Spontaneous magnetization in a dense neutron gas and a dense plasma of particles and antiparticles; magnetohydrodynamic waves in dense neutron matter

A. I. Akhiezer, N. V. Laskin, and S. V. Peletminskiĭ

Kharkov Physicotechenical Institute, 310108 Kharkov, Ukraine
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We show that a dense degenerate neutron gas can go into a magnetically ordered state and that the phase transition is second-order. We also show that the emergence of spontaneous magnetization explains the characteristic values of the density and magnetic field of neutron stars (pulsars). Using the nonrelativistic magnetohydrodynamics approximation, we establish the dispersion relation for magnetohydrodynamic waves propagating in dense magnetically ordered neutron matter. Finally, we study the problem of the origin of spontaneous magnetization in a dense electron–positron plasma. © *1996 American Institute of Physics.* © *1996 American Institute of Physics.* [S1063-7761(96)00506-9]

1. INTRODUCTION

In a dense degenerate neutron gas, an example of which is a neutron star, spontaneous magnetization can be caused by the nuclear interaction of neutrons. This interaction acts in a way similar to the Weiss molecular field, an exchange interaction leading to a ferromagnetic state of an electron gas. In this sense we can speak of the ferromagnetic state of dense neutron matter. Such a mechanism for the formation of magnetization of neutron matter explains the value of the observed magnetic field and density of a neutron star.

If magnetization is present, the value of the magnetic field can be estimated by the formula

$$B\sim\mu_n/a^3, \tag{1.1}$$

where $\mu_n=1.93\mu_{\text{nucl}}$ is the neutron's magnetic moment, $\mu_{\text{nucl}}\equiv e\hbar/2m_pc$ is the nuclear magneton, m_p is the proton mass, and a is the average neutron separation.

As is known, the concept of neutron matter has meaning when the density of matter ρ is about 10^{12} g/cm^3, which corresponds to a neutron density $n\sim10^{36}$ cm^{-3} (see Ref. 1). Assuming that $a\sim10^{-12}$ cm, we get

$$B\sim10^{13}\ \text{G},$$

which agrees with the astrophysical data on pulsar magnetic fields.[2] Note that if the gas density were higher and we would have $a\sim10^{-13}$ cm, the estimate would be $B\sim10^{16}$ G, which is much higher than the observed field strength. This suggests that the average neutron separation in a pulsar is closer to $a\sim10^{-12}$ cm than to $a\sim10^{-13}$ cm. (The distance of 10^{-13} cm constitutes the nuclear force range.)

In this paper we investigate the origin of spontaneous magnetization in neutron matter and study magnetohydrodynamic waves in such a medium in the nonrelativistic approximation. We show that in dense magnetically ordered neutron matter modified acoustic and spin waves can propagate.

We also use the nonrelativistic approximation to study the possibility of spontaneous magnetization in a dense electron–positron plasma.

2. SPONTANEOUS MAGNETIZATION OF A DENSE NEUTRON GAS

To describe a degenerate nonrelativistic neutron gas we use the main ideas of Landau's Fermi-liquid theory.[3] According to this theory, an equilibrium single-particle density matrix $\hat{f}$ can be found by solving the following nonlinear equation:

$$\hat{f}=\{\exp(Y_0(\hat{\varepsilon}(f)-\mu))+1\}^{-1}, \tag{2.1}$$

where μ is the chemical potential, $Y_0=1/T$ is the reciprocal temperature, and $\hat{\varepsilon}(f)$ is the quasiparticle energy, which is a functional of the single-particle density matrix. This energy is determined by the formula

$$\hat{\varepsilon}_{k_1k_2}(f)=\partial E(f)/\partial\hat{f}_{k_2k_1}, \tag{2.2}$$

where $k_i=\mathbf{p},\sigma$ stands for the set of the quantum numbers of a neutron ($\mathbf{p}$ is the neutron momentum and σ is the neutron spin), and $E(f)$ is the Fermi-liquid energy functional. The energy functional in Fermi-liquid theory is the analog of the system Hamiltonian. To describe a phase transition in a neutron Fermi liquid qualitatively we use the following ideas in building this functional. The interaction energy of two neutrons has the following form:

