Collapse of spectral structures caused by stimulated radiative polarization exchange

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A theory is developed that explains the relaxation of an atom interacting with a random electromagnetic field. The radiative relaxation matrix incorporating particle transitions as well as the spontaneous and stimulated transfers of optical and magnetic coherence is calculated. Finally, the problem of radiative collapse of spectral doublets and magnetooptical resonances is discussed. © 1996 American Institute of Physics. [S1063-7761(96)00908-0]

1. INTRODUCTION

The collapse of inhomogeneously broadened spectral structures is fairly widespread. The first area of research where this phenomenon was observed was, apparently, NMR spectroscopy, where it became known as motional narrowing of spectral lines (caused by molecular motion; see, e.g., Ref. 1). In the optical range collapse was discovered in the form of the Dicke decrease in the contribution of the Doppler effect to the linewidth8–10 and as the collapse of the rotational structure of the spectra of IR absorption and Raman scattering of light in dense gases and in liquids.11,12 The radiophysics analog of the collapse phenomenon is the transition from the "instrumental" broadening of the oscillation spectrum to the "natural."13

Phenomenologically, collapse can be explained by frequency modulation as a result of a rapid change in frequency or, in spectral terms, by intense exchange of polarizations between the spectral components of the inhomogeneously broadened structure. From this viewpoint the above examples fall into a single pattern, the only difference being the exchange mechanism. Dicke narrowing and the collapse of the rotational structure are caused by exchange of polarizations in collisions. A similar reason may be considered the collisions of atoms with the walls of a vessel much smaller than the wavelength.14,15 The narrowing of NMR lines is related to the translational motion of molecules, which ensures the averaging of inhomogeneities in the external molecular interactions.16,17 Such averaging is the temporal equivalent of the spectral exchange pattern.

There is also one more collapse mechanism that has never been discussed before, spectral polarization exchange stimulated by "noise" radiation with a broad spectrum (e.g., thermal radiation). The present paper analyzes the radiative collapse mechanism.

Radiative relaxation includes three processes. First, the well-known spontaneous and stimulated transitions in atoms, processes introduced by N. Bohr and A. Einstein and accompanied by photon emission and absorption. These processes shorten the coherence lifetime and, hence, broaden the spectral lines, but do not lead to a collapse of the lines.10–14

Second, collapse is caused by the transfer of polarization (or coherence) and not of particles. Spontaneous transfer of magnetic coherence (the correlations between the magnetic sublevels of the states) has been known for more than 30 years,15–17 while the existence of spontaneous transfer of optical coherence was predicted not so long ago by the present author.18–20 The spontaneous processes of coherence transfer lead to various interference phenomena, but they cannot be the reason for collapse either. The thing is that spontaneous transfer is of a "one-way" nature, since on the energy scale of stationary states it proceeds "downward" and no inverse spontaneous processes are possible. But collapse requires that there be mutual exchange of polarizations between the components of the structure, i.e., polarization transfer must occur "upward" and "downward." Only stimulated processes guarantee that there is mutual radiative exchange of polarizations. It is obvious that spontaneous polarization transfer, as any spontaneous process, must have a stimulated analog. Nevertheless, the existing theories of radiative relaxation do not mention stimulated polarization transfer.

Section 2 is devoted to a theory of radiative relaxation that incorporates both processes; spontaneous coherence transfer and stimulated coherence transfer. Section 3 examines stimulated radiative collapse of the simplest doublet spectral structure. Section 4 analyzes the role of radiative relaxation in the structure of magnetooptical resonances.

2. THE RADIATIVE RELAXATION MATRIX

We write the quantum kinetic equation for the one-particle density matrix $\rho$ in the form

$$\frac{\partial}{\partial t} \rho = H \rho + [V - \rho H],$$

(2.1)

where $V$ is the velocity of the atom, $V$ is the Hamiltonian of interaction with coherent fields, and $H$ and $S$ are the collision integral and the radiative relaxation matrix. This section is devoted to the matrix $R$. Its calculation is done in the Appendix, while here we give only the results of calculations.

We start with the simplest diagram of four levels depicted in Fig. 1. The labels $m, n_1, n$, and $n_1$ number the stationary states of an isolated atom. The transitions $m \rightarrow n$ and $n_1 \rightarrow n$ are assumed allowed in the dipole approximation, while the transitions $m_1 \rightarrow n_1$ and $m \rightarrow n$ can be either allowed or forbidden. The Bohr transition frequencies are marked off on the horizontal axis in the lower half of Fig. 1. The following obvious relationships hold:

$$\omega_{m_n} - \omega_{n_1} = \omega_{m_1} - \omega_{m_n} = \Delta.$$  

(2.2)
FIG. 1. A four-level system and the diagrams of spontaneous and stimulated polarization transfers.

The "noise" radiation with a broad spectrum \( \Delta \omega \) is described by an average spectral bulk density \( U \). The spectrum expands over a frequency range that encompasses the \( \omega_{m,n} \) doublet (Fig. 1).

The matrix \( R \) can be represented by two terms, \( R = R^{(1)} + R^{(2)} \),

\[
R^{(1)} = \begin{pmatrix}
\frac{1}{2} (1 + \nu_m) \rho_{m,n} \rho_{m,n} + \rho_{m,n} \frac{1}{2} (1 + \nu_m)
\end{pmatrix},
\]

\[
R^{(2)} = A_{m,n} V_{m,n}^2, \quad R_{m,n} = \frac{1}{2} (1 + \nu_m) \rho_{m,n} \rho_{m,n} + \rho_{m,n} \frac{1}{2} (1 + \nu_m),
\]

\[
R^{(1)}_{m,n} = V_{m,n} \sqrt{1 - \Delta \omega}, \quad R^{(2)}_{m,n} = A_{m,n} V_{m,n}^2,
\]

where \( \nu_m \) is the rate of spontaneous decay of the states \( j = m, n, m', n' \). The radiative outgoing frequencies \( \nu_j \) are given by the following formulas:

\[
\nu_{m,n} = B_{m,n} U, \quad \nu_{m,n} = B_{m,n} U, \quad \nu_{m,n} = B_{m,n} U.
\]

The incoming frequencies are given by the following formula:

\[
v_{m,n} = \sqrt{1 + \Delta \omega} \rho_{m,n} A_{m,n} V_{m,n}^2 K, \quad \nu_{m,n} = \sqrt{1 + \Delta \omega} \rho_{m,n} A_{m,n} V_{m,n}^2 K,
\]

where \( A_{m,n} \) and \( A_{m,n} \) are the Einstein coefficients for the stimulated transition \( \Delta \omega \). Thus, the radiative outgoing frequencies in \( R^{(1)} \) are given by the arithmetic mean of the outgoing frequencies for the populations of levels \( i \) and \( j \).

