Microscopic critical-state model for a hard superconductor

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A microscopic critical-state model of a hard superconductor is presented, in which the state of the system is described using the coordinates of individual vortices. After several model assumptions regarding the character of the pinning are introduced, the microscopic problem of the penetration of a field into a hard superconductor allows an exact solution. The interactions of the vortices with one another, as well as with their images, Meissner currents, and pinning centers, are considered. The existence of the Bean–Livingston barrier is taken into account. The coordinates of the vortices are calculated in both increasing and nonmonotonically varying external fields. The stability of the solution against small displacements of the vortices from their equilibrium positions is investigated. Strong pinning is investigated in detail. In this case, after going over to a macroscopic description in sufficiently strong fields, the model turns out to be completely equivalent to the phenomenological description within the macroscopic nonlocal critical-state model. A new effect, which is lost in the macroscopic description, has been detected in weak fields, viz., the formation of a macroscopic vortex-free region in a decreasing external magnetic field.
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1. INTRODUCTION

The phenomenological critical-state model (see, for example, Refs. 1 and 2) has been used successfully for a long time to describe electrodynamically hard superconductors. According to this model, screening currents with a density equal to the critical value J_c are induced near the surface of a superconductor in response to any variation of the external field. The magnetic induction and the vortex density in regions more distant from the surface remain the same as before the external field was varied. Such an electrodynamic model was based on the idea of vortex pinning. It was suggested that the vortex system relaxes to a critical state in which the pinning force acting on a vortex is counterbalanced by the magnetic driving force. The latter force is the sum of the interactions of the vortices with one another, their images, Meissner currents, and extrinsic currents. The vortex system is described within this model using a continuous function, viz., the magnetic induction $B(x)$, which is related to the vortex density $n(x)$ by the local relation $B=n\Phi_0$. Generally speaking, this relation ceases to be valid if $n(x)$ varies sharply on scales of the order of the London penetration depth λ. A macroscopic model that takes nonlocal effects into account was developed in Refs. 3–6.

The purpose of the present work is to devise a simple microscopic critical-state model of a hard superconductor. Unlike the macroscopic model, in which averaging must be performed over spatial scales much greater than the intervortex spacing a, in the present paper we shall describe a vortex system using the coordinates of the individual vortices.

On the one hand, the proposed model is a microscopic formulation of the nonlocal critical-state model,[3–6] and, on the other hand, it makes it possible to investigate several effects that are lost in a macroscopic treatment. In the general case, this is a complicated many-particle problem, in which the vortices undergoing long-range interactions are situated in a random pinning potential.

In our treatment we use a very simple model of the pinning of the vortex system. We do not deal with the relationship between the critical current density and the actual structure of the pinning centers, which is the main concern of various versions of the collective pinning model.[7,8] Our purpose is to investigate the distribution of the vortices in a superconductor, particularly when the gradients of the vortex density $n(x)$ are large.

The assumptions under which the microscopic problem of the penetration of a field into a hard superconductor allows an exact solution are formulated in Sec. 2. The coordinates of the vortices are calculated in Sec. 3 for the case of an increasing external field. The vortex distribution when the external field begins to decrease after achieving a maximum value is investigated in Sec. 4. The stability of the solution against small displacements of the vortices from their equilibrium positions is considered. It is shown that the solution becomes unstable as the external field H_0 weakens if the distance from the surface to the nearest vortices is less than a certain finite value. In weak enough fields, this results in a new effect—the formation of a macroscopic vortex-free region. We note that no macroscopic vortex-free region appears in a soft superconductor when H_0 decreases.[9,10] The transition to the soft-superconductor limit is investigated in Sec. 5. The main purpose of that section is to demonstrate that our model reproduces the known results[9,10] in that limit. Strong pinning is investigated in Sec. 6. In this case it is easy to go over to the continuous limit, i.e., to a description of the system using the vortex density. In relatively strong fields such a macroscopic description is completely equivalent to the phenomenological description within the macroscopic nonlocal critical-state model.[3–6] In weaker fields a generalized nonlocal critical-state model that takes into account the

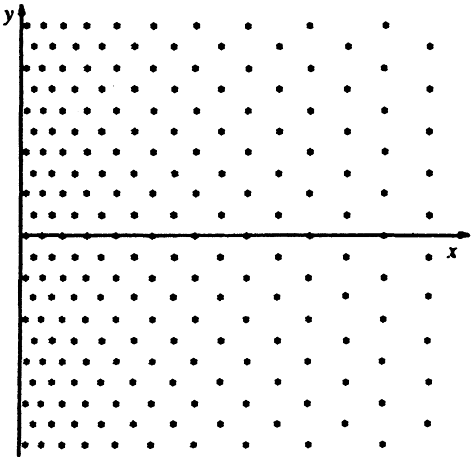

FIG. 1. Geometry of the problem. The positions of the vortices in the critical state are shown.

existence of a near-surface vortex-free region in a decreasing external field must be employed.

2. FORMULATION OF THE MICROSCOPIC CRITICAL-STATE MODEL

In this section, we devise a microscopic critical-state model of a hard superconductor which allows an exact solution, i.e., determination of the coordinates of each vortex. This problem can be solved exactly only under significant constraints affecting both the geometry of the problem and the character of the pinning. We now formulate these constraints. We investigate the simplest geometry: a superconducting half-space ($x>0$), penetrated by a magnetic field applied parallel to the surface ($\mathbf{H}_0 \| \mathbf{e}_z$) (see Fig. 1). We assume that the pinning force f^{pin} acting on a vortex is isotropic and can vary in the interval $(-p^{\text{pin}}, p^{\text{pin}})$, where p^{pin} is the critical pinning force, which is constant throughout the superconductor. Here the coordinate of a vortex varies if and only if the electromagnetic force exerted on it by the Meissner currents, the currents of other vortices, and their images exceeds p^{pin}. We assume that a vortex can be pinned at any point in the superconductor.

The assumptions made allow us to reduce the problem posed to a one-dimensional problem. In fact, the vortices are arranged in a system of rows due to the uniformity of the problem along the y axis. The Gibbs free energy expressed in terms of the coordinates of these rows x_i, which are numbered from the superconductor boundary, takes the form (see Ref. 11)

$$G_{\text{em}}(x_1,x_2,\dots,x_N)=\frac{\Phi_0^2}{16\pi\lambda}\sum_{i,j=1}^{N}\frac{1}{b_l}\left\{\exp\left(-\frac{|x_i-x_j|}{\lambda}\right)-\exp\left(-\frac{x_i+x_j}{\lambda}\right)\right\}+\frac{H_0\Phi_0}{4\pi}\sum_{i=1}^{N}\times\exp\left(-\frac{x_i}{\lambda}\right)+G_s(x_1), \tag{1}$$

where b_l is the distance between vortices in a row, N is the total number of rows in the system, and the function G_s is defined by the formula

$$G_s(x_1)=\frac{\Phi_0^2}{16\pi^2\lambda^2}\ln\left[1-\exp\left(-\frac{4\pi x_1}{b_l}\right)\right]. \tag{2}$$

The first sum in the expression for G_{em} corresponds to the interaction of the vortices with one another and with their images, and the second term corresponds to the interaction of the rows with the Meissner currents. The term G_s corresponds to the strong nonexponential attraction of the first row of vortices to its image (it was not taken into account in Ref. 11). The term $G_s(x_1)$ is significant only when the distance from the first row to the surface is less than $b_l/4\pi$. This occurs, for example, when a new row enters the superconductor. In this case the surface interaction G_s makes it possible to take into account the Bean–Livingston barrier,[12] which prevents the entry of vortices. As will be seen, G_s can always be neglected except when the entry of a row of vortices must be considered.

