Fundamental quantization procedure for gravitation: Dynamic method

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A new, mathematically correct quantization procedure for gravitation theory is derived on the basis of dynamic quantization. A classical canonical formalism in tetrad-connection variables is developed within the procedure in the four-dimensional theory of pure gravitation, and regularized quantized fields corresponding to classical tetrad and connection fields are constructed. It is shown that the regularized fields satisfy generally covariant equations of motion, and an iterative procedure for solving these equations is proposed. [S1063-7761(96)00211-9]

1. INTRODUCTION

A mathematically correct quantization procedure for the theory of gravitation in four-dimensional space-time which was derived on the basis of dynamic quantization is presented in this paper. The dynamic quantization method that we developed has been successfully applied to the theory of gravitation interacting with a Dirac field in $(2+1)$-dimensional space. Regularization, which conserves the general covariance of the theory, has been performed, and a perturbation theory has been devised.[1,2] The reasoning behind the dynamic quantization method is Dirac's theory for the canonical quantization of systems with constraints, particularly generally covariant systems.

Let us briefly describe the essence of dynamic quantization. As we know,[3] in generally covariant theories the Hamiltonian is an arbitrary linear combination of the constraints χ_α of the first kind. If $|\mathcal{M}\rangle$ is some physical state, then $\chi_\alpha|\mathcal{M}\rangle=0$. Let $|\mathcal{N}\rangle=a^+|\mathcal{M}\rangle$ be another physical state, where a^+ is a creation operator. Since the Hamiltonian annuls all physical states, $[\chi_\alpha,a^+]=0$. We assume that the theory has an infinite number of creation (a_N^+) and annihilation (a_N) operators that convert some states into others and exhaust all the local physical degrees of freedom in the system. All the operators $\{a_N,a_N^+\}$ are integrals of motion, since they commute with the Hamiltonian. Hence it follows that any set of operator pairs $\{a_N,a_N^+\}''$ can be regarded as a set of second-class constraints in the Dirac sense.[3] This fact creates the possibility, in principle, of performing regularization in the theory under consideration. This technique for performing regularization is the basis of the dynamic quantization method. The system is regularized by imposing an infinite series of second-class constraints

$$a_N=0, \quad a_N^+=0, \quad |N|>N_0. \tag{1.1}$$

Thus, only a finite number of degrees of freedom corresponding to the operators a_N^+ and a_N with $|N|<N_0$ remain in the theory. The finite set of remaining operators $\{a_N,a_N^+\}'$ corresponds to a set of physical states that adequately describe the system under study. As a result, each Poisson bracket is replaced by a corresponding Dirac bracket. It is critically important that under such regularization the equations of motion conserve their classical form (see Sec. 4), which signifies conservation of the general covariance in the regularized theory. This fact plays a significant role, since the equations of motion underlie all the calculations in the regularized theory in the dynamic quantization method. In addition, by leaving only a "small" number of physical degrees of freedom and states in the theory, we obtain a fundamentally new possibility for developing a perturbation scheme with respect to the number of degrees of freedom.

Without attempting to perform a complete survey of the other approaches to the canonical quantization of gravitation theory, let us focus our attention on the results of one of the most rapidly developing ones, which can be represented by Ref. 4 (see also the references therein). The possibility, in principle, of correct nonperturbative quantization has been established within the technique for quantizing gravitation developed in Ref. 4. The physical states of the theory are subject to explicit description and form a normed space. A theory of linear operators, which include first-class constraints or the Hamiltonian, has been devised in this space. The problem of constructing physical states that annul the Hamiltonian has been solved. In this case the commutation relations between the constraints do not contain undesirable Schwinger terms. This filled us with enthusiasm, since, in our opinion, the correctness of our method, which leads to the same general results, has been indirectly confirmed.

The dynamic quantization method was devised using the principles of canonical quantum theory. Therefore, the machinery for classical Hamiltonian mechanics must first be developed in the theory under consideration to apply it. This problem is solved in Sec. 2. Section 3 presents the formal derivation of the quantum theory. Since formal quantization is mathematically incorrect in gravitation theory, the presentation in this section has an heuristic character. A logically and mathematically correct systematic quantum theory of gravitation is devised in Sec. 4. Section 5 proposes an iterative procedure for solving the Heisenberg equations that corresponds to the dynamic method.

2. CANONICAL FORMALISM

Let us consider the action of four-dimensional pure gravitation in the Palatini form:

$$A=-\frac{1}{8\kappa^2}\int d^4x\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{abcd}e^c_\lambda e^d_\rho R^{ab}_{\mu\nu}=\int dx^0\mathscr{L}, \quad (2.1)$$

$$R^{ab}_{\mu\nu}=\partial_\mu\omega^{ab}_\nu-\partial_\nu\omega^{ab}_\mu+\omega^a_{\mu g}\omega^{gb}_\nu-\omega^a_{\nu g}\omega^{gb}_\mu.$$

Here the e^a_μ are tetrads, so that $g_{\mu\nu}=\eta_{ab}e^a_\mu e^b_\nu$ is a metric tensor in the local coordinates $x^\mu=(x^0,x^i)$; $\mu,\nu,\ldots=0,1,2,3$ are the coordinate indices; $a,b,c,\ldots=0,1,2,3$ are the local Lorentzian indices; $\eta_{ab}=\mathrm{diag}(1,-1,-1,-1)$ is the Lorentzian metric; and $\omega^{ab}_\mu=-\omega^{ba}_\mu$ is a connection in the orthogonal reference frame e^a_μ so that the covariant derivative of the vector $\xi^a=e^a_\mu\xi^\mu$ has the form

$$\nabla_\mu\xi^a=\partial_\mu\xi^a+\omega^{ab}_\mu\xi_b.$$

Furthermore, $\varepsilon^{\mu\nu\lambda\rho}(\varepsilon^{0123}=1)$ and $\varepsilon_{abcd}(\varepsilon_{0123}=1)$ are totally antisymmetric unit pseudotensors belonging to the coordinate and Lorentz reference frames, respectively.

Let us proceed to devising a canonical formalism for the Palatini action (2.1) in a form that is convenient for our purposes. This problem has been treated repeatedly in the canonically conjugate variables ω^{ab}_i and

$$\mathscr{P}^i_{\ ab}\equiv\frac{\partial\mathscr{L}}{\partial\dot{\omega}^{ab}_i}=-(2\kappa^2)^{-1}\varepsilon_{ijk}\varepsilon_{abcd}e^c_je^d_k \quad (2.2)$$

(see, for example, Ref. 5). The dot indicates the derivative with respect to the time $t=x^0$. However, because the system (2.1) contains second-class constraints, the Poisson brackets in the variables $\{\omega^{ab}_i,\mathscr{P}^i_{\ ab}\}$ are very complicated. In addition, the equations of motion and the constraints are quite cumbersome in these variables. On the other hand, both the equations of motion and the constraints appear exceptionally simple in the variables $\{\omega^{ab}_i,e^c_j\}$. This circumstance is extremely important to us, since the equations of motion play a major role for calculations in the dynamic quantization method. Another reason for preferring the tetrad-connection variables is that the property of supersymmetry is formulated in just these variables in the theory of supergravitation. Therefore, it seems more efficient to us to develop a Hamiltonian formalism in the variables $\{\omega^{ab}_i,e^c_j\}$, which have a direct physical meaning.