$$U(|\mathbf{r}_1-\mathbf{r}_2|)=U_1(|\mathbf{r}_1-\mathbf{r}_2|)+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2U_2(|\mathbf{r}_1-\mathbf{r}_2|), \tag{2.3}$$

where U_1 and U_2 are functions of neutron separation, and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are Pauli matrices (here we ignore tensor forces). Allowing for (2.3), we can write the energy functional in the form

$$E(f)=E_0(f)+\frac{1}{2}\int d\mathbf{r}_1d\mathbf{r}_2n(\mathbf{r}_1)U_1(|\mathbf{r}_1-\mathbf{r}_2|)n(\mathbf{r}_2)$$
$$+\frac{1}{2}\int d\mathbf{r}_1d\mathbf{r}_2\boldsymbol{\sigma}(\mathbf{r}_1)U_2(|\mathbf{r}_1-\mathbf{r}_2|)\boldsymbol{\sigma}(\mathbf{r}_2), \tag{2.4}$$

where $n(\mathbf{r})$ and $\boldsymbol{\sigma}(\mathbf{r})$ are the ordinary and spin neutron densities, related to the single-particle density matrix by the formulas

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$$n(\mathbf{r})=\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2}\mathrm{Tr}_{(\sigma)}\hat{f}_{\mathbf{p}_1,\mathbf{p}_2}\exp[i\mathbf{r}\cdot(\mathbf{p}_1-\mathbf{p}_2)], \quad (2.5)$$

$$\boldsymbol{\sigma}(\mathbf{r})=\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2}\mathrm{Tr}_{(\sigma)}\hat{f}_{\mathbf{p}_1,\mathbf{p}_2}\hat{\boldsymbol{\sigma}}\exp[i\mathbf{r}\cdot(\mathbf{p}_1-\mathbf{p}_2)], \quad (2.6)$$

and $E_0(f)$ is the neutron kinetic-energy functional:

$$E_0(f)=\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2}\mathrm{Tr}_{(\sigma)}\varepsilon_0(\mathbf{p})\hat{f}_{\mathbf{p}_1,\mathbf{p}_2}.$$

Since both $n(\mathbf{r})$ and $\boldsymbol{\sigma}(\mathbf{r})$ are slowly varying functions of $\mathbf{r}$, the energy functional can be written as

$$E(f)=E_0(f)+\frac{1}{2}\zeta_1\int d\mathbf{r}_1 n^2(\mathbf{r}_1)+\frac{1}{2}\zeta_2\int d\mathbf{r}_1\boldsymbol{\sigma}^2(\mathbf{r}_1),$$

where

$$\zeta_1=\int d\mathbf{r}U_1(|\mathbf{r}|), \quad \zeta_2=\int d\mathbf{r}U_2(|\mathbf{r}|).$$

Hence, using Eq. (2.2), we can write the quasiparticle Hamiltonian as

$$\hat{\varepsilon}_{k_1k_2}=\varepsilon_{0k_1k_2}+\zeta_1 n\hat{I}_{k_1k_2}+\zeta_2\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_{k_1k_2},$$

where $\sigma_{k_1k_2}=\sigma_{\sigma_1\sigma_2}\delta_{\mathbf{p}_1\mathbf{p}_2}$. (Here and in what follows we assume that a state is spatially uniform, so that the ordinary and spin densities are independent of x.)