We now derive the equation describing the balance of forces acting on a vortex. Let there be N rows of vortices that have already entered the superconductor. Below, the number of vortices is self-consistently defined as a function of the external field H_0. Equating the magnetic force $-\partial G_{\text{em}}/\partial x_n$ that results from the interaction of vortices with one another, their images, and the Meissner currents, to the pinning force f^{pin}, we obtain

$$f_n^{\text{pin}}=-\frac{\partial G_{\text{em}}}{\partial x_n}=\frac{\Phi_0^2}{8\pi b_l\lambda^2}\left[\sum_{j=1}^{n-1}\exp\left(\frac{x_j-x_n}{\lambda}\right)-\sum_{j=n+1}^{N}\exp\left(\frac{x_n-x_j}{\lambda}\right)-\sum_{j=1}^{N}\exp\left(-\frac{x_j+x_n}{\lambda}\right)\right]+\frac{H_0\Phi_0}{4\pi\lambda}\exp\left(-\frac{x_n}{\lambda}\right),\qquad 1\le n\le N, \tag{3}$$

where f_n^{pin} is the pinning force on a vortex in the nth row. We henceforth assume that the distance b_l between vortices in a row, when the row forms on the boundary, does not vary when it penetrates further into the half-space as the external field increases. In addition, we assume that b_l is identical for all rows. The latter is a model assumption.

The system (3) is a nonlinear system of a macroscopically large number N of coupled equations. However, it has an exact solution. To obtain the solution we utilize the approach proposed in Ref. 13, where the problem of the formation of the critical state in a nonuniform Josephson junction was investigated. Despite the differing physical formulation, the equations obtained are identical. A solution of the system when all pinning forces acting on the vortices are known *a priori* and are equal to their maximum values was proposed in Ref. 13. Such a situation arises only during the initial increase in external field from zero to H_0. Only this case was considered by Bryksin and Dorogovtsev. The solution proposed in Ref. 13 can be generalized to an arbitrary magnetic history and applied to the problem (3) at hand.

Following Ref. 13, we introduce the auxiliary variables

$$a_1=\exp\left(-\frac{x_1}{\lambda}\right),\qquad a_2=\exp\left(-\frac{x_2-x_1}{\lambda}\right),\ldots,$$

$$a_i=\exp\left(-\frac{x_i-x_{i-1}}{\lambda}\right),\ldots,a_N=\exp\left(-\frac{x_N-x_{N-1}}{\lambda}\right); \qquad (4)$$

$$K_1=0,\qquad K_{i+1}=a_{i+1}(1+K_i), \qquad (5)$$

$$G_{N+1}=0,\qquad G_i=a_i(1+G_{i+1}). \qquad (6)$$

In these variables the system (3) takes the form

$$f_n=K_n-G_{n+1}+a_na_{n-1}\ldots a_1\cdot(h_0-G_1),\qquad 1\leqslant n\leqslant N, \qquad (7)$$

where $f_n=8\pi b_l\lambda^2 f_n^{\text{pin}}/\Phi_0^2$ and $h_0=2\lambda b_l H_0/\Phi_0$. The force f_n can vary over the range $-p\leqslant f_n\leqslant p$, where $p=8\pi b_l\lambda^2 p^{\text{pin}}/\Phi_0^2$. We use the dimensionless notation h everywhere below for the magnetic field. The dimensional analog H can be obtained using the relation $H=h\Phi_0/2\lambda b_l$.

It is convenient to use the following equation, which was obtained by summing the equations of the system (7) numbered from k to i (see Appendix A):

$$\sum_{m=k}^{i} f_m=(1+G_{k+1})(G_{k+1}+f_k)-G_{i+1}(G_{i+1}+f_i+1),$$

$$1\leqslant i\leqslant N,\qquad 1\leqslant k\leqslant i. \qquad (8)$$

The main advantage of this equation is that if the pinning forces f_m are known for $k\leqslant m\leqslant i$, we can specify all the G_k, except G_1, as functions of the single quantity G_{i+1}. To determine G_1 we must use Eq. (7) with $n=1$. Taking into account that $K_1=0$ and that $a_1=G_1/(1+G_2)$, we obtain

$$f_1=-G_2+\frac{G_1}{1+G_2}(h_0-G_1). \qquad (9)$$

Combining Eqs. (8) and (9), we obtain the relation for determining the variable G_1:

$$\sum_{m=1}^{i} f_m=G_1(h_0-G_1)-G_{i+1}(G_{i+1}+f_i+1). \qquad (10)$$

In Ref. 13 the summation in the system of equations (7) was carried out from k to the last row N. In this case the last term drops out of (8). The solution method proposed in Ref. 13 was restricted to the case in which all the pinning forces are known. The system of equations (8) and (10) generalizes Eq. (12) in Ref. 13 and makes it possible to consider an arbitrary critical state appearing during nonmonotonic variation of the external field. The algorithm for taking the history into account is similar to that used in the nonlocal macroscopic model.[3–6]

As the external field varies, the superconductor separates into critical and subcritical regions. In the former, the coordinates of the vortices vary with the field, and the absolute value of the pinning force is equal to the maximum permissible value. In the subcritical region, which is located beyond the critical region, the coordinates of the vortices remain the same as before the external field was varied, and the absolute value of the pinning force is less than p^{pin}. Thus, in the critical region, the pinning forces acting on the vortices are known, but the coordinates of the vortices are not specified *a priori*. Conversely, in the subcritical region, the coordinates of the vortices are known, but the value of the pinning force is adjusted self-consistently to counterbalance the variation of the magnetic force.

We now formulate these conditions mathematically for the variables introduced here G_i, a_i, and K_i. Let the boundary between the critical and subcritical regions pass between rows i_0-1 and i_0. The pinning force on the rows of vortices from the first to the (i_0-1)th row is known *a priori*, and is equal to $\pm p$. In the subcritical region, the coordinates of the vortices do not vary with the field, i.e., the values of $x_{i_0+1}-x_{i_0}$, $x_{i_0+2}-x_{i_0+1},\ldots,$ x_N-x_{N-1} are conserved. Now using (4) and (6), we conclude that $a_{i_0+1},\ldots,a_N$ and $G_{i_0+1},\ldots,G_N$ remain unchanged as the external field varies. Finally, the last condition which must be added for an unequivocal formulation of the problem follows from the constancy of the position of the i_0th row $(x_{i_0}=x_{i_0}-x_{i_0-1}+x_{i_0-1}-x_{i_0-2}+\ldots+x_2-x_1+x_1)$:

$$a_1a_2\ldots a_{i_0}=\frac{G_1}{1+G_2}\,\frac{G_2}{1+G_3}\cdots\frac{G_{i_0}}{1+G_{i_0+1}}=\tilde{a}_1\tilde{a}_2\ldots\tilde{a}_{i_0}, \qquad (11)$$

where the tilde over each a_i indicates that the respective quantity was calculated before the external field was varied.

In the next sections, the proposed approach is implemented to calculate the coordinates of rows of vortices in systems with a specified magnetic history.

3. DISTRIBUTION OF THE COORDINATES OF ROWS OF VORTICES IN AN INCREASING EXTERNAL FIELD

Let us now solve an actual problem. Consider the critical state formed during an initial increase in the external field from zero to h_0. In this case only a critical region bordering the space free of vortices appears. The pinning force of maximum absolute value acts on all the vortices; therefore, we must set $i=N$ in Eqs. (8) and (10). Utilizing the fact that $G_{N+1}=0$, we have

$$(N-k+1)p=(1+G_{k+1}^{\text{inc}})(G_{k+1}^{\text{inc}}+p), \qquad (12)$$

$$Np=G_1^{\text{inc}}(h_0-G_1^{\text{inc}}), \qquad (13)$$

where the superscript means that the solutions refer to the initial increase in the external field. Solving the last two quadratic equations, we obtain

$$G_{k+1}^{\text{inc}}=-(p+1)/2+\gamma_k^{\text{inc}},\qquad 1\leqslant k\leqslant N, \qquad (14)$$

$$G_1^{\text{inc}}=h_0/2-\gamma_0^{\text{inc}}, \qquad (15)$$

where γ_k^{inc} and γ_0^{inc} are defined by

$$\gamma_k^{\text{inc}}=\tfrac{1}{2}\sqrt{(p+1)^2+4(N-k)p}, \qquad (16)$$

$$\gamma_0^{\text{inc}}=\tfrac{1}{2}\sqrt{h_0^2-4Np}. \qquad (17)$$

The minus sign preceding γ_0^{inc} in (15) was chosen in order that the solution be stable against small perturbations: the solution with the opposite sign is unstable (see Appendix B). Using the expressions obtained for G_k, we find

$$a_1^{\rm inc}=\exp\left(-\frac{x_1}{\lambda}\right)=\frac{G_1^{\rm inc}}{1+G_2^{\rm inc}}=\frac{h_0-2\gamma_0^{\rm inc}}{1-p+2\gamma_2^{\rm inc}}, \quad (18)$$

$$a_k^{\rm inc}=\exp\left(-\frac{x_k-x_{k-1}}{\lambda}\right)=\frac{G_k^{\rm inc}}{1+G_{k+1}^{\rm inc}}=\frac{-1-p+2\gamma_{k-1}^{\rm inc}}{1-p+2\gamma_k^{\rm inc}}. \quad (19)$$