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A_{m,n} V_{m,n}^2 = \frac{1}{2} (1 + \nu_m) \rho_{m,n} \rho_{m,n} + \rho_{m,n} \frac{1}{2} (1 + \nu_m),
\]

\[
v_{m,n} = \sqrt{1 + \Delta \omega} \rho_{m,n} A_{m,n} V_{m,n}^2 K,
\]

\[
\nu_{m,n} = \sqrt{1 + \Delta \omega} \rho_{m,n} A_{m,n} V_{m,n}^2 K.
\]

The matrix elements \( R_{m,n} \) and \( R_{m,n} \) can be obtained from Eqs. (2.4)–(2.7) by obvious label substitutions.

The above model of nondegenerate states is described by the simple kinetic scheme (2.4) and (2.5), which yields a graphic picture of the role of radiative relaxation. The results of calculations done in this model are only qualitative. For quantitative analysis and, more so, for the case where polarization effects and external fields are taken into account, we must allow for the degeneracy of real states.

Let us examine states with total angular momenta \( J_j \), \( j = m, n, m', n' \). In the JM-representation, the quantities \( p_{ij} \) and \( R_{ij} \) and the frequencies are matrices with respect to magnetic quantum numbers:

\[
(p_{ij})_{MM'} = \rho_{ij} M M', \quad \nu_{ij} = B_{ij} U,
\]

\[
\nu_{ij} = B_{ij} U, \quad \nu_{ij} = B_{ij} U.
\]

where the \( B_{ij} \) are the Einstein coefficients for the stimulated transition \( i \to j \). Thus, the radiative outgoing frequencies in \( R^{(1)} \) are given by the arithmetic mean of the outgoing frequencies for the populations of levels \( i \) and \( j \).

The above model of nondegenerate states is described by the simple kinetic scheme (2.4) and (2.5), which yields a graphic picture of the role of radiative relaxation. The results of calculations done in this model are only qualitative.
Similar equations hold for \( R_{\alpha} \) and \( R_{\beta} \). Comparison of the formulas (2.4) and (2.5) with (2.9) and (2.10) suggests that the radiative transfers of optical and magnetic coherences follow similar laws.

Equations (2.6), (2.7), (2.8), and (2.9) are valid if the change in the density matrix \( \rho \) in the course of the radiation correlation time \( \tau_r = 1/\Delta \omega \) is negligible.10,11,16 Under certain conditions this imposes restrictions on the atomic characteristics, say, \( \Delta \omega \ll \Gamma_1, \Gamma_2, \Gamma_3 \), etc.

The outgoing terms in \( R \) are well-known1,11 and provide no new information. Our main goal is to study the incoming terms \( R_{\alpha \alpha}^{(1)}, R_{\beta \beta}^{(1)}, R_{\alpha \beta}^{(1)}, \) and \( R_{\beta \alpha}^{(1)} \) which describe the exchange of polarizations. Notwithstanding its total randomness, an external perturbation is able to transfer both particles and coherence. The explanation lies in the fact that the same field oscillator mixes the atomic wave functions, both \( \phi_\alpha, \phi_\beta \), so that the random phases of the field oscillators do not manifest themselves in \( R \). What is important here is the difference \( \Delta \) of the Bohr frequencies of the “atomic oscillators” that interact with the field oscillator. When \( \Delta \) is large, the incoming terms rapidly oscillate and prove to be unimportant. This fact, apparently, serves as a psychological basis for ignoring the radiative transfer of optical coherence in previous research. Note that there are no oscillations in the \( R_{\alpha \alpha}^{(0)} \) matrices when magnetic coherence is transferred in the absence of a static field. The ideas developed here also refer to the spontaneous and stimulated parts of \( R^{(2)} \).

If the atom is in an external static field (magnetic or electric), the \( \omega_{ij} \) depend on \( M \) and \( M' \) according to well-known laws,21 and \( \Delta \) is a matrix with respect to the magnetic quantum numbers. In particular, in the case of a magnetic field, for \( v(mMm'M') = \sum |m_1 M_1 \rangle \langle m_1 M_1 | \) we have

\[
\Delta = \omega_{mMm'M'} = \omega_{mn},
\]

(2.11)

while for \( v(mMm'M') = \sum |m_1 M_1 \rangle \langle m_1 M_1 | \) we have

\[
\Delta = \left[ g_\alpha(M_1 - M_1) - g_\beta(M_1' - M_1') \right] \mu B.
\]

(2.12)

Here \( B \) and \( g \) are the strength of the magnetic field and the \( g \) factor of the state \( j \). We see that a magnetic field, as expected, destroys magnetic coherence transfer. In relation to optical coherence transfer a magnetic field may play an opposite role: the term proportional to \( B \) in Eq. (2.11) may balance the difference in the Bohr frequencies and decrease the oscillations in \( R^{(2)} \). A similar situation exists for a cascade of magnetic coherence in the presence of hyperfine level splitting.21

Note the structural similarity of the collision integral \( S \) in the model of relaxation constants (collisions do not change the atomic velocity) and the matrix \( R \). The outgoing term \( R^{(1)} \) consists of two terms: one \( p \) is multiplied by \( \nu \) from the left, and in the other the multiplication is from the right. The diagonal \( R_{\alpha \alpha}^{(1)} \) and \( R_{\beta \beta}^{(1)} \) and off-diagonal \( R_{\alpha \beta}^{(1)} \) matrix elements contain the same matrices \( \nu_{\alpha} \) and \( \nu_{\beta} \). The incoming terms contain a “supermatrix” that acts on both variables of the density matrix. The collision integral has the same properties.

According to Eqs. (2.4), (2.5), (2.9), and (2.10), radiative relaxation is not accompanied by a change in atomic velocity. If recoil is taken into account, the incoming term describes velocity mixing,18,22 and radiative relaxation resembles collisional relaxation even more.

The formal similarity between \( R \) and \( S \) deserves attention also because of the seeming dissimilarity of the physical conditions of collisional and radiative perturbations of an atom. In deriving the kinetic equation with a collision integral of the Boltzmann type it is assumed that a short collision time is followed by a fairly long period of free motion. In the case of radiative relaxation, however, the external radiation represents a random stationary process without prolonged pauses. Notwithstanding such dissimilarity in conditions, radiative and collisional relaxation are described by similar terms in the kinetic equation.