Thus, we have the following equations for determining the single-particle density matrix:

$$\hat{f}=\{\exp(Y_0(\hat{\varepsilon}(f)-\mu'-\zeta_2\boldsymbol{\sigma}(f)\cdot\hat{\boldsymbol{\sigma}}))+1\}^{-1}, \quad (2.7)$$

$$n(f)=\frac{1}{V}\sum_{\mathbf{p}}\sum_{\sigma}\mathrm{Tr}\hat{f}_{\mathbf{p}}, \quad (2.8)$$

$$\sigma(f)=\frac{1}{V}\sum_{\mathbf{p}}\sum_{\sigma}\mathrm{Tr}\hat{\boldsymbol{\sigma}}\hat{f}_{\mathbf{p}}, \quad (2.9)$$

where

$$\mu'=\mu-\zeta_1 n(f).$$

Introducing the magnetic moment density of the neutron liquid

$$\mathbf{M}=\mu_n\boldsymbol{\sigma}(f),$$

we can write the basic equations (2.7)–(2.9) as

$$M=\frac{\sqrt{2}\mu_n m_n^{3/2}}{\pi^2\hbar^3}\times\left\{\int_0^\infty\frac{d\varepsilon\sqrt{\varepsilon}}{\exp[(\varepsilon-\mu'-\nu\mu_n M)/T]+1}-\int_0^\infty\frac{d\varepsilon\sqrt{\varepsilon}}{\exp[(\varepsilon-\mu'+\nu\mu_n M)/T]+1}\right\},$$

$$n=\frac{\sqrt{2}m_n^{3/2}}{\pi^2\hbar^3}\times\left\{\int_0^\infty\frac{d\varepsilon\sqrt{\varepsilon}}{\exp[(\varepsilon-\mu'-\nu\mu_n M)/T]+1}+\int_0^\infty\frac{d\varepsilon\sqrt{\varepsilon}}{\exp[(\varepsilon-\mu'+\nu\mu_n M)/T]+1}\right\}, \quad (2.10)$$

where we have allowed for the nonrelativistic dispersion law of the neutron gas, $\varepsilon(p)=p^2/2m_n$, with m_n the neutron mass, and $\nu=\zeta_2/\mu_m^2$. Note that $\zeta_2=a^3U_2$, where a is the nuclear force range ($a\sim 10^{-13}$ cm), and the quantity U_2 characterizing the intensity of nuclear forces is about 40 MeV. This provides an estimate for the parameter ν, namely $\nu\sim 3\times 10^3$.

Introducing the dimensionless magnetization density $x=8\alpha^2\nu^3\mu_n M/9\pi^4 m_n c^2$, the dimensionless chemical potential $\beta=8\alpha^2\nu^2\mu'/9\pi^4 m_n c^2$, the dimensionless density $\rho=8\alpha^3\nu^3\lambda_n^3 n/9\pi^4$, and the dimensionless temperature $\tau=8\alpha^2\nu^2 T/9\pi^4 m_n c^2$ (here $\alpha=e^2/\hbar c$, and $\lambda_n=\hbar/m_n c$ is the neutron Compton wavelength), we can write the system of equations (2.10) as

$$x=\tau^{3/2}\left(\Psi\left(\frac{\beta+x}{\tau}\right)-\Psi\left(\frac{\beta-x}{\tau}\right)\right),$$

$$\rho=\tau^{3/2}\left(\Psi\left(\frac{\beta+x}{\tau}\right)+\Psi\left(\frac{\beta-x}{\tau}\right)\right), \quad (2.11)$$

where

$$\Psi(z)=\frac{3}{2}\int_0^\infty\frac{d\zeta\sqrt{\zeta}}{e^{\zeta-z}+1}.$$

Clearly, the function $\Psi(z)$ defined in this way exhibits the following asymptotic behavior as $z\to\pm\infty$:

$$\Psi(z)=\begin{cases} z^{3/2}+\frac{\pi^2}{8}z^{-1/2}, & z\to+\infty,\\ \frac{3}{2}\sqrt{\pi}e^z, & z\to-\infty. \end{cases} \quad (2.12)$$

In addition to the trivial solution $x=0$, Eqs. (2.11) have a nontrivial one $x\neq 0$. To find the phase curve separating the regions $x=0$ and $x\neq 0$, we set $x=0$ in Eqs. (2.11). Then the parametric equation of the phase curve has the form

$$\tau=1/4(\Psi'(z))^2, \quad \rho=\Psi(z)/4(\Psi'(z))^3, \quad (2.13)$$

where the parameter z is linked to the chemical potential and temperature by the formula $z=\beta/\tau$. Using the asymptotic behavior (2.12) of $\Psi(z)$, we have

$$\tau=\begin{cases} \rho, & \rho\to\infty,\\ \frac{1}{\pi}\sqrt{\frac{2}{3}}(\rho-\rho_c)^{1/2}, & \rho\to\rho_c, \end{cases}$$

where $\rho_c=2/27$. The phase curve is depicted in Fig. 1.