To complete the solution of the problem, the total number of rows N that have entered the system must be determined. Consider the successive entry of rows. As we have already noted above, this requires consideration of the strong nonexponential attraction between vortices formed on the surface and their own images: $-\partial G_s/\partial x_1(x_1=0)$. This surface force prevents the entry of a newly formed row into the superconductor until it has been cancelled by the magnetic force created by other rows of vortices, their images, and the Meissner currents. At a certain external field, the surface force is cancelled by the magnetic force and the current row of vortices enters the superconductor. As a result, we have the following equation for the field $h_{\rm inc}^{N+1}$ for the entry of row $N+1$:

$$\left.\frac{\partial G_{\rm em}}{\partial x_1}\right|_{x_1=0}=-2G_1^{\rm inc}(N)+h_{\rm inc}^{N+1}-h_s=0, \quad (20)$$

where $h_s=2b_l\lambda H_s/\Phi_0$, and $H_s=\Phi_0/4\pi\xi\lambda$ is the thermodynamic field. In Eq. (20), $G_1^{\rm inc}(N)$ is given by

$$G_1^{\rm inc}(N)=[h_{\rm inc}^{N+1}-\sqrt{(h_{\rm inc}^{N+1})^2-4Np}]/2.$$

Solving (20), we find the field

$$h_{\rm inc}^{N+1}=\sqrt{h_s^2+4Np}. \quad (21)$$

This expression can be written in the dimensional form

$$H_{\rm inc}^{N+1}=\sqrt{H_s^2+8\pi N\Phi_0 J_c/b_l c}, \quad (22)$$

where we have introduced the notation $J_c=cp^{\rm pin}/\Phi_0$. Over the range of external fields $h_{\rm inc}^N<h_0<h_{\rm inc}^{N+1}$, the number of rows in the system equals N. Now it is easy to obtain the relation for the number of rows for a given external field h_0:

$$N=\left[\frac{h_0^2-h_s^2}{4p}\right]+1, \quad (23)$$

where the integer part of the expression enclosed in square brackets is taken. If the external field $h_0<h_s$, there are no vortices in the superconductor.

Equations (16)–(19) and (23) completely define the coordinates of all rows of vortices in the superconductor during the initial increase in the field. For arbitrary p, the expressions are very cumbersome and difficult to analyze analytically. Therefore, we present here only numerically calculated plots of the dependence of the row density, defined as

$$n_k^*(x_k)=\frac{1}{x_k-x_{k-1}}, \quad (24)$$

on the distance x_i to the superconductor surface for different values of the pinning force (Fig. 2). To construct these plots we use the exact expression for the vortex row density

$$n_k^{*-\rm inc}(x_k)=\left(\lambda\ \ln\frac{1-p+\sqrt{(1+p)^2+4(N-k)p}}{-1-p+\sqrt{(1+p)^2+4(N+1-k)p}}\right)^{-1}, \quad (25)$$

which follows directly from the solution that we obtained.

The following features of the distribution of n_k^* can be seen in the figure. Near the surface there is a vortex-free region, whose thickness is considerably greater than the distance between rows. The vortex density vanishes abruptly at the vortex front. The position dependence of the density is almost always nearly linear.

The case of an initial increase in the external field considered in this section is very simple. All pinning forces are known *a priori*, their absolute values being equal to the maximum values. More interesting is the case in which the field begins to decrease after the initial increase. A subcritical region in which the pinning forces are not known *a priori*, then appears. The next section is devoted to an investigation of this case.

4. DECREASING EXTERNAL FIELD

We now consider the critical state formed when the field varies nonmonotonically. We assume that the external field begins to decrease after reaching $h_{\rm max}$. The vortex coordinates do not vary until the change in the magnetic force becomes sufficient for depinning of the first row.

Thus, there is a range of external fields, in which under any nonmonotonic variation of h_0 the coordinates of the rows are found as before from Eqs. (16)–(19) with $h_0=h_{\rm max}$ and the G_k maintain the values corresponding to the maximum field $h_{\rm max}$: $G_k=G_k^{\rm inc}(h_{\rm max})$. This state is maintained down to a certain value of the external field $h_{\rm dec}^{(1)}$, at which the absolute value of the pinning force on the first row reaches its maximum value (the subscript "dec" in $h_{\rm dec}^{(1)}$ indicates that the external field is decreasing, and the superscript 1 shows that depinning of the first row is being considered). According to (9), in the external field $h_0=h_{\rm max}$ we have

$$p=-G_2^{\rm inc}(h_{\rm max})+a_1^{\rm inc}[h_{\rm max}-G_1^{\rm inc}(h_{\rm max})].$$

In the field $h_0=h_{\rm dec}^{(1)}$ the pinning force on the first row reaches a critical value of opposite sign, but the coordinate of this row still does not vary. Thus, we have

$$-p=-G_2^{\rm inc}(h_{\rm max})+a_1^{\rm inc}[h_{\rm dec}^{(1)}-G_1^{\rm inc}(h_{\rm max})].$$

Subtracting the last two equalities from one another, we obtain

$$\Delta h=h_{\rm max}-h_{\rm dec}^{(1)}=\frac{2p}{a_1^{\rm inc}(h_{\rm max})}. \quad (26)$$

When the field decreases below $h_{\rm dec}^{(1)}$, initially the first and then the second and ensuing rows of vortices are depinned and move toward the surface. We first examine a range of fields in which no row has yet exited the superconductor. It is fairly easy to find the coordinates of the vortices at discrete values of the external field $h_{\rm dec}^{(i_0)}$, at which the pinning force acting on row i_0 has already achieved its critical value, but the coordinate of that row has not yet varied.

We now calculate the coordinates of the vortex rows in the fields $h_{\rm dec}^{(i_0)}$. The problem can be conveniently solved in the following manner: we first determine the coordinates of the vortices for a given value of i_0, utilizing the fact that they remain unchanged for $i \geqslant i_0$ and the pinning forces are equal to the critical values for $i \leqslant i_0$. We note that the two conditions hold simultaneously for $i = i_0$: the coordinate of the row is unchanged, and the pinning force is equal to the critical value. This makes it possible to calculate the coordinates of all rows and then to find the external field at which such a distribution of the vortices is realized.