As is the case with the collision integral, the radiative outgoing and incoming frequencies depend to a great extent on the symmetry of the perturbation. When the radiation is spherically symmetric, i.e., unpolarized and nondirectional, Eqs. (A21) and (A22) imply that

\[
\nu = \frac{v}{\prod_n} \delta_{ij} \mu_i B_{\mu j},
\]

(2.13)

\[
B_{\mu j} = \left| \sum_i C_{\mu ij} \right|^2.
\]

(2.14)

The spontaneous incoming rate is given by the following expression:18–20

\[
A = \frac{2 \pi \omega_{ij}}{\left( \Delta \right)} \frac{\prod_n}{\mu_i B_{\mu j}} \sum_j \frac{\left| \sum_i C_{\mu ij} \right|^2}{\prod_j}
\]

(2.15)

Thus, in the isotropic case the outgoing frequencies are specified by the rates of stimulated transitions of particles, are diagonal in the magnetic quantum numbers, and are independent of these numbers. On the other hand, there is similarity between the dependence of the spontaneous and stimulated incoming frequencies on the magnetic quantum numbers.
numbers. The products of vector addition coefficients reflect interference between the $m,M \rightarrow mM$ and $n,M \rightarrow nM$ transitions. As is known,\textsuperscript{16,24} when the perturbation of an atom is isotropic, the convenient representation for the density matrix is the $\tilde{q}$-representation, which is specified by the following relationships:

\begin{equation}
\rho(mn\tilde{q}) = \sum_{MM'} (-1)^{M'+M}(J_\text{m}M_\text{j}J_\text{m}'M_\text{j}'\tilde{q})\rho(mMnM'),
\end{equation}

\begin{equation}
\rho(mMnM') = \sum_{\tilde{q}} (-1)^{M'+M}(J_\text{m}M_\text{j}J_\text{m}'M_\text{j}'\tilde{q})\rho(mn\tilde{q}).
\end{equation}

The quantities $\rho(mn\tilde{q})$ are called the polarization moments of the level $(n=m)$ or polarization moments $(n \neq m)$ of rank $\kappa$. Since in the isotropic case the $\varphi(JM|JM')$ are diagonal in $M$ and are independent of $M$, in the $\tilde{q}$-representation they are diagonal in $\tilde{q}$, are independent of $\tilde{q}$, and coincide with the values given by (2.13). The incoming frequencies are given by the following formulas:

\begin{equation}
\varphi(mn\tilde{q})|m,n;\kappa,\tilde{q};\tilde{q}_1\rangle = \delta_\tilde{q}_1,\delta_{\tilde{q}_1},\varphi(mn|m,n,\kappa),
\end{equation}

\begin{equation}
A(mn\tilde{q})|m,n;\kappa,\tilde{q};\tilde{q}_1\rangle = \delta_\tilde{q}_1,\delta_{\tilde{q}_1},A(mn|m,n,\kappa),
\end{equation}

\begin{equation}
K_\kappa = (-1)^{J_\text{m}+M_\text{j}+J_\text{m}'+M_\text{j}'}J_\text{m}J_\text{m}'J_\text{n}J_\text{n}' \frac{\sqrt{2J_\text{m}+1}}{\sqrt{2J_\text{m}'+1}}
\end{equation}

\begin{equation}
\times [\begin{pmatrix} J_\text{m} & J_\text{n} \\ J_\text{m}' & J_\text{n}' \end{pmatrix}^\kappa],
\end{equation}

i.e., the result is diagonal in $\kappa\tilde{q}$ and independent of $q$. For $R(\text{ij})\text{q}$ and $R(\text{ij})\text{q}^{(2)}$, the following is true:

\begin{equation}
\varphi(mn\tilde{q})|m,n;\kappa,\tilde{q};\tilde{q}_1\rangle = \delta_\tilde{q}_1,\delta_{\tilde{q}_1},\varphi(mn|m,n,\kappa),
\end{equation}

\begin{equation}
A(mn\tilde{q})|m,n;\kappa,\tilde{q};\tilde{q}_1\rangle = \delta_\tilde{q}_1,\delta_{\tilde{q}_1},A(mn|m,n,\kappa),
\end{equation}

\begin{equation}
K_{\kappa} = (-1)^{J_\text{m}+M_\text{j}+J_\text{m}'+M_\text{j}'}\left[\begin{array}{c} J_\text{m} \\ J_\text{m}' \end{array}\right]^{(2)}\left[\begin{array}{c} J_\text{n} \\ J_\text{n}' \end{array}\right]^{(1)}
\end{equation}

Equations (2.21) are known from magneto-optical resonance theory\textsuperscript{16-17} and are written here to make the picture complete. The frequencies of direct and reverse transitions obey the following relationships:

\begin{equation}
(2J_\text{m}+1)\omega_\text{m} = (2J_\text{m}+1)\omega_\text{m}',
\end{equation}

\begin{equation}
(2J_\text{m}+1)\omega_\text{m} = (2J_\text{m}+1)\nu_\text{m}',
\end{equation}

\begin{equation}
\varphi(mn|m,n,\kappa) = \varphi(mn|m,n,\kappa), \quad \nu_\text{m,m} = \nu_\text{m,m}.'
\end{equation}

An analysis of the explicit expressions for 6j symbols shows that $K_{\kappa}$ monotonically decreases as $\kappa$ grows:

\begin{equation}
\nu_\text{m,m} < \nu_\text{m,m}'.
\end{equation}

Note that as $\kappa$ increases, $K_{\kappa}$ changes sign at $J_\text{m} = J_\text{m}'$. The factor $K_{\kappa}$ depends on four angular momenta, and its properties as a function of $\kappa$ are complicated. For $J_\text{m} = J_\text{m}', \quad \nu_\text{m,m} = J_\text{m} \pm 1, \quad \nu_\text{m,m}' = J_\text{m} \pm 1,$

\begin{equation}
\nu_\text{m,m} = J_\text{m} \pm 1, \quad \nu_\text{m,m}' = J_\text{m} \pm 1,
\end{equation}

the factor $K_{\kappa}$ monotonically decreases as $\kappa$ grows. But if we take $J_\text{m} = J_\text{m}', \quad \nu_\text{m,m} = J_\text{m} \pm 1, \quad \nu_\text{m,m}' = J_\text{m},$

\begin{equation}
\nu_\text{m,m} = J_\text{m} \pm 1, \quad \nu_\text{m,m}' = J_\text{m},
\end{equation}

then for fairly large values of $J_\text{m}$ and $J_\text{m}'$ the dependence of $K_{\kappa}$ on $\kappa$ may be nonmonotonic.