We see that for $\rho<\rho_c$ there can be no magnetically ordered states. Let us fix the density ρ. A certain transition

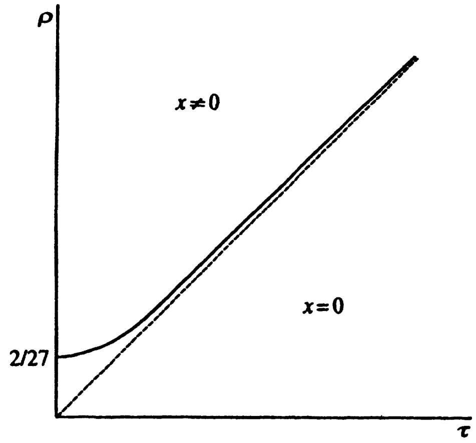

FIG. 1.

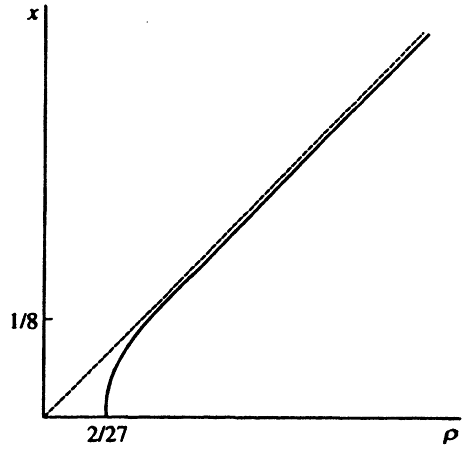

FIG. 2.

temperature $\tau=\tau_c(\rho)$ corresponds to this density. Then at $\tau\sim\tau_c$ the magnetization is given by the formula

$$x^2=A(\rho)(\tau_c(\rho)-\tau), \tag{2.14}$$

where $A(\rho)$ is a function of density defined in the following way:

$$A(\rho)=3\tau_c\frac{\Psi^2-3\Psi\Psi''}{\Psi'\Psi'''-3\Psi''^2}\bigg|_{z=z_c},$$

and the parameters τ_c and z_c as functions of density can be found from Eqs. (2.13). The temperature dependence (2.14) agrees with the self-consistent field theory.

Now we examine the case $\tau\ll 1$. What is interesting here is that for $\tau\ll 1$ we have the following estimate for real temperatures: $T\ll 10^8$ K. (The temperature at the surface of neutron stars is $T\ll 10^6$ K.)

If in what follows we assume $\beta>0$, the integrand in the definition of the function $\Psi[(\beta+x)/\tau]$ can be replaced with an "unsmeared" Fermi step; as for the function $\Psi[(\beta-x)/\tau]$, its asymptotic behavior depends on the sign of $\beta-x$. Noting that according to (2.12)

$$\Psi\left(\frac{\beta+x}{\tau}\right)\bigg|_{\tau\to 0}\to\left(\frac{\beta+x}{\tau}\right)^{3/2},$$

we arrive at following system of equations:

$$x=(\beta+x)^{3/2}-\tau^{3/2}\Psi\left(\frac{\beta-x}{\tau}\right),$$

$$\rho=(\beta+x)^{3/2}+\tau^{3/2}\Psi\left(\frac{\beta-x}{\tau}\right).$$

Eliminating the variable β from these equations yields

$$\frac{\rho-x}{2}=\tau^{3/2}\Psi\left(\frac{((\rho+x)/2)^{2/3}-2x}{\tau}\right). \tag{2.15}$$