By definition,

$$\mathscr{L}=\frac{\partial\mathscr{L}}{\partial\dot{q}}\dot{q}-H,$$

where q denotes the generalized coordinates. In our case the Hamiltonian has the form

$$H=\int d^3x\left\{-\frac{1}{2}\omega^{ab}_0\chi_{ab}+\frac{1}{2\kappa^2}e^c_0\phi_c\right\}, \quad (2.3)$$

$$\chi_{ab}=\frac{l}{\kappa^2}\varepsilon_{abcd}\varepsilon_{ijk}e^c_i\nabla_je^d_k, \quad \phi_c=\frac{1}{2}\varepsilon_{abcd}\varepsilon_{ijk}e^d_kR^{ab}_{ij}.$$

Here $\nabla_\mu e^c_\nu=\partial_\mu e^c_\nu+\omega^{cb}_\mu e_{b\nu}$, and ω^{ab}_0 and e^c_0 are arbitrary functions, which play the role of Lagrange multipliers. We represent the action in the form

$$A=-\int d^4x\frac{1}{4\kappa^2}\varepsilon_{abcd}\varepsilon_{ijk}e^c_je^d_k\dot{\omega}^{ab}_j-\int dt\ H. \quad (2.4)$$

From the condition $\delta A=0$ we find two equations for $\dot{\omega}^{ab}_i$ and $\dot{e}^c_j$:

$$\frac{1}{2\kappa^2}\varepsilon_{abcd}\varepsilon_{ijk}e^d_k\dot{\omega}^{ab}_i+\frac{\partial H}{\partial e^c_j}=0, \quad (2.5)$$

$$\frac{1}{2\kappa^2}\varepsilon_{abcd}\varepsilon_{ijk}\dot{e}^c_je^d_k-\frac{\partial H}{\partial\omega^{ab}_i}=0. \quad (2.6)$$

Equation (2.6) has a solution only under the additional condition

$$\lambda^{ij}=(g^{ik}\varepsilon_{jlm}+g^{jk}\varepsilon_{ilm})e_{ak}\nabla_le^a_m=0, \quad (2.7)$$

where $g^{ik}g_{kj}=\delta^i_j$. Equation (2.7) should be regarded as a second-class constraint. This is seen from the fact that the six equations (2.7) reduce the number of independent variables $\omega^{ab}_i(x)$ at each point x from 18 to 12. The number of independent variables $e^a_i(x)$ is also equal to 12. From (2.6) under the condition (2.7) we find

$$\nabla_0e^a_i=\nabla_ie^a_0. \quad (2.8)$$

In addition, the constraints $\chi_{ab}\approx0$ and Eq. (2.7) give

$$\nabla_ie^a_j-\nabla_je^a_i=0(\mathrm{mod}\ \chi_{ab}). \quad (2.9)$$

As we know, a connection can be uniquely expressed on the basis of Eqs. (2.8) and (2.9) in terms of tetrads and their derivatives. Equation (2.5) defines $\dot{\omega}^{ab}_i$ to within the term $\varepsilon_{jkl}e^a_ke^b_ls_{ij}$, $s_{ij}=s_{ji}$. The equations of motion can ultimately be determined after requiring conservation of the constraint (2.7) in time. We write out the equations of motion for a connection:

$$R^{ab}_{0i}=\frac{1}{2}\varepsilon^{dab}_{c}\varepsilon_{jkl}\tilde{e}_{0d}e_{if}e_{0g}e^{(f}_lR^{c)g}_{jk}-\frac{1}{2}\left(\tilde{e}^{[a}_0e^{b]}_le^c_i+\frac{1}{2}e^{[a}_le^{b]}_i\tilde{e}^c_0\right)\varepsilon_{cdfg}\varepsilon_{jkl}e^d_0R^{fg}_{jk}\ \mathrm{mod}\ (\chi_{ab},\phi_c). \quad (2.10)$$

Here and in the following $\tilde{e}_{0a}=-g^{-1}\varepsilon_{abcd}e^b_1e^c_2e^d_3$, $g=\det g_{ij}$, and $(a\ b)$ (or $[a\ b]$) denotes symmetrization (or antisymmetrization) with respect to the pair of indices in the brackets.

We note that Eq. (2.10) and the constraints $\phi_c=0$ are contained in the equations

$$\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{abcd}e^c_\lambda R^{ab}_{\mu\nu}=0. \quad (2.11)$$

According to the definition of a Poisson bracket, the equations of motion (2.8) and (2.10) can be written in the form

$$\dot{A}=[A,H].$$

Here $[\ \ldots,\ldots]$ is the Poisson bracket of A and H, and A is any function of the dynamic variables.

Equations (2.5) and (2.6), as well as the condition $[e^a_i(x),\lambda^{jk}(y)]=0$, uniquely specify the following Poisson brackets:

$$[e^a_i(x),e^b_j(y)]=0, \quad (2.12)$$

$$[\omega_j^{bc}(x),e_i^a(y)]=\delta(x-y)\kappa^2\{2\tilde{e}_0^{[b}e_i^{c]}e_j^a+e_i^{[b}e_j^{c]}\tilde{e}_0^a + e_j^{[b}\tilde{e}_0^{c]}e_i^a\}(x). \qquad (2.13)$$

More lengthy calculations are needed to find connection-connection Poisson brackets. Since connection-connection Poisson brackets will not be used explicitly below, they are not written out here.

To complete the description of the Hamiltonian formalism, we show that with consideration of the second-class constraints (2.7) χ_{ab} and ϕ_c are first-class constraints.