The orthonormality of the 6j symbols\textsuperscript{24} implies that

\begin{equation}
\varphi(mn|m,n,\kappa) = \sqrt{\frac{(2J_\text{m}+1)(2J_\text{m}'+1)}{J_\text{m}J_\text{m}'\kappa!}} \varphi(mn|m,n,\kappa'),
\end{equation}

Equations (2.25) and (2.29) are also valid for collision frequencies\textsuperscript{3,6} which emphasizes once again the similarity of the properties of the collision integral and the stimulated radiative relaxation matrix.

Above we studied the case of isotropic radiation. If the perturbing radiation is anisotropic, the outgoing and incoming frequencies in the $\tilde{q}$-representation are not diagonal in $\tilde{q}$ (see Eqs. (A24)-(A27)). In other words, anisotropic radiative relaxation mixes the polarization moments of different orders.

3. COLLAPSE OF A SPECTRAL DOUBLET

In the absence of polarization transfer the spectrum of absorption (emission, scattering, gain) of the four-level system of Fig. 1 consists of four lines with central frequencies $\omega_{\text{j}m}$. When $|A|$ is small, the lines are grouped into two doublets: $\omega_{\text{m},\text{m')),}\omega_{\text{m},\text{m}'}$, and $\omega_{\text{m},\text{n')),}\omega_{\text{m},\text{m}'}$, and $\omega_{\text{m},\text{m}'}).$ Let us examine the contour of the $\omega_{\text{m},\text{m'}},\omega_{\text{m},\text{m'})}$ doublet in the case where the isotropic radiation is in resonance with the other doublet, $\omega_{\text{m},\text{m')},}\omega_{\text{m},\text{m'})$, and stimulates polarization transfer between the $m-n$ and $m-n$ transitions.

In an approximation that is linear in the intensity of the monochromatic field and in which the levels $m, n, n, m$, and $n$ are isotropically populated, solving the absorption-spectrum problem requires knowing the off-diagonal elements of the density matrix $\rho(jj\tilde{q})$ for $\kappa = 1$ (the dipole approximation). At this point it is advisable to introduce the following simplifying notation:

\begin{equation}
\rho_{mn,\kappa} = \rho_{mn,\kappa}'.
\end{equation}
\[ \rho_p = \rho(m n l q), \quad \Gamma_m = \frac{\Gamma_n + \Gamma_l}{2}, \quad \nu_m = \frac{\nu_n + \nu_l}{2}, \]

where \( \rho_p \) and \( \rho_{plq} \) can be obtained from Eq. (2.1) in which the matrix elements of the interaction \( V \)

\[ V = \sum_{\sigma} (-1)^{\sigma} \left( \langle m n M M' \rangle \right) |d_{m \sigma}| \left( \langle p \rangle \right) \]

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\[ \rho_p \rho_{plq} = \rho(M N M' N'), \quad \Gamma = \frac{\Gamma_M + \Gamma_{M'}}{2}, \quad \nu = \frac{\nu_M + \nu_{M'}}{2}, \]

\[ A = A(m n | m n, 1), \quad \nu = v(m n | m n, 1). \]

The system of equations for \( \rho_{p} \) and \( \rho_{plq} \) can be obtained from Eq. (2.1) in which the matrix elements of the interaction \( V \) are

\[ V(M N M' N') = \sum_{\sigma} (-1)^{\sigma} \left( \langle m n M M' \rangle \right) |d_{m \sigma}| \left( \langle p \rangle \right) \]

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\[ \rho_{plq} = \rho(m n l q), \quad \Gamma = \frac{\Gamma_1 + \Gamma_2}{2}, \quad \nu = \frac{\nu_1 + \nu_2}{2}, \]

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\[ A = A(m n | m n, 1), \quad \nu = v(m n | m n, 1). \]
When the exchange rate is low, or the Lorentzians are centered at the frequencies $\Sigma = 0$ and $R = 0$, we have halfwidths $\nu + \nu$ and $\nu + \nu$. In the expressions for the amplitudes $C_1$ and $C_2$, we discard the term proportional to $F$ but keep the combination $AN$, which describes a spontaneous polarization. A detailed analysis of $\alpha(\Omega)$ will be carried out with the simplifying assumptions

$$\nu = \nu_1, \quad R = R_1, \quad a = a_1, \quad N = N_1,$$

in which case the interference of the doublet components manifests itself most vividly. When the conditions specified in (3.15) are met, we have

$$\gamma_\nu = \nu + \nu_1, \quad C_{1,2} = aN \{ 1 \pm (\nu + \nu_2)/2 \},$$

and

$$d = \sqrt{\nu + \nu_1 - \Delta^2}. \quad (3.16)$$

If we have $\nu \neq 0$ but $I - I < \Delta^2/4$, then $d$ is imaginary, the Lorentzians in (3.12) are centered at different frequencies, and the distance between them is

$$|\gamma_\nu - \gamma_\nu| = 2 \Delta d = \sqrt{\nu^2 - 4 \nu_1}.$$

As $|\nu|$ grows, the components of the doublet shift toward one another, and if

$$|\nu| \approx (\gamma_\nu + \gamma_\nu)/2 \approx \nu_0.$$

The doublet collapses: both Lorentzians are at the central frequency $\nu = \Delta/2$ but have different halfwidths $\gamma_\nu$ and $\gamma_\nu$, and the Lorentzian with the smaller halfwidth $\gamma_\nu$ has a greater amplitude $C_1$, while the Lorentzian with the larger halfwidth $\gamma_\nu$ has a smaller (in absolute value) negative amplitude $C_2$. Figure 2 depicts $\gamma_\nu$ (an arc of a circle) and $\nu \gamma_\nu$ (a hyperbola) as functions of $|\nu|$. One can clearly see how fast the Lorentzian shifts toward one another and how the difference of their halfwidths $|\gamma_\nu - \gamma_\nu|$ increases in a fairly narrow interval near $\nu_0$. A remarkable case here is the limit

$$|\nu| \approx (\gamma_\nu + \gamma_\nu)/2 \approx \nu_0,$$

where the difference in the Lorentzians is at its maximum:

$$\gamma_\nu = \nu + |\nu_1|, \quad \gamma_\nu = \nu - |\nu_1|, \quad C_1 = aN, \quad C_2 = -aN \Delta^2/\Delta^2.$$  