To find the solution $x=x(\rho,\tau)$ of this equation for $\tau\to 0$, we first assume the parameters x and ρ to be such that

$$\left(\frac{\rho+x}{2}\right)^{2/3}-2x>0. \tag{2.16}$$

In this case Eq. (2.15) for $\tau\to 0$ becomes

$$2x=\left(\frac{\rho+x}{2}\right)^{2/3}-\left(\frac{\rho-x}{2}\right)^{2/3}. \tag{2.17}$$

According to Eq. (2.17), the inequality (2.16) holds if $\rho\leqslant 1/8$. Thus, the density dependence of magnetization corresponding to Eq. (2.17) is attained in the region $0<\rho\leqslant 1.8$. As Fig. 2 shows, spontaneous magnetization emerges for $\rho>\rho_c=2/27$. For $\rho<\rho_c=2/27$ there is only the trivial solution $x=0$, which correspond to the normal, or disordered, phase. Here the density ρ and the chemical potential β are related by the standard formula $\rho=2\beta^{3/2}$.

As Eq. (2.17) implies, near $\rho\geqslant\rho_c=2/27$ we have

$$x=\frac{1}{\sqrt{3}}(\rho-\rho_c)^{1/2},$$

which suggests that there is a second-order phase transition in density.

For densities $\rho>1/8$ condition (2.16) breaks down, with the result that for $\rho>1/8$ another condition must be met:

$$\left(\frac{\rho+x}{2}\right)^{2/3}-2x<0.$$

Hence, in analyzing Eq. (2.15) we must use the asymptotic behavior of $\Psi(z)$ as $z\to-\infty$ (see (2.12)). We then get

$$\frac{\rho-x}{2}=\tau^{3/2}\frac{3}{2}\sqrt{\pi}\exp\left\{-\frac{1}{\tau}\left(2x-\left(\frac{\rho+x}{2}\right)^{2/3}\right)\right\}. \tag{2.18}$$

As this formula implies, within the range of parameters considered here, $\rho\to x$ as $\tau\to 0$. Hence we have

$$x=\rho-3\sqrt{\pi}\tau^{3/2}\exp\left\{-\frac{1}{\tau}(2\rho-\rho^{2/3})\right\}.$$

Since, by assumption, $2\rho-\rho^{2/3}>0$, the above formula is valid when $\rho>1/8$. Equation (2.18) clearly implies that $\partial x/\partial\rho|_{\rho=1/8}\sim 1$.

Thus, for $\tau\to 0$, the density dependence of the magnetization has, according to (2.17) and (2.18), the form depicted in Fig. 2.

The density of a neutron star is $n \sim 10^{36}\,\mathrm{cm}^{-3}$, which corresponds to the value $\rho \sim 1/8$. Here $x=\rho$, and therefore we have

$$M=\frac{9}{8}\pi^4\frac{m_n c^2 x}{\alpha^2 \nu^3 \mu_n}\sim 10^{13}\,\mathrm{G}.$$

Thus, we see that a degenerate neutron gas can pass into a magnetically ordered state in response to variations of its density and that this is a second-order phase transition in density for $\rho \geqslant \rho_c$. The values of the magnetization and density of a neutron star predicted by our model agrees with astrophysical data.

3. SPONTANEOUS MAGNETIZATION OF AN ELECTRON–POSITRON PLASMA

The method used in Sec. 2 can be applied to the problem of spontaneous magnetization in a dense electron–positron plasma. We believe that the exchange interaction between the particles of such a plasma acts as an ordering interaction. We start with the nonrelativistic plasma. As shown below, such a system can be described by a system of equations of the form (2.10) in which the neutron magnetic moment μ_n should be replaced with the Bohr magneton μ_B and m_n should be replaced with the electron mass m. Here the factor ν can be determined from the structure of the Hamiltonian of the electron–positron interaction in the weakly relativistic case.[4] (Only in this case can we introduce the concept of an interaction potential between particles.) Precisely, the potential energy of the exchange interaction between electrons V_{ex}^e (or positrons V_{ex}^p) is given by the formula

$$V_{ex}^e=V_{ex}^p=-\frac{2\pi}{3}\frac{e^2\hbar^2}{m^2c^2}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\delta(\mathbf{r}_1-\mathbf{r}_2).$$