It follows from (2.3), (2.8), and (2.10) that the $\chi_{ab}(x)$ generate local Lorentz transformations at the point x. In particular,

$$[\chi_{ab}(x),e_i^c(y)]=-\delta(x-y)(\delta_a^c e_{bi}-\delta_b^c e_{ai})(x),$$

$$[\chi_{ab}(x),\omega_i^{cd}(y)]=-2\partial_i\delta(x-y)\delta_a^{[c}\delta_b^{d]}-2\delta(x-y)\times\{\delta_a^{[c}\omega_{b_i}^{d]}+\delta_a^{[d}\omega_i^{c]b}\}(x).$$

Any vector quantity has a Poisson bracket with χ_{ab} similar to those written out. Therefore, the equation of motion for χ_{ab} has the form

$$\dot{\chi}^{ab}=2\omega_{0c}^{[a}\chi^{b]c}-\frac{1}{\kappa^2}e_0^{[a}\phi^{b]}.$$

Thus,

$$\partial_\mu\chi^{ab}=0(\mathrm{mod}\ \chi_{ab},\phi_c). \qquad (2.14)$$

Differentiating the curvature tensor (2.1) with respect to the space-time coordinates, we find

$$\nabla_\lambda R_{\mu\nu}^{ab}+\nabla_\mu R_{\nu\lambda}^{ab}+\nabla_\nu R_{\lambda\mu}^{ab}=0, \qquad (2.15a)$$

where

$$\nabla_\lambda R_{\mu\nu}^{ab}=\partial_\lambda R_{\mu\nu}^{ab}+\omega_{\lambda c}^a R_{\mu\nu}^{cb}+\omega_{\lambda c}^b R_{\mu\nu}^{ac}.$$

Next, differentiating Eqs. (2.8) and (2.9) and utilizing (2.14), we obtain

$$R_{\mu\nu}^{ab}e_{b\lambda}+R_{\nu\lambda}^{ab}e_{b\mu}+R_{\lambda\mu}^{ab}e_{b\nu}=0(\mathrm{mod}\ \chi_{ab},\phi_c). \qquad (2.15b)$$

The relations (2.15) indicate that the Bianchi identities hold in the canonical formalism.

We now establish that

$$\dot{\phi}_c=0\ \ (\mathrm{mod}\ \chi_{ab},\phi_c). \qquad (2.16)$$

Equations (2.14) and (2.16) indicate that χ_{ab} and ϕ_c are first-class constraints. From the definition of the ϕ_a we have

$$\nabla_0\phi_a\sim\varepsilon_{abcd}\varepsilon_{ijk}(\nabla_0 e_i^b R_{jk}^{cd}+e_i^b\nabla_0 R_{jk}^{cd})$$
$$=\varepsilon_{abcd}\varepsilon_{ijk}[\nabla_i e_0^b R_{jk}^{cd}-e_i^b(\nabla_j R_{k0}^{cd}+\nabla_k R_{0j}^{cd})]. \qquad (2.17)$$

The last equality is based on Eq. (2.8) and the identity (2.15a). Using (2.9) and (2.15), we represent the right-hand side of (2.17) up to terms containing χ_{ab} or ϕ_c in the following manner:

$$\nabla_i[\varepsilon_{abcd}\varepsilon_{ijk}(e_0^b R_{jk}^{cd}-2e_j^b R_{0k}^{cd})](\mathrm{mod}\ \chi_{ab},\phi_c). \qquad (2.18)$$

The quantity in the square brackets in (2.18) can be written in the form $\delta\mathscr{L}/\delta e_i^a$ which is equal to zero by virtue of the equations of motion. The equality (2.16) is thereby established.

3. FORMAL QUANTIZATION

When we go from classical to quantum mechanics, the classical Poisson bracket must be replaced by a quantum bracket. It is usually assumed that the quantum Poisson brackets for fundamental variables differ from the classical analogs only in the multiplier i, which appears on the right-hand sides of (2.12) and (2.13) in our case. The Heisenberg equations $i\dot{A}=[A,H]$ for the variables e_i^a and ω_i^{ab} maintain their classical form to within the arrangement of the operators.

It follows from transposition relations like (2.12) and (2.13) that the set of variables $\{e_i^a(x)\}$ is a complete set of commuting variables. The possible values of these variables satisfy the condition

$$-\infty<e_i^a(x)<+\infty.$$

We write out the formula for a connection operator. Since a formal quantum theory is considered in this section, the question of correctly ordering the operators in the equations presented here is meaningless. Using the classical Poisson brackets (2.12) and Eq. (2.7), we find

$$\omega_j^{bc}(x)=\kappa^2\{2\tilde{e}_0^{[b}e_i^{c]}e_j^a+e_i^{[b}e_j^{c]}\tilde{e}_0^a + e_j^{[b}\tilde{e}_0^{c]}e_i^a\}\pi_a^i(x)+(2g)^{-1}\varepsilon_{klm}\varepsilon_{inp}e_n^b e_p^c\partial_l e_m^d \times\{g_{ij}e_{dk}-g_{jk}e_{di}-g_{ik}e_{dj}\}. \qquad (3.1)$$

Here the operator field $\pi_a^i(x)$ satisfies the commutation relations:

$$[\pi_a^i(x),e_j^b(y)]=i\delta_j^i\delta_a^b\delta(x-y),\quad [\pi_a^i(x),\pi_b^j(y)]=0. \qquad (3.2)$$

The second term in (3.1) is uniquely specified by Eq. (2.7).

We now move on to the solution of the problem of finding the conserved operators that embody all the physical degrees of freedom of the system.

In generally covariant theories it is also reasonable to refer to conserved operators as gauge-invariant operators. It is far more convenient to solve the problem posed of deriving complete sets of gauge-invariant operators axiomatically, as was done in Ref. 2 during the quantization of gravitation in three-dimensional space-time.

We introduce the following natural hypotheses or axioms regarding the structure of the space F of physical states of the theory.

Axiom 1. All the states of the theory that have physical meaning are obtained from the ground state $|0\rangle$ using the boson creation operators a_N^+:

$$|n_1,N_1;\ldots;n_s,N_s\rangle=(n_1!\ldots n_s!)^{-1/2}(a_{N_1}^+)^{n_1}\ldots(a_{N_s}^+)^{n_s}|0\rangle, \qquad (3.3)$$

$$a_N|0\rangle=0.$$

The states (3.3) form an orthonormalized basis of the space F of physical states of the theory.

The numbers $n_1, \ldots, n_s$ take positive integer values and are called occupation numbers.

Axiom 2. The states (3.3) satisfy the conditions

$$\chi_{ab}(x)|n_1,N_1;\ldots;n_s,N_s\rangle=0, \tag{3.4}$$

$$\phi_a(x)|n_1,N_1;\ldots;n_s,N_s\rangle=0.$$

Axiom 3. The state $e_i^a(x)|n_1,N_1;\ldots;n_s,N_s\rangle$ contains a superposition of all the states of the theory, in which one of the occupation numbers differs in absolute value by unity, while the others coincide with the occupation numbers of the state (3.3).

Here the operators a_N^+ and their conjugates a_N have the usual commutation properties:

$$[a_N,a_M]=0, \quad [a_N,a_M^+]=\delta_{NM}. \tag{3.5}$$

For the case of interest to us of a compact x space without the generality restriction it can be assumed that the index N, which labels the creation and annihilation operators, belongs to a discrete finite-dimensional lattice. A norm is easily introduced into the space of indices.

It follows from (3.3) and (3.4) that

$$[H,a_N^+]=0, \quad [H,a_N]=0, \tag{3.6}$$

where the Hamiltonian operator H is assigned according to (2.3).

Thus, the gauge-invariant operators of interest to us have been formally indicated. They comprise the set of annihilation and creation operators $\{a_N,a_N^+\}$, which exhaust all the physical degrees of freedom of the system. We note that the commutation relations (3.6) are a consequence of the general covariance of the theory. A set of operators with the properties (3.6) that exhausts the physical degrees of freedom of the system does not exist in the other theories.