The component $\Gamma = \nu - |\nu_1|/2$ of the halfwidth $\gamma_\nu$ is related primarily to spontaneous processes. In contrast to collisional collapse, where the spontaneous halfwidth remains constant, in radiative collapse the contribution of spontaneous processes changes. Instead of the natural halfwidth $\Gamma = \nu + \nu_1/2$, the halfwidth $\gamma_\nu$ contains the difference $\Gamma = \nu_1/2$, which is interpreted as the spontaneous analog of the difference $\nu - |\nu_1|$. Clearly, the term $\Delta^2/\Delta^2$ is important only in the limit $\nu \approx |\nu_1|$. The component $\Gamma = \nu_1/2$ of the halfwidth $\gamma_\nu$ should be interpreted as the analog of the diffusion halfwidth. This term is significant when the spontaneous outgoing process is nearly perfectly balanced by the spontaneous incoming process ($\Gamma = \nu_1/2 \approx \Gamma$). In the theory of spectral line broadening the contribution of the spontaneous decay of states is always assumed to be a universal quantity, i.e., a quantity that any additional perturbation of the radiating atom is unable to change. The radiative collapse mechanism proves to be a fundamental exception in the general physical sense. Indeed, the universality of the spontaneous part of the linewidth is related to the irreversible nature of the relaxation of the dipole moment caused by spontaneous decay and to the fact that the later is statistically independent from other reasons for relaxation. In the

$$\gamma_\nu = \nu + |\nu_1|, \quad \gamma_\nu = \nu - |\nu_1|, \quad C_1 = aN, \quad C_2 = -aN \Delta^2/\Delta^2.$$  

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The first three terms on the right-hand side of this equation are related to spontaneous processes, and the remaining terms are related to stimulated processes. According to Refs. 19 and 20, the inequality

$$aN\Gamma + aN\Gamma - \sqrt{aN}\bar{A}N > 0$$

can be met, with the result that at $\Gamma = 0$, $\Gamma = 0$ we have $\alpha(I) < 0$. In determining the sign of the sum of the last four terms on the right-hand side of Eq. (2.31) it is advisable to use (3.21), in view of which

$$aN\Gamma + aN\Gamma - \sqrt{aN}\bar{A}N = 0$$

The quantity in square brackets is positive. The remaining term may be either positive or negative, but for high saturation the difference $N - N_1$ is inversely proportional to $\Gamma$ and $1/\nu_1$, so that this term must be of the order of the spontaneous term.

Now let $\Delta = 0$, $\nu < 0$, and $\bar{A} < 0$. For the center of the line we have

$$\sigma(\Omega) = \frac{\lambda^2}{4\pi T} [aN\Gamma + aN\Gamma - \sqrt{aN}\bar{A}N\Gamma \mp aN\nu]\frac{aN\nu}{aN\nu}$$

(3.22)

If $\nu \Gamma$, $\nu \Gamma$, and $|\bar{A}| = |\bar{A}|$, we can say the same things about the expression in braces in (3.23) as we did about the right-hand side of (3.22). Thus, stimulated radiative relaxation does not exclude the possibility of gain (or negative absorption) without population inversion.

4. EFFECT OF STIMULATED RADIATIVE RELAXATION ON MAGNETOOPTICAL RESONANCE

Let us examine magnetooptical resonances on the $m-n$ transition and take into account radiative relaxation, which mixes the levels $m$ and $n$. We are dealing, therefore, with two excited levels that are split by a magnetic field $H$ and interact with broad-band radiation quasi-resonant to the $m-n$ transition. The radiation is assumed isotropic, so that Eqs. (2.13) (2.22) are applicable. The problem consists primarily in calculating the polarization moments $p(r|\Omega)$ of the levels that determine magnetooptical resonances in absorption, refraction, spontaneous emission, etc. (10, 11)

To simplify matters we introduce the following notation:

$$\mu(m|\nu) = \rho_{m\nu}, \quad \rho(m|\nu) = \rho_{m\nu}, \quad \rho_{m\nu} = \rho_{m\nu}$$

(4.1)

The system of kinetic equations for $\rho_{m\nu}$ and $\rho_{m\nu}$ has the following form:

$$(\Gamma_m + \nu)\rho_{m\nu} - \nu\rho_{m\nu} = \rho_{m\nu}$$

(4.2)

The stimulated outgoing frequencies $\nu_\nu$ and $\nu_\nu$ are given by Eq. (2.13), and the spontaneous and stimulated incoming frequencies ($\nu_\nu$) are given by Eqs. (2.20) (2.23) to within level notation, with

$$\nu_\nu = \nu_\nu = (2J_\nu + 1)\nu_\nu, \quad \nu_\nu = \nu_\nu$$

(4.3)

The products $\Omega_j$ in Eqs. (4.2) describe magnetic splitting:

$$\Omega_j = g_j\mu H$$

(4.4)

where $g_j$ is the $g$ factor of the levels $j = m, n$. The right-hand sides of system (4.2), $\rho_{m\nu}$ and $\rho_{m\nu}$, are the level pump rates, with the pump radiation generally being anisotropic. The solutions of the system of equations (4.2) have the standard form

$$\rho_{m\nu} = \frac{(\Gamma_m + \nu - i\Omega_j)\rho_{m\nu} + \nu\rho_{m\nu}}{D_{\nu\nu}}, \quad \rho_{m\nu} = \frac{(\nu_m + \nu - i\Omega_j)\rho_{m\nu} + \nu\rho_{m\nu}}{D_{\nu\nu}}$$

(4.5)

The second part of the problem consists in calculating the experimentally measurable quantities via $\rho_{m\nu}$. Here we

S. G. Rautian
are interested in the absorption coefficient \( \alpha(\omega, H) \) of the monochromatic field (with frequency \( \omega \) and wave vector \( k \)) that is in resonance with the \( m-n \) transition:

\[
\alpha(\omega, H) = \frac{\lambda^2}{4 \pi} A_{m-n}(2J_m + 1) \times \text{Re} \left( \frac{1}{1 - i\Omega}\sum_{\sigma q} \langle a_{m\sigma q} a_{m\sigma q} - a_{m\sigma q} a_{m\sigma q} \rangle \delta(\omega) \right),
\]

where the angle brackets stand for averaging over the velocities \( v \), and \( i(\omega) \) is the normalized field polarization tensor.

In (4.8) we did not allow for Zeeman line splitting, which is by assumption negligible in comparison to the Doppler width.