Since the coordinates of the vortices in the i_0th row and thereafter remain the same in the field $h_{\rm dec}^{(i_0)}$ as they were in $h_{\max}$, it is obvious that

$$a_k^{\rm dec}(h_{\rm dec}^{(i_0)}) = a_k^{\rm inc}(h_{\max}),$$

$$G_k^{\rm dec}(h_{\rm dec}^{(i_0)}) = G_k^{\rm inc}(h_{\max}), \qquad k > i_0. \tag{27}$$

The values of the variables $G_k^{\rm dec}(h_{\rm dec}^{(i_0)})$ for $2 \leqslant k \leqslant i_0$ can be found from Eq. (8) by setting $i = i_0$:

$$-(i_0-k+1)p = [1+G_{k+1}^{\rm dec}(h_{\rm dec}^{(i_0)})][G_{k+1}^{\rm dec}(h_{\rm dec}^{(i_0)})-p] - G_{i_0+1}^{\rm inc}(h_{\max})[G_{i_0+1}^{\rm inc}(h_{\max})-p+1]. \tag{28}$$

Solving (28), we find

$$G_{k+1}^{\rm dec}(h_{\rm dec}^{(i_0)}) = -\frac{1-p}{2} + \gamma_k^{\rm dec}(h_{\rm dec}^{(i_0)}), \qquad 1 \leqslant k < i_0, \tag{29}$$

where

$$\gamma_k^{\rm dec}(h_{\rm dec}^{(i_0)}) = \tfrac{1}{2}\sqrt{(1+p)^2+4p^2+4(N-i_0)p-4(i_0-k)p-4p\sqrt{(p+1)^2+4(N-i_0)p}}. \tag{30}$$

The sign preceding $\gamma_k^{\rm dec}(h_{\rm dec}^{(i_0)})$ in (29) was chosen in order that the solution be stable (for further details see Appendix B). Since in the discrete field under consideration the coordinates of the i_0th row are still the same as when the external field increased to $h_{\max}$, the condition (11) holds. This condition enables us to obtain the value of the last undetermined variable $G_1^{\rm dec}(h_{\rm dec}^{(i_0)})$:

$$G_1^{\rm dec}(h_{\rm dec}^{(i_0)}) = \prod_{m=2}^{i_0} \frac{G_m^{\rm inc}(h_{\max})[1+G_m^{\rm dec}(h_{\rm dec}^{(i_0)})]}{G_m^{\rm dec}(h_{\rm dec}^{(i_0)})[1+G_m^{\rm inc}(h_{\max})]}. \tag{31}$$

We now know all the values of the variables $G_k^{\rm dec}(h_{\rm dec}^{(i_0)})$, i.e., the coordinates of all the vortex rows in a certain field $h_{\rm dec}^{(i_0)}$. We now determine the strength of the field $h_{\rm dec}^{(i_0)}$ in which the respective state is realized. Using (9), we have

$$h_{\rm dec}^{(i_0)} = \frac{[G_1^{\rm dec}(h_{\rm dec}^{(i_0)})]^2+[G_2^{\rm dec}(h_{\rm dec}^{(i_0)})-p][1+G_2^{\rm dec}(h_{\rm dec}^{(i_0)})]}{G_1^{\rm dec}(h_{\rm dec}^{(i_0)})} = \frac{[G_1^{\rm dec}(h_{\rm dec}^{(i_0)})]^2+(N-2i_0)p+p(p+1)-p\sqrt{(1+p)^2+4(N-i_0)p}}{G_1^{\rm dec}(h_{\rm dec}^{(i_0)})}. \tag{32}$$

Equations (27) and (29)–(31) specify the coordinates of the vortex rows in the discrete set of external fields (32).

The problem becomes considerably more complicated in the intermediate fields $h_0 \neq h_{\rm dec}^{(i_0)}$. The investigation of this case has not yielded any new results, and we shall therefore not present it.

We are next interested in the following problem: when is the solution (27) and (29)–(32) constructed with a certain fixed value of N stable? It bears on the conditions under which vortices exit the superconductor, and in turn helps one formulate specific boundary conditions on the superconductor surface in going over to a macroscopic description (see Sec. 6).

As shown in Appendix B, the stable solutions correspond to a specific choice of the root of the quadratic equation (9) when G_1 is defined as a function of the external field h_0:

$$G_1^{\rm dec}(h_0) = h_{\rm dec}/2 \pm \gamma_0^{\rm dec}, \tag{33}$$

where

$$\gamma_0^{\rm dec} = \frac{1}{2}\sqrt{h_{\rm dec}^2 - 4(G_2^{\rm dec}-p)(1+G_2^{\rm dec})}. \tag{34}$$

Conversely, in the solution (27)–(32) constructed above we determine the field h_0 from the known value of G_1. Here it is unclear whether the solution constructed corresponds to the stable case (the minus sign in (33)) or the unstable case (the plus sign). In fact, the same formula (32) is obtained for $h_{\rm dec}(G_1^{\rm dec}, G_2^{\rm dec})$ in both cases of $G_1^{\rm dec}$.

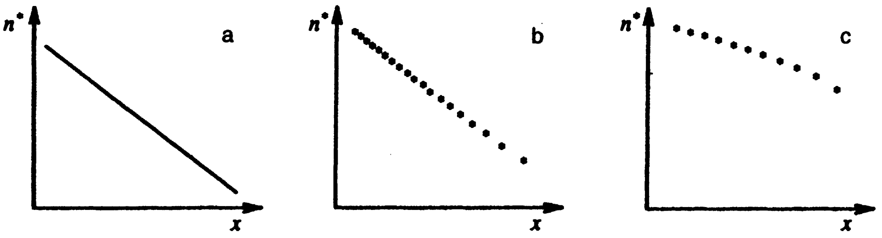

FIG. 2. Vortex row density n^* for various values of the pinning force: a) $p=100$, b) $p=1$, c) $p=0.01$.

Thus, the investigation of the conditions for the exit of vortices from the superconductor reduces to determining the external field at which the stable solution is replaced by the unstable solution. Figure 3 graphically illustrates just how the transition from the stable solution to the unstable solution occurs. The parabola depicted in this figure corresponds to the entire set of solutions of $G_1(h_0)$. The left-hand branch of the parabola corresponds to the stable solutions, and the right-hand branch corresponds to the unstable solution. If a stable solution (G_1, h_0) (the filled point in Fig. 3) is realized in a certain external field for a certain fixed total number of rows in the superconductor, the point (G_1, h_0) will move downward along the parabola as h_0 is subsequently reduced. In a certain external field the point (G_1, h_0) reaches the minimum. In weaker fields no stable solution exists for the respective value of N, and the row nearest the surface exits the superconductor. The number of vortex rows in the superconductor then changes, and the point (G_1, h_0) abruptly moves over to a site on the stable branch of the parabola.

Let us find the point at which the stable solution vanishes. If the solution is stable, substituting $h_{\rm dec}(G_1^{\rm dec}, G_2^{\rm dec})$ from (32) into (33) with a minus sign should obviously yield an identity. After performing this substitution, we have

$$G_1^{\rm dec}=\frac{(G_1^{\rm dec})^2+(G_2^{\rm dec}-p)(1+G_2^{\rm dec})}{2G_1^{\rm dec}}$$

$$-\frac{\sqrt{[(G_1^{\rm dec})^2-(G_2^{\rm dec}-p)(1+G_2^{\rm dec})]^2}}{2G_1^{\rm dec}}.$$

The latter expression becomes the identity $G_1^{\rm dec}=G_1^{\rm dec}$, if

$$(G_1^{\rm dec})^2\leqslant(G_2^{\rm dec}-p)(1+G_2^{\rm dec}).$$

This can be brought to the form

$$a_1^{\rm dec}=\exp\left(-\frac{x_1^{\rm dec}}{\lambda}\right)\leqslant a_1^{\rm unst}=\sqrt{\frac{G_2^{\rm dec}-p}{1+G_2^{\rm dec}}}\leqslant 1. \qquad (35)$$

Thus, we conclude that states with $a_1^{\rm dec}\leqslant a_1^{\rm unsat}$ cannot be realized, since they will be unstable. Thus, as the external field is reduced, a vortex-free region appears near the surface:

$$0<x<x_{\rm unst}=-\lambda\ \ln a_1^{\rm unst}. \qquad (36)$$

In strong fields its thickness coincides in order of magnitude with the distance between vortices. However, in weak enough fields (see Sec. 6) this region becomes comparable to λ, and it must be taken into account in the macroscopic nonlocal critical-state model.

After any number of vortex rows have exited the superconductor, the calculation algorithm described above must, generally speaking, be repeated with the new number of vortex rows N.

It is difficult to analyze the formulas obtained in the preceding sections for arbitrary values of p. Therefore, below we shall examine the two limiting values $p\to 0$ and $p\gg 1$.