We present some consequences of Axioms 1–3.

Let $|N\rangle=a_N^+|0\rangle$. It follows from Axiom 3 that

$$e_i^a(x)|N\rangle=\frac{1}{\sqrt{2}}e_{Ni}^a(x)|0\rangle+|N;e_i^a(x)\rangle, \tag{3.7}$$

$$\langle 0|N;e_i^a(x)\rangle=0.$$

Here the fields $e_{Ni}^a(x)$ are linearly independent and do not contain creation and annihilation operators:

$$[e_{Ni}^a(x),a_M]=0, \quad [e_{Ni}^a(x),a_M^+]=0. \tag{3.8}$$

Since the operator $e_i^a(x)$ is Hermitian, the following expansion exists as a consequence of (3.7):

$$e_i^a(x)=\frac{1}{\sqrt{2}}\sum_N (a_N e_{Ni}^a(x)+a_N^+\bar{e}^{\,a}_{Ni}(x))+\tilde{e}^{\,a}_i(x). \tag{3.9}$$

The field $\tilde{e}^{\,a}_i(x)$ does not contain the operators a_N and a_N^+ in the first power, but it contains a zeroth-order contribution with respect to these operators, which we denote by $e_i^{a(0)} \times(x)$.

Information regarding the configuration of the fields $e_{Ni}^a(x)$ can be obtained by studying the matrix elements of several invariant operators relative to the states (3.3). For example, let us consider the quantity $V=\int d^3x\sqrt{-g}$, which is invariant with respect to permutations of the coordinates in three-dimensional space. The expression on the right-hand side of (2.8) is known to give the variation of a tetrad in response to infinitesimal variation of the coordinates. On the other hand, the right-hand side of (2.8) is equal to the Poisson bracket of a tetrad and the Hamiltonian. Since the Hamiltonian annuls physical states, we have

$$\langle 0|\left(\int d^3x\sqrt{-g}\right)|N\rangle=\langle 0|e^{-i\varepsilon H}\left(\int d^3x\sqrt{-g}\right)e^{i\varepsilon H}|N\rangle.$$

This equality readily yields the relation

$$\langle 0|:\left\{\int d^3x\sqrt{-g}g^{ij}(e_j^a\nabla_i\xi_a+\nabla_i\xi_a\cdot e_j^a)\right\}:|N\rangle=0, \tag{3.10}$$

which is valid for any field $\xi^a(x)$.

To proceed further, we assume that Heisenberg tetrad-connection fields can be represented in the form of formal series in the operators a_N and a_N^+ For tetrad fields the beginning of this expansion is assigned according to (3.9). For a connection operator the naturalness of this hypothesis follows from Eq. (3.1). We use $\omega_i^{ab(0)}(x)$ to denote the zeroth-order (in the operators a_N and a_N^+) contribution to the connection. This contribution can be expressed in terms of the fields $e_i^{a(0)}(x)$ using (2.8) and (2.9).

Now, from (3.10) we obtain the conditions for the fields e_{Ni}^a:

$$\nabla_i^{(0)}(\sqrt{-g^{(0)}}g^{ij(0)}e_{Nj}^a)=0. \tag{3.11}$$

The superscript (0) means that all the operators and fields bearing it depend only on the zeroth approximation of the tetrad fields $e_i^{a(0)}$ and the connection fields $\omega_i^{ab(0)}$.

We use $e_i^{a(s)}(x)$ and $\omega_i^{ab(s)}(x)$, where $s=0,1,\ldots$, to denote the contributions that are of order s with respect to the creation and annihilation operators to the tetrad and connection fields, so that

$$e_i^a(x)=\sum_{s=0}^{\infty} e_i^{a(s)}(x),$$

$$e_i^{a(s)}(x)=\sum_{N_1\ldots N_s} a_{N_1}\ldots a_{N_s}e^a_{N_1\ldots N_s i}(x)$$

$$+\sum_{M_1}\sum_{N_1\ldots N_{s-1}} a^+_{M_1}a_{N_1}\ldots a_{N_{s-1}}$$

$$\times e^a_{M_1;N_1\ldots N_{s-1}i}(x)+\ldots\ . \tag{3.12}$$

Here the creation and annihilation operators are normally ordered. There are similar formulas for a connection field. All the information on the evolution of the system with time is contained in the fields $e^a_{N_1\ldots N_s i}(x)$, $\omega^{ab}_{N_1\ldots N_s i}(x)$, etc. $(s=0,1,\ldots)$. The set of these fields is denoted by $\Phi_{.\,l}(x)$.

The group of motions or the gauge group (a set of exponential functions of the Hamiltonian) is denoted by $\mathcal{G}$. This group acts in the space of the fields $\Phi_{.\,l}(x)$. The Lie algebra of the group $\mathcal{G}$ is the set of operators $\{\chi_{ab}(x),\phi_c(x)\}$. The operators $\chi_{ab}(x)$ and $\phi_c(x)$ are first-class constraints in the Dirac sense [see (A7)]. Therefore, they can be represented as vector fields on $\mathcal{G}$, whose components depend in the general case on $\Phi_{.\,l}(x)$.

The group $\mathscr{G}$ contains the subgroup G of local Lorentz transformations, which can easily be described. Let the matrix field $S^a_b(x)$ satisfy the condition

$$S^a_c(x)\,\eta^{cd}S^b_d(x)=\eta^{ab}.$$

The group formed by the set of elements $\{S^a_b(x)\}$ is denoted by G. Each element of G corresponds to a transformation of the tetrad and connection fields:

$$e'^a_i(x)=S^a_b(x)e^b_i(x),$$

$$\omega'^{ab}_i(x)=S^a_c(x)S^b_d(x)\omega^{cd}_i(x)+S^a_c(x)\,\eta^{cd}\partial_i S^b_d(x).$$

The operator

$$\chi(\omega_0)=\frac{1}{2}\int d^3x\,\omega^{ab}_0\chi_{ab},\quad \omega^{ab}_0\to 0,$$

is a right-invariant vector field on G, which transforms the point $S^a_b(x)$ to an infinitely close point $S^a_b(x)-\omega^a_{0c}(x)S^c_b(x)$.