The exchange potential energy between an electron and positron is determined by

$$V_{ex}^{ep}=-V_{ex}^e+\frac{\pi}{2}\frac{e^2\hbar^2}{m^2c^2}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\delta(\mathbf{r}_1-\mathbf{r}_2)$$
$$=-\frac{7\pi}{6}\frac{e^2\hbar^2}{m^2c^2}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\delta(\mathbf{r}_1-\mathbf{r}_2).$$

Hence the self-consistent exchange energies of the electron–electron and positron–positron interactions in the Fermi-liquid model are

$$V_{--}=-\frac{\pi}{3}\frac{e^2\hbar^2}{m^2c^2}\int d\mathbf{r}x\boldsymbol{\sigma}_-(\mathbf{r})\cdot\boldsymbol{\sigma}_-(\mathbf{r}),$$

$$V_{++}=-\frac{\pi}{3}\frac{e^2\hbar^2}{m^2c^2}\int d\mathbf{r}x\boldsymbol{\sigma}_+(\mathbf{r})\cdot\boldsymbol{\sigma}_+(\mathbf{r}),$$

and the energy of the electron–positron exchange interaction is

$$V_{+-}=\frac{7\pi}{6}\frac{e^2\hbar^2}{m^2c^2}\int d^3x\boldsymbol{\sigma}_+(\mathbf{r})\cdot\boldsymbol{\sigma}_-(\mathbf{r}),$$

where $\boldsymbol{\sigma}_+(\mathbf{r})$ and $\boldsymbol{\sigma}_-(\mathbf{r})$ are, respectively, the electron and positron spin densities, related to the electron and positron density matrices $\hat{f}_{-\mathbf{p}_1,\mathbf{p}_2}$ and $\hat{f}_{+\mathbf{p}_1,\mathbf{p}_2}$ by the following formula:

$$\boldsymbol{\sigma}_\pm(\mathbf{x})=\sum_{\mathbf{p}_1\mathbf{p}_2}\exp[i\mathbf{x}\cdot(\mathbf{p}_1-\mathbf{p}_2)]\,\mathrm{Tr}_{(\sigma)}\hat{\boldsymbol{\sigma}}\hat{f}_{\pm\mathbf{p}_1,\mathbf{p}_2}$$

(see Eq. (2.9)).

Hence for the electron exchange energy $\varepsilon'_{-\mathbf{p}_1,\mathbf{p}_2}=\partial V/\partial f_{-\mathbf{p}_1,\mathbf{p}_2}$ and the positron exchange energy $\varepsilon'_{+\mathbf{p}_1,\mathbf{p}_2}=\partial V/\partial f_{+\mathbf{p}_1,\mathbf{p}_2}$ (where we have introduced the notation $V=V_{++}+V_{+-}+V_{--}$) we have

$$\varepsilon'_{+\mathbf{p}_1,\mathbf{p}_2}=-\frac{2\pi}{3}\frac{e^2\hbar^2}{m^2c^2}\delta_{\mathbf{p}_1,\mathbf{p}_2}\left(\boldsymbol{\sigma}_+-\boldsymbol{\sigma}_--\frac{3}{4}\boldsymbol{\sigma}_-\right)\cdot\hat{\boldsymbol{\sigma}}_+,\tag{3.1}$$

$$\varepsilon'_{-\mathbf{p}_1,\mathbf{p}_2}=-\frac{2\pi}{3}\frac{e^2\hbar^2}{m^2c^2}\delta_{\mathbf{p}_1,\mathbf{p}_2}\left(\boldsymbol{\sigma}_--\boldsymbol{\sigma}_+-\frac{3}{4}\boldsymbol{\sigma}_+\right)\cdot\hat{\boldsymbol{\sigma}}_-.\tag{3.2}$$