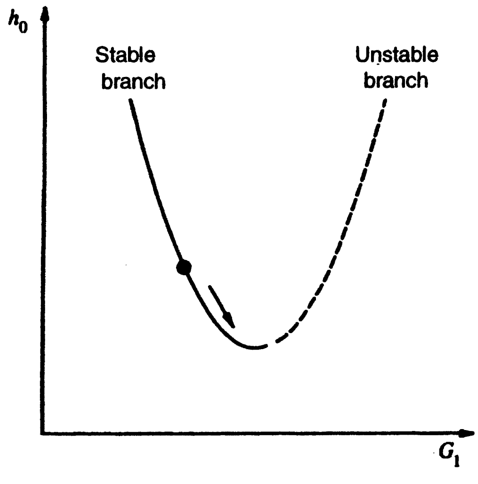

FIG. 3. Stable and unstable branches of the solution of the problem. The arrow shows the evolution of the stable solution as the external field decreases.

5. SOFT-SUPERCONDUCTOR LIMIT

Let us examine the formal transition to the soft-superconductor limit, $p\to 0$. We prove that in this limit the solutions obtained reproduce all known results.[9,10]

We first investigate the case of an increasing external field. Using Eq. (23), we find that the number of rows in a superconductor in fields $h_0>h_s$ tends to infinity: $N(h_0>h_s, p\to 0)\to\infty$. However, the product Np is then finite and proportional to the magnetic induction $B_0=\Phi_0/ab_l$, where a is the distance between rows. Letting p tend to zero in (18) and (19) and taking into account that the product Np is finite, we obtain

$$\exp\left(-\frac{a}{\lambda}\right)=\frac{-1+\sqrt{1+4Np}}{1+\sqrt{1+4Np}},$$

$$\exp\left(-\frac{x_1^{\rm inc}}{\lambda}\right)=\frac{h_0-\sqrt{h_0^2-4Np}}{1+\sqrt{1+4Np}}, \qquad (37)$$

where $x_1^{\rm inc}$ is the distance from the first row to the boundary. It is clear from (37) that the distance between rows becomes much smaller than λ when $\sqrt{NP} \gg 1$. In this case, expanding $\exp(-a/\lambda)$ into a series and neglecting unity in comparison with Np, from Eq. (37) we obtain

$$\sqrt{\frac{\Phi_0}{B_0}} \sim a = \frac{\lambda}{\sqrt{Np}}, \qquad \exp\left(-\frac{x_1^{\rm inc}}{\lambda}\right) = \frac{h_0 - \sqrt{h_0^2 - 4Np}}{2\sqrt{Np}}. \tag{38}$$

Eliminating Np from the last expression and transforming to dimensional variables, we obtain the known results (see Refs. 9 and 10)

$$H_0 = \sqrt{H_s^2 + \left(\frac{\Phi_0}{ab_l}\right)^2} = \sqrt{H_s^2 + B_0^2},$$

$$\cosh\left(\frac{x_1^{\rm inc}}{\lambda}\right) = \frac{H_0}{B_0}. \tag{39}$$

When the external field increases, it follows from these equations that a near-surface vortex-free region $0 < x < x_l^{\rm inc}$ exists in a soft superconductor, and that the magnetic induction B_0 does not coincide with the external field H_0.

When the external field decreases and p tends to zero, the boundary between the critical and subcritical regions goes to infinity, and all $G_{k \neq 1}^{\rm dec}$ become equal. To determine the thickness of the vortex-free region $x_l^{\rm dec}$, we write the chain of consequences

$$\exp\left(-\frac{x_1^{\rm dec}}{\lambda}\right) = a_1^{\rm unst} = \sqrt{\frac{G_2^{\rm dec} - p}{1 + G_2^{\rm dec}}} \xrightarrow{p \to 0} \sqrt{\frac{G_2^{\rm dec}}{1 + G_3^{\rm dec}}} = \sqrt{a_2}.$$

This relation implies that

$$x_1^{\rm dec} = a/2. \tag{40}$$

Clearly, a macroscopic vortex-free region does not appear in a decreasing field. The lack of this region leads to the fact (see, for example, Ref. 10) that the magnetic induction within a soft superconductor in the macroscopic approximation simply coincides with the external field at the surface.

Thus, in the soft-superconductor limit ($p = 0$), our model reduces to the model of Ref. 10. Here we shall not investigate the case of weak pinning in detail, since it was studied in Ref. 13.

6. STRONG-PINNING LIMIT

In this section we are interested in the case $p \gg 1$. The treatment is simplified considerably in this case, since the transition to the continuous limit is fairly easy. In the continuous limit, the system of vortices is described using the continuous vortex density n, rather than the coordinates of the individual vortices. The transition to a macroscopic description is thus accomplished, and the model presented above should transform into a critical-state model. In this process we do not average over spatial scales greater than the London penetration depth λ. Therefore, the microscopic model should transform into the nonlocal critical-state model,[3–6] rather than the local Bean model.[1]

The main difference between the nonlocal model and the local model is that, generally speaking, the relationship between the magnetic induction and the vortex density $B = n\Phi_0$ does not hold in the former. These quantities are related by

$$B - \lambda^2 \frac{d^2 B}{dx^2} = n\Phi_0. \tag{41}$$

The following features of the vortex density distribution were obtained within the macroscopic nonlocal critical-state model.[3–6] It has discontinuities at the boundary between the critical and subcritical regions, as well as at the front of the vortex distribution. There is a range of external fields over which variation of the field does not lead to variation of the vortex density. When the Bean–Livingston surface barrier is taken into account, a near-surface vortex-free region appears.

The purpose of this section is to demonstrate that all the features just indicated are obtained from the exact microscopic solution, as well as to investigate the boundary conditions for a decreasing external field, which cannot be obtained via the macroscopic approach.

We introduce the row density n^* (24) and the vortex density $n = n^*/b_l$. (Everywhere below we present the final formulas in two forms: for n^* and for n.) In the general case, these are functions of the discrete variable x_k; however, in the limit under consideration it is possible to replace the discrete function by a continuous function and determine its derivative.

The distribution of the coordinates of the vortices during an increase in the field from zero to $h_0 > h_s$ can be described by Eqs. (16)–(19) and (23). We then obtain the relation for the row density:

$$\exp\left(-\frac{1}{n_k^* \lambda}\right) = \frac{-1 - p + 2\gamma_{k-1}^{\rm inc}}{1 - p + 2\gamma_k^{\rm inc}}. \tag{42}$$

Since according to (16) the inequality $2\gamma_k^{\rm inc} \geq p \gg 1$ always holds, we can go to the strong-pinning limit by expanding this expression in the small parameter $1/\gamma_k$. We ultimately obtain

$$n_k^* \approx \gamma_k^{\rm inc}/\lambda \tag{43}$$

for all k. At the fromt of the vortex distribution x_N, the density is given by

$$n_N^* \approx \frac{p}{2\lambda}, \qquad n(x_N) = \frac{4\pi}{c\Phi_0} J_c \lambda, \tag{44}$$

where the critical current density J_c is defined in (22).

We now consider the variation of the row density from row to row, and show that it is small everywhere in comparison with n_k^*. In fact, we have

$$\Delta n_k^* = n_k^* - n_{k-1}^* = \frac{\gamma_k^{\rm inc} - \gamma_{k-1}^{\rm inc}}{\lambda} = -\frac{p}{2\lambda \gamma_k^{\rm inc}}. \tag{45}$$

The following condition clearly holds up to the front itself:

$$\frac{|\Delta n_k^*|}{n_k^*} \sim \frac{p}{\gamma_k^2} \sim \frac{1}{p} \ll 1.$$

Thus, a continuous function of the vortex row density can be introduced over the interval (x_1, x_N). In addition, since $n_N^* \gg \Delta n_N^*$, the density jumps to zero at the vortex front. The magnitude of this jump is specified by Eq. (44).