The vector field on $\mathscr{G}$ corresponding to the operator

$$\phi(e_0)=\frac{1}{2\kappa^2}\int d^3x e^c_0\phi_c,\quad e^c_0\to 0,$$

generates the next displacement of the tetrad fields [see (2.3) and (2.8)]:

$$e^a_i\to e^a_i+\nabla_i e^a_0.$$

Now it is not difficult to represent the field $\pi^i_a(x)$, which is conjugate to the tetrad field $e^a_i(x)$ [see (3.2)], in the lowest approximation with respect to the operators a^+_N and a_N. For this purpose we must supplement the set of the fields $e^a_{Ni}(x)$ in (3.9) to obtain the complete orthonormalized set $\{e^a_{Ni}(x)\}$. This gives

$$\int d^3x\sqrt{-g^{(0)}}\bar{e}^{\,a}_{Mi}g^{ij(0)}e_{Naj}=\kappa^2q^2_M\eta_{MN},\tag{3.13}$$

where $\eta_{MN}=0$, if $M\neq N$, and $\eta_{NN}=1$ or -1 for a space- or time-like field e^a_{Ni}, respectively. Here q^2_M is a normalization factor, which has the dimensions of length. Besides the orthonormality condition (3.13) there is a completeness condition:

$$\sum_N \kappa^{-2}q_N^{-2}\eta_{NN}e^a_{Ni}(x)\bar{e}^{\,b}_{Nj}(x)=\frac{1}{\sqrt{-g^{(0)}}}g^{(0)}_{ij}\eta^{ab}\delta(x-y)-\nabla^{(0)}_i(x)\nabla^{(0)}_j(y)D^{ab(0)}(x,y),\tag{3.14}$$

$$-\nabla^{(0)}_i\sqrt{-g^{(0)}}g^{ij(0)}\nabla^{(0)}_jD^{ab(0)}(x,y)=\eta^{ab}\delta(x-y).$$

The set of the operators a^+_N and a_N should also be supplemented so that the commutation relations (3.5) are replaced by the commutation relations

$$[a_M,a^+_N]=\eta_{MN}.\tag{3.15}$$

Taking into account Eqs. (3.9), (3.14), and (3.15), we obtain the following representation for the operator π^i_a

$$\pi^i_a(x)=i(\sqrt{-g^{(0)}}g^{ij(0)})(x)\left\{\frac{1}{\sqrt{2}\kappa^2}\sum_N q_N^{-2}(a_Ne_{Naj}(x)-a^+_N\bar{e}_{Naj}(x))+\nabla^{(0)}_j\int d^3z\,D^{c(0)}_a(x,z)\frac{\delta}{\delta\xi^c(z)}\right\}.\tag{3.16}$$

The second term in (3.16) is a vector field on $\mathscr{G}$, so that

$$\left[\int d^3z\,e^c_0(z)\frac{\delta}{\delta\xi^c(z)},e^a_i(x)\right]=\nabla_ie^a_0(x).\tag{3.17}$$

It is seen from Eqs. (3.1), (3.9), and (3.14)–(3.17), first, that to lowest order in the operators a^+_N and a_N the tetrad and connection fields satisfy the quantum Poisson brackets (2.12) and (2.13) and, second, that the connection fields, like the tetrad fields, contain all these operators in first order with respect to a^+_N and a_N.

In the approach described at the end of this section [beginning from Eq. (3.13)] the tetrad and connection fields clearly have more degrees of freedom than are necessary on the basis of kinematic arguments. In fact, besides the degrees of freedom that correspond to the gauge group $\mathscr{G}$ and are included in the fields $\Phi_{\mathscr{J}}(x)$, there is an overfull system of degrees of freedom that are included in the set of all the creation and annihilation operators. However, this is not a problem, and it even facilitates further progress for the following reasons. First, all these creation and annihilation operators are gauge-invariant, and, second, when regularization is performed, almost all the creation and annihilation operators are eliminated, and only the "minimum" necessary number of operators remain. On the other hand, the gauge group remains unharmed.

The presence of extra degrees of freedom above the kinematically required number in the tetrad and connection fields is similar to the analogous phenomenon in a more common example. Consider any gauge theory containing a Dirac field ψ. The gauge group transforms the Dirac field according to the formula $\psi(x)\to S(x)\psi(x)$. Therefore, it can be assumed that the field ψ contains the degrees of freedom of the gauge group $S(x)$ in addition to the kinematically required fermion degrees of freedom.

We now have all the necessary tools to devise a regularized theory.

4. REGULARIZATION

The presentation in the preceding section had a formal character, since divergences were not taken into consideration. The formulas written out in Sec. 3 have a heuristic character. In this section we perform regularization and thereby impart strict meanings to all the operators and equations used.

The commutation relations (3.5) and (3.6) are important because any set of pairs of the operators a_N and a^+_N can be treated as a set of second-class constraints according to Dirac.[3] This makes it possible to perform regularization in the following manner.

We identify a finite set $\{a_N,a^+_N\}'$ of pairs of annihilation and creation operators and number them in such a manner that $|N|<N_0$. Operators from the set $\{a_N, a_N^+\}'$ satisfy the commutation relations (3.5). Since physical information is contained in the wave functions $e_{Ni}^a(x)$, this choice is actually determined by the choice of the set $\{e_{Ni}^a(x)\}'$ of linearly independent wave functions corresponding to the step of operators $\{a_N, a_N^+\}'$. The choice of the functions in $\{e_{Ni}^a(x)\}'$ is determined by the physical conditions of the problem. For example, if the x-space is a torus, periodic traveling waves, whose wave numbers are restricted in absolute value, can be taken as the wave functions of this set. Such a choice of $\{e_{Ni}^a(x)\}'$ corresponds to the problem of gravitational waves.

Regularization of the theory involves equating to zero all the pairs of creation and annihilation operators except the selected pairs, i.e., all the pairs with $|N|>N_0$

$$a_N=0, \quad a_N^+=0, \quad |N|>N_0. \tag{4.1}$$

An infinite system of second-class constraints is thereby imposed. As a result, the commutation relations (2.12), (2.13), etc., should be replaced by the corresponding Dirac commutation relations, and the regularized equations of motion should be investigated.

Let us prove an important theorem for our method, which imparts meaning to the entire dynamic quantization procedure.

Theorem: Imposition of the second-class constraints (4.1) does not alter the forms of the Heisenberg equations and conserves their classical character.

Proof: Let $|\mathcal{M}'\rangle, |\mathcal{N}'\rangle, \ldots$ be the basis vectors (3.3) constructed using the restricted set of operators $\{a_N, a_N^+\}'$, and let F' be a Fock space with these basis vectors. The imposition of constraints (4.1) means that the space of physical states F is confined to the regularized subspace $F'\subset F$. Only the matrix elements of the form $\langle\mathcal{M}'|A|\mathcal{N}'\rangle$ are considered for any operator A in the regularized theory. Therefore, the matrix elements of the quantum Dirac bracket[1)] of A and B corresponding to the constraints (4.1) are represented in the form

$$\langle\mathcal{M}'|[A,B]^*|\mathcal{N}'\rangle=\sum_{\mathcal{L}'} (\langle\mathcal{M}'|A|\mathcal{L}'\rangle\langle\mathcal{L}'|B|\mathcal{N}'\rangle - \langle\mathcal{M}'|B|\mathcal{L}'\rangle\langle\mathcal{L}'|A|\mathcal{N}'\rangle). \tag{4.2}$$