The fact that the dependence of \( \rho_{m+} \) and \( \alpha(\omega, H) \) on the magnetic field \( H \) is of a resonant nature is due to the form of the determinants \( \Delta_{m\sigma} \). In the absence of stimulated relaxation \( (\gamma_j = \gamma_k = 0) \), the magnetooptical resonances are described by the following Lorentzians:

\[
\Gamma_a = 3(-1)^{1+q} + i \left( \frac{J_a}{1} + J_a \right),
\]

\[
\Gamma_a = 3(-1)^{1+q} + i \left( \frac{J_a}{1} + J_a \right),
\]

where the width of each contour depends on the property of only one level, \( (\gamma_j = \gamma_k = 0) \). The magnetic resonances (4.8) and (4.9) assume the values \( \kappa = 0, 1, 2 \), while \( q \) varies in the interval \( 1 \leq q \leq 2 \). The terms with \( q = 0 \) are independent of \( H \). Hence the resonant dependence on \( H \) is contained in six pairs of terms: \( \kappa = 1 \) and \( q = 1, 2 \).

According to the properties of \( \delta j \) symbols, the order \( \kappa \) of the polarization moments in Eqs. (4.8) and (4.9) assumes the values \( \kappa = 0, 1, 2 \), while \( q \) varies in the interval \( 1 \leq q \leq 2 \). The terms with \( q = 0 \) are independent of \( H \). Hence the resonant dependence on \( H \) is contained in six pairs of terms: \( \kappa = 1 \) and \( q = 1, 2 \).

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The absorbed coefficient \( \alpha(\omega, H) \) can be represented by the following sum of Lorentzians:

\[
\alpha(\omega, H) = \sum_{\kappa} \frac{C_{\kappa}(\omega)}{\Gamma_{\kappa} + i\Omega}.
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FIG. 3. Halfwidths $\Gamma_{\alpha\kappa}$ and $\Gamma_{\alpha}$ of the Lorentzian components of the magnetooptical resonance as functions of the frequency $\tilde{\nu}$.

Here we are dealing only with the quadrant where $K_1 > 0$ and $\beta > 0$ in Fig. 3). In turn the definitions (4.15) imply that $b^2 > \gamma^2$ or $b^2 < \gamma^2$ depending on whether

$$\frac{\Gamma_{\alpha} - \Gamma_{\kappa}}{\tilde{\nu}_\alpha - \tilde{\nu}_\kappa} < \sqrt{1 + \gamma^2},$$

or

$$\frac{\Gamma_{\alpha} - \Gamma_{\kappa}}{\tilde{\nu}_\alpha - \tilde{\nu}_\kappa} > \sqrt{1 + \gamma^2} (b^2 < \gamma^2).$$

(4.17)

Both cases may occur. Interestingly, the conditions (4.17) and (4.18) do not contain the factor $K_{1\kappa}$, i.e., they are valid for all values of $\kappa$: the hyperbolae for $\kappa = 1$ and $\kappa = 2$ are distinct but have the same asymptotic lines.

Let us examine the simple case where $g_{\alpha} = g_{\kappa}$ and $v_{\alpha} = v_{\kappa} = \tilde{\nu}$. The hyperbola (4.14) assumes the form

$$(\tilde{\nu}_{\alpha} - \tilde{\nu})^2 - (v_{\alpha} + A_{\alpha}/2)^2 = \tilde{\nu}^2 - A_{\alpha}^2/4.$$  

(4.14a)

In conditions of a well-developed collapse we have

$$\Gamma_{\alpha\kappa} = \Gamma_{\alpha} + \frac{|A_{\alpha}|}{2} + v_{\alpha} - |v_{\alpha}|, \quad \frac{A_{\alpha}}{2} = (\Gamma_{\alpha} - \Gamma_{\kappa})^2 / 8 |v_{\alpha}|,$$

$$\Gamma_{\alpha} = \Gamma_{\alpha\kappa} + \frac{|A_{\alpha}|}{2} + v_{\alpha} + |v_{\alpha}|,$$

(4.19)

where $|v_{\alpha}| > |A_{\alpha}|, |\Gamma_{\alpha\kappa} - \Gamma_{\alpha}|$. Well-developed collapse is characterized by a partial balance between the spontaneous and stimulated components of the halfwidth $\Gamma_{\alpha\kappa}$ and by a considerable increase in $\Gamma_{\alpha\kappa}$. The condition $v_{\alpha} = v_{\kappa}$ means that $J_{\alpha} = J_{\kappa}$. If we use the explicit expression for a 6j symbol, we find that

$$1 - |v_{\alpha}|/v_{\alpha} = 1 - |A_{\alpha}|/A_{\alpha\kappa} = 1 - |K_{\alpha}| = 1 - (1 - \kappa(\kappa + 1)/2J_{\alpha}(J_{\alpha} + 1)),$$

(4.20)

Thus, radiative transfer of magnetic coherence always balances departure, but the extent of such balancing decreases as $\kappa$ grows and increases with $J_{\alpha}$. Note also the extensive analogy between the collapse of magnetooptical resonances and that of a spectral doublet (Sec. 3).

Callas and Chatka\textsuperscript{25} described experiments that demonstrate the effect of radiation emitted by a plasma on magnetooptical resonances (see also Refs. 15 and 17). Their interpretation of the results was based on the idea that the radiation changes the rates of excitation of the polarization moments $J_{\alpha\kappa}$ in the notation adopted in (4.21). Such an effect must certainly exist. It is possible, however, that to a certain extent changes in the relaxation matrix also contribute.

5. DISCUSSION

The main achievement of the general theory developed in Sec. 2 is the prediction of radiation transfer of optical and magnetic coherence. The applied method of derivation coincides at crucial points with the methods used earlier. Hence the achievements of the theory are not related to the method but rather to the statement of the problem, i.e., by drawing on the analogy between stimulated and spontaneous transfer processes, on the one hand, and on the analogy between the transfer of magnetic and optical coherence, on the other.\textsuperscript{16-20}

Explaining optical coherence transfer means introducing a complicated diagram of the levels participating in the radiative process, a diagram consisting of at least four levels (Fig. 1). Of course, transfer between stationary states involving any characteristics of the system leads to a broadening of the set of "essential" states. However, in relation to optical coherence such "broadening" seemed to be unnecessary. In other words, analysis points to certain limitations in the applicability of the canonical two-level system even in the resonance approximation. Up till now the limited nature of the two-level system was related only to the resonance approximation.

When the pump radiation is isotropic, the radiative relaxation matrix in the $\alpha\kappa$-representation is diagonal in $\alpha\kappa$, in accordance with general ideas.\textsuperscript{16,24} The outgoing frequencies are proportional to the second Einstein coefficients, while the incoming frequencies for polarization are the geometric mean of two Einstein coefficients for transitions from the combining levels. In accordance with the above-mentioned analogy, the rates of spontaneous and stimulated polarization transfers follow similar patterns in their dependence on the moments of the states and the rank $\kappa$. When the pump radiation is anisotropic, stimulated radiative relaxation "mixes" polarization moments of different orders.