We now obtain a differential equation for the vortex row density. Having defined the derivative of the row density with respect to the coordinate x as

$$\frac{dn^*}{dx} = \frac{\Delta n_k^*}{x_k - x_{k-1}} = n_k^* \Delta n_k^* \tag{46}$$

and using (43) and (45), we obtrain

$$\frac{dn^*}{dx} = -\frac{p}{2\lambda^2}, \qquad \Phi_0 \frac{dn}{dx} = -\frac{4\pi}{c} J_c. \tag{47}$$

The latter equation describes a linear decrease in the vortex density with increasing depth in the superconductor with the coefficient $4\pi J_c / c\Phi_0$. Thus, the previously introduced quantity J_c becomes the critical current density of the superconductor in the macroscopic critical-state model.[3]

Let us now consider the near-surface vortex-free region. We must determine the thickness of the vortex-free region x_1^{inc} and the value of the vortex density at its boundary $n(x_1^{\text{inc}})$. We investigate the situation in which the number of vortex rows in the system is large, $N \gg 1$. Neglecting unity in comparison with N, we find $4Np \approx h_0^2 - h_s^2$. Using this equality and (43), we obtain

$$n^*(x_1^{\text{inc}}) = \frac{\sqrt{p^2 + h_0^2 - h_s^2}}{2\lambda},$$

$$\Phi_0 n(x_1^{\text{inc}}) = \sqrt{H_0^2 - H_s^2 + \left(\frac{4\pi}{c} J_c \lambda\right)^2}. \tag{48}$$

From (18) we find the thickness of the vortex-free region:

$$x_1^{\text{inc}} = \lambda \ln\left\{ \sqrt{\frac{H_0 + H_s}{H_0 - H_s} + \left[\frac{4\pi J_c \lambda}{c(H_0 - H_s)}\right]^2} + \frac{4\pi J_c \lambda}{c(H_0 - H_s)} \right\}. \tag{49}$$

Let us now consider the case of a decreasing external field. Equation (26) for the range of fields over which all vortices remain pinned as H_0 varies takes the following form in dimensional quantities:

$$\Delta H = \frac{8\pi}{c} J_c \lambda \exp\left(-\frac{x_1^{\text{inc}}}{\lambda}\right). \tag{50}$$

This expression coincides with the corresponding relation obtained via the macroscopic nonlocal critical-state model.[6]

When the field decreases further, a critical region appears near the surface. The coordinates of the vortices in it deviate from the values for $h_{\max}$, and the absolute value of the pinning force takes its maximum value. As $p \to \infty$, the interval between the discrete fields $h_{\text{dec}}^{(i_0)}$ at which depinning of the vortex rows occurs tends to zero. For this reason we analyze only the solutions corresponding to $h_{\text{dec}}^{(i_0)}$.

We first obtain the differential equation specifying $n(x)$ in the critical region. Using (29), we can write

$$\exp\left(-\frac{x_k - x_{k-1}}{\lambda}\right) = \exp\left(-\frac{1}{n_k^{\text{dec}}\lambda}\right) = \frac{p - 1 + 2\gamma_{k-1}^{\text{dec}}}{p + 1 + 2\gamma_k^{\text{dec}}}. \tag{51}$$

Comparing the expressions for γ_k^{dec} and $\gamma_{k-1}^{\text{dec}}$ (see (20)), we obtain

$$\gamma_{k-1}^{\text{dec}} = \sqrt{(\gamma_k^{\text{dec}})^2 - p} \approx \gamma_k^{\text{dec}} - p/2\gamma_k^{\text{dec}}. \tag{52}$$

We assume that as before, the condition $x_i - x_{i-1} \ll \lambda$ holds everywhere and that $\gamma_k^{\text{dec}} \sim p$. We shall see below that these requirements remain valid even in weak fields; therefore, this assumption does not impose any additional constraints on h_0, except for the one requirement that $h_0 \geqslant 0$. All subsequent mathematical manipulations are completely analogous to those in an increasing field. Performing them, we obtain

$$n_k^{*-\text{dec}} = \gamma_k^{\text{dec}}/\lambda, \tag{53}$$

$$\frac{dn^{*-\text{dec}}}{dx} = \frac{p}{2\lambda^2}, \qquad \Phi_0 \frac{dn^{\text{dec}}}{dx} = \frac{4\pi}{c} J_c. \tag{54}$$

The last equation specifies the vortex density distribution in the critical region. In the subcritical region the density remains the same as that produced during the initial increase in the field. In order to completely determine the distribution $n(x)$ in the continuous limit, the matching conditions for the vortex density on the boundary between the critical and subcritical regions and the boundary conditions on the superconductor surface must be known.

Let us find the condition relating the vortex densities on opposite sides of the boundary x_0 between the critical and subcritical regions. We utilize the fact that at $x < x_0$ the density n^* is equal to $n^{*-\text{dec}}$, while at $x > x_0$ it equals $n^{*-\text{inc}}$. On the basis of (16) and (30), we arrive at the relation

$$\gamma_{i_0}^{\text{dec}} = \sqrt{(\gamma_{i_0}^{\text{inc}} - p)^2} = |\gamma_{i_0}^{\text{inc}} - p|. \tag{55}$$

Using (43), (53), and (55), we ultimately obtain

$$n^{*-\text{dec}}(x_0) = n^{*-\text{inc}}(x_0) - p/\lambda,$$

$$n^{\text{dec}}(x_0) = n^{\text{inc}}(x_0) - 8\pi J_c \lambda / c\Phi_0. \tag{56}$$

The absolute value is omitted in (56), since the appearance of a subcritical region at $H_0 > 0$ requires that $H_{\max} > 8\pi J_c \lambda / c$, which implies, in turn, that $n^{\text{inc}}(x_0^{\text{dec}}) > 8\pi J_c \lambda / c\Phi_0$.

We now consider the boundary conditions at the superconductor surface. If we have a situation in which no vortex has yet exited the superconductor, to calculate $n(x)$ it is sufficient to know how the values of the vortex density on the boundary between the critical and subcritical regions $x = x_0^{\text{dec}}$ are related. It is easy to verify that this condition, together with the condition for conservation of the total number of vortices, uniquely specifies the solution of the problem (see Ref. 6).

If the external field decreases so much that vortices begin to exit the superconductor, another condition is needed in addition to that specifying the density change at the boundary between the critical and subcritical regions. The latter depends on whether a macroscopic near-surface vortex-free region $0 < x < x_1^{\text{dec}}$ exists or not. If there is no such region, we obtain the following boundary condition for the vortex density by virtue of the continuity of the magnetic induction at the superconductor surface:

$$n^{\text{dec}}(x = 0,\ x_1^{\text{dec}} \approx 0) = H_0 / \Phi_0. \tag{57}$$

If a macroscopic vortex-free region exists, the additional condition is the relationship between the vortex density at the boundary of this region, its thickness, and the critical current density. As we have already noted, when the field decreases below $h_{\rm dec}^{(1)}$, the vortices nearest the surface gradually begin to approach the surface. As shown in Sec. 4 and Appendix B, the state of the vortex system becomes unstable when the first row of vortices is at a distance smaller than $x_{\rm unst}$ (36) from the surface. The first row of vortices then exits the superconductor, and a near-surface vortex-free region forms between the sample surface and the point $x_{\rm unst}$. If the thickness of this region is much greater than the distance between rows, it becomes macroscopic; otherwise, it can be omitted from the macroscopic description. We note that the stability of the vortex rows near the surface cannot be correctly analyzed in the macroscopic nonlocal critical-state model,[3–6] and therefore such a vortex-free region does not appear in it.

We now obtain the conditions specifying the thickness of the vortex-free region and show that it becomes macroscopic in weak enough fields. Using (36), we obtain

$$a_1^{\rm unst}=\exp\left(-\frac{x_{\rm unst}}{\lambda}\right)\approx\sqrt{\frac{n^{*-\rm dec}(x_{\rm unst})-p/2\lambda}{n^{*-\rm dec}(x_{\rm unst})+p/2\lambda}}$$

$$=\sqrt{\frac{\Phi_0 n^{\rm dec}(x_{\rm unst})-(4\pi/c)J_c\lambda}{\Phi_0 n^{\rm dec}(x_{\rm unst})+(4\pi/c)J_c\lambda}}. \qquad (58)$$

It is clear from the latter expression that as the external field and, consequently, the vortex density near the surface decrease, the distance $x_{\rm unst}$ increases and diverges when $\Phi_0 n^{\rm dec}(x_{\rm unst})\to 4\pi J_c\lambda/c$. We note that this divergence should be eliminated by cutting off $x_{\rm unst}$ at scales of order λ, since the terms of order unity were neglected in comparison to p during the derivation of (58). Thus, it is clear from everything said that consideration of the near-surface vortex-free region is essential at least in weak fields.