According to the definition of a quantum Dirac bracket, the operators a_N and a_N^+ with $|N|>N_0$ contained in the operators A and B from (4.2) are normally ordered and then set equal to zero. The Poisson bracket $[A,B]$ is formally distinguished from the Dirac bracket (4.2) by the fact that in the calculation of the matrix elements $\langle\mathcal{M}'|[A,B]|\mathcal{N}'\rangle$ using a formula similar to (4.2) the summation is carried out over all the intermediate states (3.3). We assume that the operator B is diagonal in the basis (3.3) and does not depend on the operators a_N and a_N^+ with $|N|>N_0$. Then it is seen from (4.2) that

$$\langle\mathcal{M}'|[A,B]^*|\mathcal{N}'\rangle=\langle\mathcal{M}'|[A,B]|\mathcal{N}'\rangle, \tag{4.3}$$

if we set $a_N=0$ and $a_N^+=0$ for $|N|>N_0$ in the matrix element on the right-hand side of (4.3). Now, we need only note that all the operators of the occupation numbers $n_N=a_N^+a_N$ commute with the Hamiltonian of the theory according to the axioms introduced. Moreover, because of the commutation relations (3.6) the Hamiltonian does not depend on the operators a_N and a_N^+ Therefore, the Hamiltonian H can be substituted for the operator B. This implies the validity of the theorem.

There is also a classical variant of this theorem.

Theorem. Imposition of the second-class constraints (4.1) does not alter the classical form of the Hamilton equations for the remaining degrees of freedom.

To prove it we write out the formula for a Dirac bracket in the classical theory.

Let $\{\chi_\alpha\}$ and $\{\kappa_n\}$ be finite or infinite sets of first- and second-class constraints, respectively. By definition, this means that

$$[\chi_\alpha,\chi_\beta]\approx 0, \tag{4.4}$$

$$[\chi_\alpha,\kappa_n]\approx 0, \tag{4.5}$$

$$[\kappa_m,\kappa_n]=c_{mn}^{-1}. \tag{4.6}$$

Following Dirac, we use the symbol $\approx$ to denote equality with respect to the absolute values of the terms containing κ_n or χ_α. We note that the matrix c_{mn}^{-1} in (4.6) is a nondegenerate matrix, which depends in the general case on the dynamic variables. The Hamiltonian H of the system is a first-class quantity:

$$[H,\chi_\alpha]\approx 0, \tag{4.7}$$

$$[H,\kappa_n]\approx 0. \tag{4.8}$$

In the classical theory the Dirac bracket of any two quantities is defined by the formula

$$[\xi,\eta]^*=[\xi,\eta]-\sum_{m,n}[\xi,\kappa_m]c_{mn}[\kappa_n,\eta]. \tag{4.9}$$

It is obvious that for any ξ and κ_n we have

$$[\xi,\kappa_n]^*=0.$$

Hence it follows that the second-class constraints κ_n can be set equal to zero before the Dirac brackets are calculated. As a consequence of (4.8) and (4.9), we have the weak equality

$$[\xi,H]\approx[\xi,H]^*. \tag{4.10}$$

The weak equality (4.10) means that the equations of motion obtained using the Poisson brackets and the Dirac brackets essentially coincide.

According to (3.6), the weak equalities (4.5) and (4.8) transform into strong equalities within our method. Therefore, it follows immediately from (4.9) that $[\xi,H]=[\xi,H]^*$ for any ξ. This means that the theorem is correct.

Corollary. The regularized theory is generally covariant.

In fact, this follows directly from the theorem just proved. The equations of motion which the fields $e_i^a(x)$ and $\omega_i^{ab}(x)$ obey in the regularized theory coincide in form with the classical equations of motion, which are generally covariant. The corollary is thereby proved.

Precisely because the regularization is ideally matched to the dynamics of the system in the method employed, we called this quantization method dynamic. We again note that the regularization does not in any way affect the gauge group, which remains the same in the regularized theory as it was in the classical theory.

To conclude this section we present a more abstract procedure for solving our problem. Although this procedure is, perhaps, less natural, it has great logical harmony and simplifies the calculations, since only regularized quantities are treated within it.

The basis of this approach is *ab initio*.

Proposition. The theory is regularized in such a manner that the following axioms hold.

Axiom R1. All the states of the theory having physical meaning are obtained from the ground state $|0\rangle$ by means of the creation operators a_N^+ with $|N|<N_0$:

$$|n_1,N_1;\dots;n_s,N_s\rangle = (n_1!\dots n_s!)^{-1/2}(a_{N_1}^+)^{n_1}\dots(a_{N_s}^+)^{n_s}|0\rangle, \qquad (4.11)$$

$$a_N|0\rangle=0.$$

The states (4.11) form an orthonormalized basis of the space F' of physical states of the theory.

Axiom R2. The states (4.11) satisfy the conditions

$$\chi_{ab}(x)|\rangle=0, \quad \phi_a(x)|\rangle=0. \qquad (4.12)$$

Axiom R3. The dynamic variables $e_i^a(x)$ transfer the state (4.11) into a superposition of states of the theory like (4.11), which contains all the states in which one of the occupation numbers differs in absolute value by unity, while the remaining occupation numbers coincide with the occupation numbers of the state (4.11).

Axiom R4. The equations of motion and the constraints for the physical fields $e_i^a(x)$ and $\omega_i^{ab}(x)$ coincide in form with the corresponding classical equations and constraints to within permutations of the operators.

Axioms R1–R3 are analogs of Axioms 1–3 in the unregularized theory. Axiom R4 replaces the theorem itself. It postulates the correct form of the equations of motion and the constraints in agreement with classical mechanics. Since it is no longer necessary to derive equations of motion, the role of the Hamiltonian is diminished.

It seems to us that under the quantization approach described here the problem of ordering the operators in the equations of motion can be solved in the following manner. We use the single symbol $\Phi(x)$ to denote $e_i^a(x)$ and $\omega_i^{ab}(x)$ and the symbol $\lambda(x)$ to denote the fields $e_0^a(x)$ and $\omega_0^{ab}(x)$. The problem of ordering the operators arises in connection with the problem of establishing the self-consistency of the theory.[3] We write out the Heisenberg equations in the general form

$$\dot{\Phi}(x)=f(\lambda,\Phi)(x). \qquad (4.13)$$

In (4.13) f is a local function of the fields $\lambda(x)$ and $\Phi(x)$ and depends linearly on the field λ. By definition, if λ is an operator field, it is positioned to the left of Φ in the function $f(\lambda,\Phi)$. According to (4.13), for the arbitrary fields λ_1 and λ_2 we have

$$\delta_i\Phi=\delta t_i f(\lambda_i,\Phi), \quad i=1,2.$$

Let us consider the quantity $(\delta_1\delta_2-\delta_2\delta_1)\Phi$, which we denote by $\delta_{[12]}\Phi$. A necessary condition for self-consistency of the theory is the possibility of ordering the operators in (4.13) so that the following weak equality holds:

$$\delta_{[12]}\Phi\approx\delta t_1\delta t_2 f(\lambda_{[12]},\Phi). \qquad (4.14)$$

Here the field $\lambda_{[12]}$ is a bilinear antisymmetric form with respect to the fields λ_1 and λ_2 and, generally speaking, depends on the field Φ. The first-class constraints χ_{ab} and ϕ_c, being vector fields on the group $\mathcal{G}$, do not contain creation and annihilation operators. The latter property is utilized explicitly in the calculations in the next section.

