c^0}{\gamma\Delta_\infty}\right) = \frac{\pi\Gamma}{4} \quad \text{if} \quad \Delta_\infty > \Gamma, \\ &\Delta_\infty \ln\left(\frac{\pi T_c^0}{\gamma(\Gamma + \sqrt{\Gamma^2 - \Delta_\infty^2})}\right) = \frac{\Gamma}{2}\left\{\arcsin\left(\frac{\Delta_\infty}{\Gamma}\right) - \frac{\Delta_\infty}{\Gamma}\sqrt{1 - \left(\frac{\Delta_\infty}{\Gamma}\right)^2}\right\} \quad \text{if} \quad \Delta_\infty < \Gamma. \end{aligned} \tag{55}$$

The quantity B was found with the help of Eqs. (30), which were used to obtain the boundary conditions for the vector potential A at the point y such that $Z = 1$. It is easy to obtain that

$$\begin{aligned} &\Gamma + 2e^2DA^2\Big|_{Z=1} = \frac{\pi T_c^0}{\gamma}\exp\left(-\frac{\pi}{4}\right), \\ &\Delta\Big|_{Z=1} = \frac{\pi T_c^0}{\gamma}\exp\left(-\frac{\pi}{4}\right), \\ &\frac{\partial A}{\partial y}\Big|_{Z=1} = -\sqrt{8\pi\nu}\left\{\Delta_\infty\left(\frac{\Delta_\infty}{2} - \frac{\pi\Gamma}{4}\right) + \left(\frac{\pi T_c^0}{\gamma}\right)^2\left(\frac{\pi}{4} - \frac{5}{6}\right)\exp\left(-\frac{\pi}{2}\right) + \frac{\Gamma^2}{3}\right\}^{1/2}. \end{aligned} \tag{56}$$

Using boundary condition (26) in Eq. (52), we immediately obtain the critical field H_s as

$$H_s = \left(B + \tilde{\Phi}_1(0)\right)^{1/2}. \tag{57}$$

At the critical point H_s, the condition $\Delta\Big|_{y=0} = 0$ is satisfied for all values of Γ. Equation (57) can be checked in the simplest case where $\Gamma \gg \Delta_\infty$. We then have

$$\begin{aligned} &A = \left(\frac{\pi T_c^0}{48e^2D\gamma}\right)^{1/2}\left(\left(\frac{\Delta_\infty}{\Gamma}\right)^2 - Z^2\right)^{1/2}, \\ &\tilde{\Phi}_1(Z) = \frac{\pi\nu}{12}\left(\frac{\pi T_c^0}{\gamma}\right)^2\left(\left(\frac{\Delta_\infty}{\Gamma}\right)^4 - Z^4\right). \end{aligned} \tag{58}$$

The solution of Eq. (52) is given by

$$A = \frac{\Delta_\infty}{\Gamma}\left(\frac{\pi T_c^0}{6e^2 D\gamma}\right)^{1/2} \frac{C\exp(y/\tilde{\lambda})}{C^2+\exp(2y/\tilde{\lambda})}, \tag{59}$$

where $0 < C < 1$ is an arbitrary constant and the penetration depth $\tilde{\lambda}$ is given by

$$\tilde{\lambda}^{-1} = (8\pi\nu e^2 D)^{1/2}\left(\frac{\pi T_c^0}{\gamma}\right)^{1/2}\frac{\Delta_\infty}{\Gamma}. \tag{60}$$

Using Eq. (59), we find the magnetic field value at the point $y = 0$ as

$$H(0) = \frac{C(1-C^2)}{(1+C^2)^2}\frac{\Delta_\infty}{\Gamma\tilde{\lambda}}\left(\frac{\pi T_c^0}{6e^2 D\gamma}\right)^{1/2}. \tag{61}$$

The condition $\partial H(0)/\partial C = 0$ determines the value of the critical field H_s. With the help of Eq. (61), we obtain

$$C^2 = 3 - 2\sqrt{2},$$
$$H_s^2 = \left(\frac{\Delta_\infty}{4\Gamma\tilde{\lambda}}\right)^2 \frac{\pi T_c^0}{6e^2 D\gamma} = \frac{\pi\nu}{3}\frac{\Delta_\infty^4}{\Gamma^2}. \tag{62}$$

This value of H_s corresponds to the point $Z = 0$ and coincides with the value given by Eqs. (57) and (58). To complete the calculation, we give the equation for the thermodynamic field H_{cm} [10],

$$-\delta\mathcal{F} = \nu\int_{2\gamma\Gamma/\pi}^{T_c^0}\frac{dT_c}{T_c}|\Delta|^2 = \frac{H_{cm}^2}{8\pi}, \tag{63}$$

where $\Delta = \Delta_\infty$ can be found from Eq. (55) with the replacement $T_c^0 \to T_c$.