The existence of direct and reverse polarization transfers and the absence of a phase shift (the incoming frequencies are real-valued) cause tight spectral structures to collapse. In the above example of a doublet, collapse imposes certain restrictions on the relationship between the doublet splitting and the outgoing and incoming frequencies, and the restrictions are typical of other collapse mechanisms as well. A remarkable feature of radiative collapse is that in the limit of a well-developed collapse the width of the narrow compo-
There is another difference between radiative collapse and collisional collapse worth noting. It is an established fact that in gas-kinetic conditions collisions mix a group of states whose energies differ little (fine splitting of atomic levels, and rotational splitting of molecular levels). On the other hand, radiation may initiate transitions between states located in an arbitrary manner on the energy scale. Hence radiative coherence exchange may lead to a collapse of a structure in which the collisional mechanism is ineffective. If we ignore the problem of how the radiative and collisional transfer mechanisms are related and remain on purely phenomenological grounds, collisions can be taken into account by introducing certain terms in the expressions for the outgoing and incoming frequencies. Collisions may enhance the collapse process, but they may also hinder it.

Similar to the case of the collapse of spectral structures, stimulated radiative relaxation leads to the collapse of magnetooptical resonances by mixing the polarization moments of the levels. In this case, too, in conditions of a well-developed collapse the widths of magnetooptical resonances may prove to be smaller than the natural spontaneous values.

Polarization induced between two levels can be considered the simplest type of coherence. This naturally leads to the problem of spontaneous and stimulated transfers of one state of an arbitrary type into another. Undoubtedly such a “generalized” transfer is possible in principle, but its effectiveness depends on the specific conditions and is determined by transfer coefficients similar to the Einstein coefficients and the conditions of resonance of the interfering coherent states of the quantum system.

In conclusion we note that the effects of stimulated radiative transfer of optical coherence manifest themselves in conditions where the rates of stimulated transitions are comparable to those of spontaneous transitions or are higher. This occurs at radiation brightness temperatures of order $\hbar \omega$. The real values of intensities are therefore different in the ultraviolet, visual, and infrared ranges of the spectrum.

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**APPENDIX**

Let us introduce the matrix $\sigma_{ab} = \sigma_a^* \sigma_b$ consisting of the column $\sigma_a$ of probability amplitudes of the stationary states of an isolated atom and the Hermitian conjugate row $\sigma_a^*$. The density matrix $\rho$ can be obtained from $\sigma$ by averaging over the random parameters:

$$p = \bar{\sigma}.$$  \hspace{2cm}  \text{(A1)}

The system of equations for finding the off-diagonal elements $\sigma_{ij}$ of the four-level system of Fig. 1 has the form

$$\dot{\sigma}_{mn} = -i \Gamma_{mn} \sigma_{mn} + \sum_k \left( G_{mnk}^\dagger \sigma_{nk} + \sigma_{km} G_{mnk}^\dagger \right),$$

$$\dot{\sigma}_{mn} = -i \Gamma_{mn} \sigma_{mn} + \sum_k \left( G_{mnk}^\dagger \sigma_{nk} + \sigma_{km} G_{mnk}^\dagger \right).$$  \hspace{2cm}  \text{(A2)}

In the dipole approximation, $X(J \rightarrow M)$ is the circular component of the field in the $\lambda$th mode.

Further calculations are done according to the following scheme. First the formal solutions of Eqs. (A2) for $\sigma_{a}^{lln}$ and $\sigma_{a}^{mn}$ are constructed, say

$$\sigma_{a}^{lln}(t) = \sum_k \left( G_{a}^{llk} e^{-i \omega_k t} \sigma_{nk}(t) + \sigma_{kn} G_{a}^{lk} e^{i \omega_k t} \right),$$

$$\sigma_{a}^{mn}(t) = \sum_k \left( G_{a}^{mnk} e^{-i \omega_k t} \sigma_{nk}(t) + \sigma_{kn} G_{a}^{km} e^{i \omega_k t} \right).$$  \hspace{2cm}  \text{(A5)}

Then those expressions for $\sigma_{a}^{lln}$ and $\sigma_{a}^{mn}$ are plugged into the equations for $\sigma_{a}^{lln}$ and $\sigma_{a}^{mn}$ and form two sets of four similar terms there. Here is one of these terms:

$$- \sum_k \left( G_{a}^{llk} e^{-i \omega_k t} \int_0^\infty e^{-\Gamma_{mn} \tau} \sigma_{mn} G_{a}^{lm} \sigma_{mk}^l \sigma_{nl}^k \right).$$  \hspace{2cm}  \text{(A6)}

Bearing in mind the averaging procedure (A1) and the fact that the phases $\omega_k$ are random, we should leave only the terms with $\lambda = \lambda_1$ in the double sum over $\lambda$ and $\lambda_1$. Summa-
tion over \( \lambda \) is replaced by integration with respect to \( \omega \), which results in the appearance of \( 2\pi \delta(\omega - \omega') \). As a result the term (A6) becomes

\[
-2\pi \rho_{\delta} G_{m_n}^{\delta} \sigma_{m_{\delta} \omega_{m_{\delta}}}^{\delta} (\omega).
\]  

(7)

where \( \rho_{\delta} \) is the spectral density of the number of modes. In the averaging procedure we adopt the "decoupling" hypothesis:

\[
G_{m_{\delta} \omega_{m_{\delta}}}^{\delta} \sigma_{m_{\omega_{m_{\delta}}} m_{\delta}}^{\delta} = G_{m_{\delta} \omega_{m_{\delta}}}^{\delta} \sigma_{m_{\delta} \omega_{m_{\delta}}}^{\delta} \rho_{\delta}.
\]  

(8)

As a result of the above calculations we get

\[
R_{m_{\delta} \omega_{m_{\delta}}}^{(1)} = \frac{1}{2} (\Gamma_m + r_m) \rho_{\delta} + \frac{1}{2} (\Gamma_{\omega_m} + r_{\omega_m})
\]  

(9)

\[
R_{m_{\delta} \omega_{m_{\delta}}}^{(2)} = N_{m_{\omega_{m_{\delta}}} m_{\delta}} \rho_{\delta} e^{\Delta t_{m_{\omega_{m_{\delta}}} m_{\delta}}}
\]  