Thus, the self-consistency equations for the equilibrium electron and positron density matrices are

$$\begin{aligned}&\hat{f}_+=(\exp\beta(\varepsilon_+^0+\hat{\varepsilon}'_+)+1)^{-1},\\&\hat{f}_-=(\exp\beta(\varepsilon_-^0+\hat{\varepsilon}'_-)+1)^{-1},\\&\varepsilon_\pm^0=\varepsilon^0-\mu_\pm,\end{aligned}\tag{3.3}$$

where ε'_+ and ε'_- have been defined by Eqs. (3.1) and (3.2), $\mu_\pm$ are the positron and electron chemical potentials, and ε^0 is the kinetic energy of positrons and electrons. These self-consistency equations yield a self-consistency equation for the electron and positron spins (see Eq. (2.9)),

$$\boldsymbol{\sigma}_\pm=\frac{1}{(2\pi\hbar)^3}\int d^3p\mathrm{Tr}\hat{\boldsymbol{\sigma}}_\pm(\exp\beta(\varepsilon_\pm^0+\hat{\varepsilon}'_\pm)+1)^{-1},\tag{3.4}$$

and an equation for determining the electron and positron chemical potentials,

$$n_\pm=\frac{1}{(2\pi\hbar)^3}\int d^3p\ \mathrm{Tr}(\exp\beta(\varepsilon_\pm^0+\hat{\varepsilon}'_\pm)+1)^{-1}.\tag{3.5}$$

We now allow for the electroneutrality condition $n_+=n_-\equiv n$. In this case the system of Eqs. (3.4) and (3.5) has the solution

$$\sigma_+=\sigma_-\equiv\sigma,\quad \mu_+=\mu_-\equiv\mu,$$

where the quantities σ and μ can be found from the following equations;

$$\sigma=\frac{1}{(2\pi\hbar)^3}\int d^3p\times\left\{\left(\exp\beta\left(\varepsilon^0-\mu-\frac{3}{4}\xi\sigma\right)+1\right)^{-1}-\left(\exp\beta\left(\varepsilon^0-\mu+\frac{3}{4}\xi\sigma\right)+1\right)^{-1}\right\},\tag{3.6}$$

$$n=\frac{1}{(2\pi\hbar)^3}\int d^3p$$
$$\times\left\{\left(\exp\beta\left(\varepsilon^0-\mu-\frac{3}{4}\xi\sigma\right)+1\right)^{-1}\right.$$
$$\left.+\left(\exp\beta\left(\varepsilon^0-\mu+\frac{3}{4}\xi\sigma\right)+1\right)^{-1}\right\}, \quad (3.7)$$

with

$$\frac{3}{4}\xi=\frac{\pi}{2}\frac{e^2\hbar^2}{m^2c^2}.$$

(This situation corresponds to ferromagnetic spin ordering.) Since near the transition temperature the value of σ must be small, we easily see that the equation for determining the critical temperature, derived from (3.6), has nontrivial solutions, since $\xi>0$.

However, Eqs (3.4) and (3.5) also have a solution corresponding to antiferromagnetic spin ordering:

$$\sigma_+=-\sigma_-\equiv\sigma, \quad \mu_+=\mu_-\equiv\mu,$$

where

$$\sigma=\frac{1}{(2\pi\hbar)^3}\int d^3p\{(\exp\beta(\varepsilon^0-\mu-\zeta\sigma)+1)^{-1}$$
$$-(\exp\beta(\varepsilon^0-\mu+\zeta\sigma)+1)^{-1}\}, \quad (3.8)$$

$$n=\frac{1}{(2\pi\hbar)^3}\int d^3p\{(\exp\beta(\varepsilon^0-\mu-\zeta\sigma)+1)^{-1}$$
$$+(\exp\beta(\varepsilon^0-\mu+\zeta\sigma)+1)^{-1}\}, \quad (3.9)$$

with $\zeta=11\xi/4=11\pi e^2\hbar^2/6m^2c^2$.