Thus, for weak fields, the macroscopic nonlocal critical state-model[3–6] must be generalized with consideration of the presence of the near-surface vortex-free region. This can be done by introducing an additional condition. Since $B(x)^{\rm dec}=\Phi_0 n^{\rm dec}(x)$ in the critical region (see, for example, Ref. 3), from the equality (58) we obtain

$$B^{\rm dec}(x_{\rm unst})\sinh\left(\frac{x_{\rm unst}}{\lambda}\right)=\frac{4\pi}{c}J_c\lambda\cosh\left(\frac{x_{\rm unst}}{\lambda}\right). \qquad (59)$$

In the vortex-free region the magnetic induction is determined by Eq. (41) with $n=0$:

$$B^{\rm dec}(x, 0<x<x_{\rm unst})$$
$$=H_0\cosh\left(\frac{x}{\lambda}\right)$$
$$-\frac{H_0\sinh(x_{\rm unst}/\lambda)-4\pi J_c\lambda/c}{\cosh(x_{\rm unst}/\lambda)}\sinh\left(\frac{x}{\lambda}\right). \qquad (60)$$

Substituting this solution, taken at the point $x=x_{\rm unst}$, into the condition (59) and solving the resulting equation relative to the thickness of the vortex-free region, we find

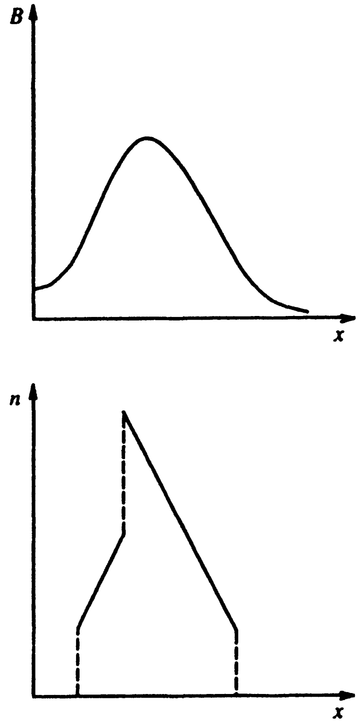

FIG. 4. Distribution of the magnetic induction and the vortex density in a superconductor when the external field decreases and there is a macroscopic vortex-free region.

$$x_{\rm unst}=\lambda\ \text{arcsinh}\left(\frac{4\pi J_c\lambda}{cH_0}\right). \qquad (61)$$

It follows from the latter that the vortex-free region is in fact macroscopic, at least in weak fields. It vanishes in sufficiently strong fields, in which $x_{\rm unst}$ is of the same order as the distance between vortices $a\sim\sqrt{\Phi_0/B}$, i.e., in fields $H\sim H^*$, where the field H^* in which the vortex-free region vanishes is specified by the following equality:

$$H^*=\left(\frac{4\pi}{c}J_c\lambda\right)^2\left(\frac{\Phi_0}{\lambda^2}\right)^{-1}. \qquad (62)$$

Thus, the discussion in this section shows that the microscopic model goes over completely to the nonlocal critical-state model[6] in the limit $p\gg 1$ in the case of an increasing external field. When the external field decreases, the microscopic model coincides with the nonlocal critical-state model at fields $H_0>H^*$. When $H_0<H^*$, the near-surface vortex-free region must be taken into account. We note that when $J_c\to 0$, we have $x_{\rm unst}\to 0$, i.e., the macroscopic vortex-free region under consideration is not present in a soft superconductor. An example of the distributions of the magnetic induction and the vortex density with consideration of the near-surface vortex-free region under discussion is presented in Fig. 4.

7. CONCLUSIONS

A microscopic discrete critical-state model for a hard superconductor has been devised in this work. In this model the state of the system is described using the coordinates of the individual vortices. In the general case, the problem of calculating the coordinates of all the vortices is very complicated. However, when certain constraints are imposed on the character of the pinning, this problem can be solved exactly, i.e., the coordinates of all the rows of vortices found within the superconducting half-space can be calculated. In addition, the stability of the solution obtained can be investigated (see Appendix B). This makes it possible to investigate several effects which cannot be studied via the macroscopic approach.

It has been found that in the strong-pinning limit, $p \gg 1$, it is easy to go over to the continuous limit. In this case all the results of the microscopic model for sufficiently strong external fields $H_0 > H^*$ coincide with the results of the macroscopic nonlocal critical-state model,[6] in which the Bean–Livingston barrier is taken into account. All of the basic results of the macroscopic nonlocal critical-state model[3–6] have been confirmed. The exact microscopic solution demonstrates the occurrence of density jumps, as well as the existence of vortex-free regions and an interval ΔH for variation of the external field, in which the coordinates of the vortices do not vary.

In weaker fields $H_0 < H^*$, the vortices exiting the superconductor as the external field decreases become unstable at a certain distance from the surface, which is macroscopically large. A macroscopic vortex-free region appears. A generalization of the macroscopic nonlocal critical-state model to the case of $H < H^*$, in which the existence of the near-surface vortex region must be taken into account in a decreasing external field, has been devised in Sec. 6.

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APPENDIX A

In this appendix we derive of Eq. (8), which makes it possible to simplify the nonlinear system of coupled equations (3). For this purpose we sum the equalities (7) with $n=k$ to $n=i \geq k$:

$$\sum_{m=k}^{i} f_m = \sum_{m=k}^{i} K_m - \sum_{m=k}^{i} G_{k+1} + (h_0 - G_1) \sum_{m=k}^{i} a_m \ldots a_1 . \tag{A1}$$

Using the recurrence relations (5) and (6), we obtain

$$K_m = a_m + a_m a_{m-1} + \ldots + a_m \ldots a_{k+1} + a_m \ldots a_{k+1} K_k \tag{A2}$$

for $m > k$ and

$$G_{m+1} = a_{m+1} + a_{m+1} a_{m+2} + \ldots + a_{m+1} \ldots a_i + a_{m+1} \ldots a_i G_{i+1} \tag{A3}$$

for $m < i$. We now consider the difference

$$\sum_{m=k}^{i} K_m - \sum_{m=k}^{i} G_{m+1} .$$

Using (A2) and (A3), we find

$$\begin{aligned} \sum_{m=k}^{i} K_m - \sum_{m=k}^{i} G_{m+1} = & \sum_{m=k+1}^{i} \sum_{j=0}^{m-k+1} a_m a_{m-1} \ldots a_{m-j} \\ & + K_k \left(1 + \sum_{m=k+1}^{i} a_m \ldots a_{k+1} \right) \\ & - \sum_{m=k}^{i-1} \sum_{j=1}^{i-m} a_{m+1} a_{m+2} \ldots a_{m+j} \\ & - G_{i+1} \left(1 + \sum_{m=k}^{i-1} a_{m+1} \ldots a_i \right) . \end{aligned} \tag{A4}$$

It is easy to prove that the first and third terms on the right-hand side of the last equation completely cancel one another. From (A1) and (A4), we then have

$$\begin{aligned} \sum_{m=k}^{i} f_m = & K_k \left(1 + \sum_{m=k+1}^{i} a_m \ldots a_{k+1} \right) - G_{i+1} \\ & \times \left(1 + \sum_{m=k}^{i-1} a_{m+1} \ldots a_i \right) + (h_0 - G_1) a_1 \ldots a_k \\ & \times \left(1 + \sum_{m=k+1}^{i} a_m \ldots a_{k+1} \right) . \end{aligned} \tag{A5}$$

Now we utilize (7) with $n=k$. We obtain

$$\begin{aligned} \sum_{m=k}^{i} f_m = & \left(1 + \sum_{m=k+1}^{i} a_m \ldots a_{k+1} \right) (G_{k+1} + f_k) - G_{i+1} \\ & \times \left(1 + \sum_{m=k}^{i-1} a_{m+1} \ldots a_i \right) . \end{aligned} \tag{A6}$$

Substituting

$$1 + \sum_{m=k+1}^{i} a_m \ldots a_{k+1} = 1 + G_{k+1} - a_{k+1} \ldots a_i G_{i+1} , \tag{A7}$$

$$1 + \sum_{m=k}^{i-1} a_{m+1} \ldots a_i = 1 + K_i - a_i \ldots a_{k+1} K_k , \tag{A8}$$

into (A6), we obtain

$$\begin{aligned} \sum_{m=k}^{i} f_m = & (1 + G_{k+1})(G_{k+1} + f_k) - G_{i+1} \\ & \times (1 + K_i - a_i \ldots a_{k+1} a_k \ldots a_1 (h_0 - G_1)) . \end{aligned} \tag{A9}$$

Finally, using (7) with $n=i$, we arrive at Eq. (8), which makes it possible to obtain the desired variables $G_k \ldots G_i$ as functions of G_{i+1}, if the pinning forces on the vortices in the rows numbered k to i are known.