(10)

where we have introduced the following notation:

\[
v_m = 2\pi \rho_{\delta} G_{m_{\delta} \omega_{m_{\delta}}}^{\delta} \sigma_{m_{\delta} \omega_{m_{\delta}}}^{\delta}, \quad r_m = 2\pi \rho_{\delta} G_{m_{\delta} \omega_{m_{\delta}}}^{\delta} \sigma_{m_{\omega_{m_{\delta}}} m_{\delta}}^{\delta}, \quad \Delta = \omega_{m_{\delta}} - \omega_{m_{\delta}}.
\]  

(11)

Here the symbol \( \otimes \) stands for a direct product. The term \( A_{m_{\delta} m_{\delta}} \) in (A10) represents spontaneous processes and is not present in our derivation scheme; it has been introduced here on the basis of the results of Refs. 18–20. Similar expressions can be obtained for the matrices \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) and \( R_{m_{\delta} \omega_{m_{\delta}}} \):

\[
R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}^{(1)} = \frac{1}{2} (\Gamma_{\omega_m} + r_{\omega_m}) \rho_{\delta} + \frac{1}{2} (\Gamma_m + r_m)
\]  

(13)

\[
R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}^{(2)} = N_{m_{\omega_{m_{\delta}}} m_{\delta}} \rho_{\delta} e^{\Delta t_{m_{\omega_{m_{\delta}}} m_{\delta}}}
\]  

(14)

\[
R_{m_{\omega_{m_{\delta}}} m_{\delta}}^{(1)} = \frac{1}{2} (\Gamma_{m} + \Gamma_{\omega_m}) \rho_{\delta} + \frac{1}{2} (\Gamma_{m} + \Gamma_{\omega_m})
\]  

(15)

\[
R_{m_{\omega_{m_{\delta}}} m_{\delta}}^{(2)} = N_{m_{\omega_{m_{\delta}}} m_{\delta}} \rho_{\delta} e^{\Delta t_{m_{\omega_{m_{\delta}}} m_{\delta}}}
\]  

(16)

\[
R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}^{(1)} = \frac{1}{2} (\Gamma_{\omega_m} + \Gamma_m) \rho_{\delta} + \frac{1}{2} (\Gamma_{\omega_m} + \Gamma_m)
\]  

(17)

\[
R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}^{(2)} = N_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \rho_{\delta} e^{\Delta t_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}}
\]  

(18)

\[
v_{m_{\omega_{m_{\delta}}} m_{\delta}} = 2\pi \rho_{\delta} G_{m_{\omega_{m_{\delta}}} m_{\delta}}^{\delta} \sigma_{m_{\omega_{m_{\delta}}} m_{\delta}}^{\delta}, \quad v_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} = 2\pi \rho_{\delta} G_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}^{\delta} \sigma_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}^{\delta}.
\]  

(19)

The matrices \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) and \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) can be obtained from \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) and \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) by replacing the label \( m \) with \( n \) and the label \( m_{\omega_{m_{\delta}}} \) with \( n_{\omega_{m_{\delta}}} \).

Above we assumed that there is only one pair of levels, \( m \) and \( n \), from which transitions to the levels \( m \) and \( n \) occur. If there are other such pairs of levels, each contributes to \( R \). For instance, for \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) this is reduced to summing over \( m_1 \) and \( n_1 \) in Eqs. (A10)–(A12). Similarly, several pairs of levels similar to \( m \) and \( n \) can contribute to \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) and \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \), a fact that can be taken into account by summing over \( m \) and \( n \) in Eqs. (A14) and (A19).

Let us write the expressions for the outgoing and incoming frequencies in the JM-representation:

\[
\nu(mM') = 2\pi \frac{d_{m_{\omega_{m_{\delta}}} m_{\delta}}}{(2\pi)^2 \sqrt{2J_{m_{\delta}} + 1}} \sum_{\omega_{m_{\delta}}} \frac{1}{
\langle J_m, J_m, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle} \times \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle}
\]

(21)

\[
\nu(mM' \mid m_1, m_1; M_1)
\]  

\[
= 2\pi \frac{d_{\omega_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}}}}{(2\pi)^2 \sqrt{2J_{m_{\delta}} + 1}} \sum_{\omega_{m_{\delta}}} \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle} \times \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle}
\]

(22)

where \( \langle \omega_{m_{\delta}} \rangle \) is the polarization tensor of the noise radiation:

\[
\langle \omega_{m_{\delta}} \rangle = \sum_{\Delta \omega_{m_{\delta}}} (1 - e^{-\Delta t_{m_{\omega_{m_{\delta}}} m_{\delta}}}) \frac{\omega_{m_{\delta}}}{E_{\omega_{m_{\delta}}} E_{\omega_{m_{\delta}}}}.
\]  

(23)

The other outgoing and incoming frequencies can be obtained from (A21) and (A22) by label substitutions. In the \( \omega_{m_{\delta}} \)-representation for \( R_{m_{\omega_{m_{\delta}}} \omega_{m_{\delta}}} \) we have

\[
\nu(mm' \mid m_1, m_1; M_1)
\]  

\[
= 2\pi \frac{d_{m_{\omega_{m_{\delta}}} m_{\delta}}}{(2\pi)^2 \sqrt{2J_{m_{\delta}} + 1}} \sum_{\omega_{m_{\delta}}} \frac{1}{
\langle J_m, J_m, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle} \times \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle}
\]

(24)

\[
\nu(mm' \mid m_1, m_1; M_1)
\]  

\[
= 2\pi \frac{d_{\omega_{m_{\omega_{m_{\delta}}} m_{\delta}}}}{(2\pi)^2 \sqrt{2J_{m_{\delta}} + 1}} \sum_{\omega_{m_{\delta}}} \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle} \times \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle}
\]

(25)

where the outgoing and incoming frequencies are given by the following relationships:

\[
\nu(mm' \mid m_1, m_1; M_1) = \sum_{\omega_{m_{\delta}}} \frac{1}{
\langle J_m, J_m, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle} \times \frac{1}{
\langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle \langle J_{m_{\delta}}, J_{m_{\delta}}, \kappa \rangle}
\]

(26)
The other elements of the matrix $R$ in the $\kappa \varphi$—representation can be derived from Eqs. (A24)–(A27) by appropriate label substitutions.

The translational equation (A27) takes the form

$$\nabla \cdot \overrightarrow{\rho} = \frac{1}{\kappa^2} \left( \frac{d^2}{d\kappa^2} \right) \sum_{\kappa_1 \kappa_2} \left( \kappa_1 \kappa_2 | \kappa_1 \kappa_2 \right) \varphi$$

(A27)

The other elements of the matrix $R$ in the $\kappa \varphi$—representation can be derived from Eqs. (A24)–(A27) by appropriate label substitutions.


Translated by Eugene Yankovsky