In the extreme case where $\Gamma \gg \Delta_\infty$, we find from Eqs. (55) and (63) that

$$\frac{dT_c}{T_c} = \frac{1}{12}d\left(\frac{\Delta}{\Gamma}\right)^2, \quad \delta\mathcal{F} = -\frac{\nu}{24}\frac{\Delta_\infty^2}{\Gamma^2}. \tag{64}$$

As before, we obtain

$$\frac{H_s}{H_{cm}} = 1, \quad \Gamma \gg \Delta_\infty. \tag{65}$$

We now prove that for an arbitrary value of the depairing fuctor Γ, we have the relation

$$H_s = H_{cm}. \tag{66}$$

Using Eqs. (55), (57), and (63), we verify that in both regions $\Delta_\infty > \Gamma$ and $\Delta_\infty < \Gamma$, the derivatives of H_s^2 and H_{cm}^2 with respect to T_c^0 coincide,

$$\frac{\partial H_s^2}{\partial T_c^0} = \frac{\partial H_{cm}^2}{\partial T_c^0}. \tag{67}$$

This proves Eq. (66). Equation (57) allows us to take the integral over T_c in expression (63) for the free energy and to obtain the critical field H_{cm} in the explicit form

$$\frac{H_{cm}^2}{4\pi\nu} = \Delta_\infty^2 - \frac{\pi\Delta_\infty\Gamma}{2} + \frac{2\Gamma^2}{3} \quad \text{if } \Delta_\infty > \Gamma, \tag{68a}$$

and

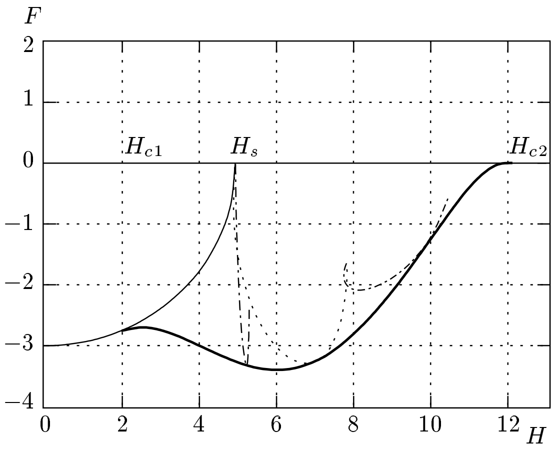
$$\frac{H_{cm}^2}{4\pi\nu} = \left[\Delta_\infty^2 - \left(\frac{\pi T_c^0}{\gamma}\right)^2 R(Z)\left(\frac{Z}{1+\sqrt{1-Z^2}} - \frac{1}{2Z^2}\left(\arcsin Z - Z\sqrt{1-Z^2}\right)\right)^{-1}\right]_{Z=\Delta_\infty/\Gamma} \quad \text{if } \Delta_\infty < \Gamma, \tag{68b}$$

where Δ_∞ and T_c^0 are related by Eq. (55).

6. CONCLUSION

We have found the general method of calculating the critical field H_s in the entire range of κ values and given the results for H_s in the temperature region near T_c and $T = 0$. For an arbitrary value of the depairing factor Γ, the quantities H_s and H_{cm} are equal in both temperature regions. The initial definition of H_s is the value of the external magnetic field at which vortices can penetrate into the superconductor without a threshold. In superconductors of the second kind, the value of this field coincides with the critical field value of the absolute instability of the Meissner state. In a superconductor of the first kind, the field of the absolute instability of the Meissner state is the overheating field. If the order parameter Δ is nonzero at the boundary of the superconductor, the energy of a vortex–antivortex pair (at least in the $\kappa \gg 1$ limit) decreases very slowly with the distance $2a$ between them in the range $2a \ll \xi$ [8]. As the result, the order parameter Δ is zero on the boundary of the superconductor at the point H_s. The point H_s is an essentially singular point because an infinite number of states with different numbers of vortices in the sample go out of this point (the number of states is of the order SH_s/Φ_0, where S is the area of the sample and $\Phi_0 = \pi/e$ is the flux quantum). The free energy of the Shubnikov phase is the envelope curve for all these states.

By the Shubnikov phase, we mean the state with the minimum value of the free energy in a given external magnetic field. The disappearance of the threshold in the Meissner state at the point H_s does not mean that vortices can freely enter the superconductor in the Shubnikov phase as the external magnetic field



Energy of the Subnikov phase (solid line); energy of the Meissner (thin line); energy of states with different values of the vortex density (dashed lines)

changes. In the figure, we present the free energy as a function of the magnetic field for the Shubnikov phase (solid line), the Meissner state energy (thin line), and the energy of states with different densities of vortices in the superconductor (dashed line).

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