APPENDIX B

The stability of the solutions of the microscopic critical-state model is investigated in this appendix.

The solution of Eqs. (8) is not unique. For the case of an increase in the field, the sign preceding γ_1^{inc} in (15) is arbitrary. The signs preceding γ_1^{dec} in (33) and preceding γ_k^{dec} in (29) remain undetermined for a decreasing field. Thus, the unstable solutions must be eliminated.

To show that a particular solution is stable, its stability against an arbitrary perturbation must be demonstrated. This is a difficult problem. We restrict ourselves to consideration of two types of perturbations: 1) displacement of any one row from its equilibrium position without altering the coordinates of any other vortices; 2) displacement of the vortex lattice as a whole, i.e., displacement of all vortices over the same distance. This approach makes it possible to eliminate some solutions known to be unstable.

Let us consider the first type of perturbation. Let the kth row be displaced relative to its equilibrium position: $x_k=x_k^{(0)}+\xi$. Here and in the following the superscript (0) corresponds to the equilibrium values of the respective quantities. The coordinates of all the other rows remain equal to their equilibrium values: $x_{i\neq k}=x_i^{(0)}$. The distance between neighboring rows x_i-x_{i-1} varies only for $i=k$ and $i=k+1$; therefore,

$$a_{i\neq k+1,k}=a_i^{(0)}. \tag{B1}$$

In addition, since the difference $x_{k+1}-x_{k-1}=x_{k+1}-x_k+x_k-x_{k-1}$ remains unchanged,

$$a_{k+1}a_k=a_{k+1}^{(0)}a_k^{(0)}. \tag{B2}$$

Thus, only a_k and a_{k+1} vary in response to the perturbation under consideration:

$$a_k=\exp\left(-\frac{x_k-x_{k-1}}{\lambda}\right)=a_k^{(0)}\exp\left(-\frac{\xi}{\lambda}\right)\approx\left(1-\frac{\xi}{\lambda}\right)a_k^{(0)},$$

$$a_{k+1}=\exp\left(-\frac{x_{k+1}-x_k}{\lambda}\right)=a_{k+1}^{(0)}\exp\left(\frac{\xi}{\lambda}\right)\approx\left(1+\frac{\xi}{\lambda}\right)a_{k+1}^{(0)}. \tag{B3}$$

Using Eqs. (B1)–(B3), it can easily be proved that

$$G_{k+1}\approx\left(1+\frac{\xi}{\lambda}\right)G_{k+1}^{(0)},\qquad G_k\approx-\frac{\xi}{\lambda}a_k^{(0)}+G_k^{(0)},$$

$$G_{k-1}\approx-\frac{\xi}{\lambda}a_{k-1}^{(0)}a_k^{(0)}+G_{k-1}^{(0)},\ldots,$$

$$G_1\approx-\frac{\xi}{\lambda}a_1^{(0)}\ldots a_k^{(0)}+G_1^{(0)},$$

$$K_k\approx\left(1-\frac{\xi}{\lambda}\right)K_k^{(0)},\qquad K_{k+1}\approx\frac{\xi}{\lambda}a_{k+1}^{(0)}+K_{k+1}^{(0)},$$

$$K_{k+2}\approx\frac{\xi}{\lambda}a_{k+1}^{(0)}a_{k+2}^{(0)}+K_{k+2}^{(0)},\ldots,$$

$$K_N\approx\frac{\xi}{\lambda}a_{k+1}^{(0)}\ldots a_N^{(0)}+K_K^{(0)}. \tag{B4}$$

None of the remaining values of G and K vary.

The solution is stable in the class of perturbations under consideration if the sign of the change in the magnetic force Δf^{em} upon displacement of a row is opposite the sign of the displacement. Since the expression for the magnetic force is the right-hand side of force balance equality (7), using (B4), we arrive at the expression

$$\Delta f_k^{\text{em}}=-\frac{\xi}{\lambda}K_k^{(0)}-\frac{\xi}{\lambda}G_{k+1}^{(0)}+[a_k^{(0)}a_{k-1}^{(0)}\ldots a_1]^2\frac{\xi}{\lambda}, \tag{B5}$$

where the subscript k on Δf indicates that this relation refers to the kth row. Since instability can appear only in the critical region (in the subcritical region a small change in the magnetic force is balanced by the pinning force), we can use (7) with $n=k$. We then have

$$\Delta f_k^{\text{em}}=-(2G_{k+1}^{(0)}\pm p)\frac{\xi}{\lambda}+[a_k^{(0)}a_{k-1}^{(0)}\ldots a_1]^2\frac{\xi}{\lambda}, \tag{B6}$$

where the plus sign refers to an increase in the external field and the minus sign refers to a decrease.

Let us now analyze the stability of the solutions against this perturbation. We first consider a decreasing external field. If, in contrast to (29), we do not predetermine the sign preceding γ_k^{dec}, we obtain two solutions, rather than one, for the equations of force balance (8):

$$G_{k+1}^{(0)}=-\frac{1-p}{2}\pm\gamma_k^{\text{dec}}. \tag{B7}$$

The solution with the minus sign preceding γ is always unstable, since the sign of Δf_k^{em} is the same as the sign of ξ. For the solution with the plus sign we have

$$\Delta f_k^{\text{em}}=-(2\gamma_k-1)\frac{\xi}{\lambda}+[a_k^{(0)}a_{k-1}^{(0)}\ldots a_1]^2\frac{\xi}{\lambda}. \tag{B8}$$

The superscript "dec" on γ_k was omitted intentionally. The fact is that the change in the magnetic force for the solutions corresponding to an increase in the external field (14) reduces to precisely the same expression, except that it contains γ_k^{inc} instead of γ_k^{dec}. For this reason, all subsequent discussions apply to both increasing and decreasing external fields. It is clear from (B8) that if $\gamma_k\geqslant 1$, the solutions are stable against the class of perturbations under consideration. This is always true in the case of strong pinning. In the case of weak pinning, the condition (B8) holds at least in the region not excessively close to the vortex front.

We now consider the second type of perturbation, which affects all the rows of vortices at once. Let all the rows in the system be displaced by the same distance: $x_i=x_i^{(0)}+\xi$. The distances between the rows do not vary. The only distance that varies is the distance between the first row and the surface, i.e., only a_1 and G_1 vary:

$$a_1\approx\left(1-\frac{\xi}{\lambda}\right)a_1^{(0)},\qquad G_1\approx\left(1-\frac{\xi}{\lambda}\right)G_1^{(0)}. \tag{B9}$$

Such a displacement of the entire vortex lattice results in the following change in the magnetic force acting on the kth row:

$$\Delta f_k^{\text{em}}=a_k^{(0)}a_{k-1}^{(0)}\ldots a_1^{(0)}(2G_1^{(0)}-h_0)\xi/\lambda. \tag{B10}$$

Using (15) and (33), we can represent $G_1^{(0)}$ for both increasing and decreasing external fields in the form $G_1^{(0)}=h_0/2\pm\gamma_0$, which gives

$$\Delta f_k^{em} = \pm 2 a_k^{(0)} a_{k-1}^{(0)} \dots a_1^{(0)} \gamma_0 \xi/\lambda. \tag{B11}$$

It is seen from (B11) that the solution is stable, if there is a minus sign preceding γ_0. The solution with the plus sign is unstable.

Eliminating the solutions known to be unstable from our discussion, we arrive at a single solution for the coordinates of the rows of vortices.

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