5. PERTURBATION THEORY

In this section we show how the field coefficients in the expansion of the tetrad and connection fields in operators a_N and a_N^+ can be found systematically. This is done first without consideration of the quantum corrections (loops). Then the result is refined with consideration of the quantum fluctuations, which is formally equivalent to expansion in the number N_0. As is shown below, formal expansion in N_0 is, in turn, equivalent to expansion in the dimensionless parameter $(\Lambda\kappa)^2$, where Λ is the cutoff momentum of the theory. If the cutoff momentum is much smaller than the Planck momentum, then $(\Lambda\kappa)^2\ll 1$.

The presentation in this section is very schematic. A detailed study of perturbation theory and consideration of concrete problems using this method should be left for special studies.

The calculations begin from the zeroth approximation, i.e., from $e_i^{a(0)}(x)$ and $\omega_i^{ab(0)}(x)$. The tetrad and connection fields in the zeroth approximation satisfy the constraints (2.8), (2.9), and (2.11) and do not depend on the creation and annihilation operators. However, in the zeroth approximation the tetrad and connection fields are operators on the group of gauge transformations $\mathcal{G}$, according to (3.1), (3.16), and (3.17). Thus, the fields $e_i^{a(0)}(x)$ and $\omega_i^{ab(0)}(x)$ satisfy the equations of motion (2.8) and (2.10), and the constraints $\chi_{ab}^{(0)}$ and $\phi_c^{(0)}$, which are composed of these fields, annul the single state $|0\rangle$ in the zeroth approximation by definition. Using reasoning similar to that used to obtain (3.10), we find that $\langle 0|\chi_{ab}^{(0)}|0\rangle$ and $\langle 0|\phi_a^{(0)}|0\rangle$ do not depend on the operator parts of the fields in the zeroth approximation. This means that under the matrix elements all fields can be regarded in the zeroth approximation as purely classical and as satisfying the equations of motion and the constraints (2.8), (2.9), and (2.11).

In the first approximation all the quantum states (4.11) from the regularized space are included in the treatment. The tetrad and connection fields are expanded in the first approximation in the following manner [see (3.9) and (3.12)]:

$$e_i^a(x)=\frac{1}{\sqrt{2}}\sum_{|N|<N_0}(a_N e_{Ni}^a(x)+a_N^+\bar{e}^{\,a}_{Ni}(x))+e_i^{a(0)}(x)$$
$$\equiv e_i^{a(0)}(x)+e_i^{a(1)}(x), \qquad (5.1)$$
$$\omega_i^{ab}(x)=\frac{1}{\sqrt{2}}\sum_{|N|<N_0}(a_N\omega_{Ni}^{ab}(x)+a_N^+\bar{\omega}^{\,ab}_{Ni}(x))+\omega_i^{ab(0)}(x)$$
$$\equiv\omega_i^{ab(0)}(x)+\omega_i^{ab(1)}(x).$$

We note that under our approach the fields e_0^a and ω_0^{ab}, which play the role of Lagrange multipliers, remain numerical:

$$e_0^{a(s)}=0,\quad \omega_0^{ab(s)}=0,\quad s=1,2,\ldots\,. \qquad (5.2)$$

Substituting the fields from (5.1) into Eqs. (2.8), (2.9), and (2.11) and taking into account Eqs. (5.2) and the fact that the fields $e_i^{a(0)}$ and $\omega_i^{ab(0)}$ satisfy all the classical equations, we obtain

$$\varepsilon_{abcd}\varepsilon_{ijk}\{R_{ij}^{ab(0)}e_k^{c(1)}+2e_k^{c(0)}\nabla_i^{(0)}\omega_j^{ab(1)}\}=0, \qquad (5.3)$$
$$\varepsilon_{ijk}(\nabla_i^{(0)}e_j^{a(1)}-e_{bi}^{(0)}\omega_j^{ab(1)})=0,$$
$$\varepsilon_{abcd}\varepsilon_{ijk}\{R_{0i}^{ab(0)}e_j^{c(1)}+e_j^{c(0)}\nabla_0\omega_i^{ab(1)}+e_0^c\nabla_i^{(0)}\omega_j^{ab(1)}\}=0, \qquad (5.4)$$
$$\nabla_0 e_i^{a(1)}-e_{b0}\omega_i^{ab(1)}=0.$$

Equations (5.3) are constraints, and Eqs. (5.4) are the equations of motion for the tetrad and connection fields. Placing Eqs. (5.3) and (5.4) between the bra and ket $\langle 0|\ldots|N\rangle$, we find

$$\varepsilon_{abcd}\varepsilon_{ijk}\{R_{ij}^{ab(0)}e_{Nk}^c+2e_k^{c(0)}\nabla_i^{(0)}\omega_{Nj}^{ab}\}=0, \qquad (5.5a)$$
$$\varepsilon_{ijk}(\nabla_i^{(0)}e_{Nj}^a-e_{bi}^{(0)}\omega_{Nj}^{ab})=0, \qquad (5.5b)$$
$$\varepsilon_{abcd}\varepsilon_{ijk}\{R_{0i}^{ab(0)}e_{Nj}^c+e_j^{c(0)}\nabla_0\omega_{Ni}^{ab}+e_0^c\nabla_i^{(0)}\omega_{Nj}^{ab}\}=0, \qquad (5.6a)$$
$$\nabla_0 e_{Ni}^a-e_{b0}\omega_{Ni}^{ab}=0. \qquad (5.6b)$$

Thus, the constraints (5.5) and the equations of motion (5.6) break down into individual equations with the assigned number N. The fields $e^a_{N_1\ldots N_si}$ and the like also have this property, if the quantum fluctuations are not taken into account. For example, we have the following analogs of Eqs. (5.5b) and (5.6b) for $e^a_{N_1N_2i}$ and $\omega^{ab}_{N_1N_2i}$

$$\varepsilon_{ijk}\{\nabla_i^{(0)}e^a_{N_1N_2j}+e_{bj}^{(0)}\omega^{ab}_{N_1N_2i}+\omega^{ab}_{N_1i}e_{N_2bj}+\omega^{ab}_{N_2i}e_{N_1bj}\}=0, \qquad (5.7)$$
$$\nabla_0\, e^a_{N_1N_2i}-e_{b0}\omega^{ab}_{N_1N_2i}=0. \qquad (5.8)$$

We note that while Eqs. (5.5) and (5.6) are homogeneous with respect to e_{Ni}^a and ω_{Ni}^{ab}, the equations like (5.7) for $e^a_{N_1N_2i}$ and $\omega^{ab}_{N_1N_2i}$ are not.

The following procedure can be used to solve Eqs. (5.5)–(5.8). The homogeneous system of equations (5.5) and (5.6) must first be solved for e_{Ni}^a and ω_{Ni}^{ab}. Then an inhomogeneous linear system of equations that includes Eqs. (5.7) and (5.8) is solved for $e^a_{N_1N_2i}$ and $\omega^{ab}_{N_1N_2i}$. This system of equations depends on the previously found fields e_{Ni}^a and ω_{Ni}^{ab}. Next, this process is extended to the higher fields $e^a_{N_1\ldots N_si}$, etc. If the quantum fluctuations are not taken into account, the equations obtained are inhomogeneous linear equations with respect to $e^a_{N_1\ldots N_si}$, etc., which depend on the fields found in the preceding steps.

The number of constraints together with Eqs. (5.5) and (5.6) equals 40, and the number of e_{Ni}^a and ω_{Ni}^{ab} sought for a fixed N equals 30. Although the system of equations (5.5) and (5.6) has an excessive number of equations, it has nonzero solutions. In fact, the equations of motion (5.6) have solutions for any values of the fields e_0^a and ω_0^{ab}. This is obvious for the original equations (2.8), (2.9), and (2.11). Therefore, the equations (5.5) and (5.6) with respect to e_{Ni}^a and ω_{Ni}^{ab} obtained from them have solutions.

The initial conditions (3.11) for the fields e_{Ni}^a at $t=t_0$ should be used to solve Eqs. (5.5) and (5.6). The field ω_{Ni}^{ab} can be uniquely expressed in terms of e_{Ni}^a and $\dot{e}_{Ni}^a$ using Eqs. (5.5b) and (5.6b). After this Eqs. (5.5a) and (5.6a) lead to linear differential equations that are second-order in e_{Ni}^a.

It is seen from the normalization condition (3.13) and Eqs. (5.5) and (5.6) that the fields e_{Ni}^a and ω_{Ni}^{ab} are proportional to the gravitation constant κ. Now we can clarify the meaning of the multiplication by κ^2 on the right-hand side of the normalization condition (3.13). The possibility of satisfying the Poisson brackets (2.13) in the lowest approximation with respect to the creation and annihilation operators (in the unregularized theory) follows from the expansion (5.1) and the proportionality of e_{Ni}^a and ω_{Ni}^{ab} to κ.

Thus, the fields e_{Ni}^a and ω_{Ni}^{ab} are found according to the following rule. The fields e_{Ni}^a and ω_{Ni}^{ab} which satisfy the conditions (3.11) at $t=t_0$ and (3.13), as well as Eqs. (5.5) and (5.6), are sought. The tetrad and connection fields composed from them according to (5.1) (where the summation is carried out over all N) must satisfy the Poisson brackets (2.13) in the lowest approximation. Then the regularized set of fields $\{e_{Ni}^a,\omega_{Ni}^{ab}\}'$ with $|N|<N_0$ is chosen from the entire set of e_{Ni}^a and ω_{Ni}^{ab}.

We note that, according to Eqs. (5.7), we have

$$e^a_{N_1N_2i}\propto\kappa^2,\quad \omega^{ab}_{N_1N_2i}\propto\kappa^2. \qquad (5.9)$$

Let us briefly consider the question of taking into account the quantum fluctuations or loops.

We use $e_{Ni}^{a(s)}$, $s=0,1,\ldots$, to denote the contributions to e_{Ni}^a that correspond to the s-loop contribution. Thus, the fields e_{Ni}^a considered above correspond to $e_{Ni}^{a(0)}$ in the new notation.

We find the following equations in the single-loop approximation in the same manner that we obtained Eqs. (5.5)–(5.8):

$$\varepsilon_{ijk}\Bigg\{\partial_i e_{Nj}^{a(1)}+\sum_{|M|<N_0}[2(\omega_{NMi}^{ab(0)}\bar{e}^{\,(0)}_{Mbj}+\bar{\omega}^{\,ab(0)}_{Mi}e^{(0)}_{NMbj})$$
$$+(\omega^{ab(0)}_{M;Ni}e^{(0)}_{Mbj}+\omega^{ab(0)}_{Mi}e^{(00)}_{M;Nbj})]\Bigg\}=0. \qquad (5.10)$$

It can easily be understood from dimensionality arguments [see (5.9)] that the order of the sum in the last equation is

$(\Lambda\kappa)^2$, where Λ is the cutoff momentum of the theory, i.e., the maximum momentum of the regularized system of functions $\{e_{Ni}^{a(0)}(x)\}'$. The expansion for systematically taking into account the quantum fluctuations is performed in the dimensionless parameter $(\Lambda\kappa)^2$. The concept of the "small number" of physical degrees of freedom now becomes clear: it is a number for which

$$(\Lambda\kappa)^2 \ll 1. \tag{5.11}$$

The condition (5.11) means that the cutoff momentum is much smaller than the Planck momentum. In this case the quantum fluctuations can be taken into account using a finite perturbation scheme, as was shown above.

6. CONCLUSIONS

Thus, we have devised a mathematically correct procedure for the canonical quantization of gravitation theory. The quantum theory devised has the following principal properties.

A. If the x-space is compact, the number of physical degrees of freedom is finite.

B. The Heisenberg equations for tetrad, connection, and other fields have the classical form (to within permutations of the operators).

C. The theory derived is generally covariant.

Unfortunately, we are forced to make a significant reservation. The mathematical correctness of the theory will be established completely only when the problem of ordering the annihilation and creation operators in the equations of motion is solved. The mathematical correctness of the theory claimed in this paper refers to the ultraviolet divergence and general covariance problems solved herein. At first glance, the problem of ordering the creation and annihilation operators in Sec. 5 is solved automatically, since the coefficients in front of the creation and annihilation operators in the equations of motion and the constraints are set equal to zero. This produces equations for finding the fields $\Phi_{\mathcal{J}}$. However, it also raises the question of the compatibility and correctness of these equations. This question calls for a special investigation.

We note that under the nonperturbative axiomatic approach the problem of ordering the creation and annihilation operators in the constraints does not exist, since the constraints do not depend on these operators in our theory. Therefore, the problem of ordering the creation and annihilation operators is significant only in the equations of motion (see the end of Sec. 4).

We note that the condition (5.11) for the existence of the perturbation scheme is not a necessary condition for mathematical correctness of the theory. In our opinion, a physically intelligible quantum theory of gravitation need not be subject to the condition (5.11). Nevertheless, it is possible that the condition (5.11) will be significant in some specific problems.

Although the theory was devised in the case of pure gravitation in this paper, the inclusion of matter in the theory cannot create fundamental difficulties in dynamic quantization at first glance. In our opinion, the study of supergravitation is of greatest interest in this respect. It also seems promising, because the supersymmetry of the theory is established most easily in the equations of motion, which play the main role in the dynamic method.

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[1)]The notation for a Dirac bracket is distinguished from the notation for a Poisson bracket by an additional raised asterisk.

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