Clearly, these equations coincide with Eqs. (2.10), with $M=2\mu\sigma$ and $\nu=\zeta/\mu^2$. Thus, the quantity ν in Eqs. (2.10), in relation to Eqs. (3.8) and (3.9) is given by the following formula:

$$\nu=22\pi/3.$$

Since Eqs. (2.10) coincide with the system of Eqs. (3.8) and (3.9), the analysis done in Sec. 2 is valid here, too, if m is substituted for m_n. The phase transition occurs, as shown in Sec. 2, when $n\geqslant\pi^4/12(\alpha\nu)^3\lambda^3$. Since for an electron–positron plasma $\nu=22\pi/3$ and the fine-structure constant $\alpha\sim1/137$, we have $\alpha\nu\sim1/6$. We see that the condition $n\geqslant\pi^4/12(\alpha\nu)^3\lambda^3$ is met for densities $n\geqslant10^{34}\,\mathrm{cm}^{-3}$.

If we are dealing with quarks rather than an electron–positron plasma, the fine-structure constant α must be replaced with the strong interaction constant $\alpha_s\sim1/10$, the interaction caused by gluon exchange. This suggests that in the nonrelativistic case a quark system can also undergo a phase transition with magnetic ordering. Note that for two interacting quarks ν can be calculated by analogy with the quantum electrodynamics calculation for the exchange interaction.[4]

Up to this point we have dealt with the nonrelativistic case. However, Eqs. (3.8) and (3.9) are also valid in the relativistic case if $\varepsilon(p)$ is interpreted as the relativistic dispersion law $\varepsilon(p)=\sqrt{p^2c^2+m^2c^4}$ and μ as the relativistic chemical potential, which differs from the nonrelativistic one by the presence of the term mc^2. Here we tacitly assume that an exchange interaction characterized by a phenomenological parameter ν can also be introduced into the relativistic region.

Note that the electron–positron plasma considered here, with a nonzero chemical potential μ, is actually in a quasi-equilibrium state since the plasma lifetime with respect to processes of annihilation and creation of electron–positron pairs,

$$\tau_a=\frac{1}{\pi r_0^2cn}, \quad T\ll mc^2$$

(where r_0 is the classical electron radius), is considerably longer in the nonrelativistic case than the Coulomb-collision relaxation time

$$\tau_r=\frac{\sqrt{2m}}{\pi e^4}\frac{T^{3/2}}{n\Phi(\eta)}.$$

Here

$$\Phi(\eta)=\ln(1+\eta)-\frac{\eta^2}{1+\eta}, \quad \eta=\frac{mT^2}{\pi e^2n},$$

i.e., $\tau_a/\tau_r=(T/mc^2)^{-3/2}\gg1$.

At $\mu=0$ the electron–positron plasma is in equilibrium with the photon gas. We do not study this case here, however, since it belongs to the relativistic region, where the possibility of introducing a parameter ν related to the exchange interaction requires a special investigation.

It is possible that the emergence of spontaneous magnetization in a dense gas of particles and antiparticles is part of the mechanism that led to separation of matter and antimatter at the early stages of the evolution of the universe.

4. MAGNETOHYDRODYNAMIC WAVES IN DENSE NEUTRON MATTER

In magnetically ordered neutron matter (Sec. 2) there can be modified acoustic and spin waves, just as there can be in a ferroelectric. Let us establish the law of dispersion for these waves in the nonrelativistic magnetohydrodynamics approximation.

In describing neutron matter by a magnetic moment per unit mass, $\boldsymbol{\mu}(\mathbf{r},t)$, and a velocity vector $\mathbf{v}(\mathbf{r},t)$, which we assume to be functions of the Eulerian coordinates $\mathbf{r}$ and time t, we denote the energy per unit mass by $F=F(\rho,\boldsymbol{\mu},\partial\mu_i/\partial x_k)$, where ρ is the neutron matter density. Then the equations of magnetohydrodynamics assume the form[5]

$$\frac{\partial\mathbf{v}}{\partial t}+v_i\frac{\partial\mathbf{v}}{\partial x_i}=\rho\mu_i\frac{\partial H_i}{\partial\mathbf{x}}+\mathbf{f}, \quad (4.1)$$

$$\frac{\partial\rho}{\partial l}+\operatorname{div}\mathbf{v}\rho=0, \quad (4.2)$$

where $\mathbf{H}$ is the magnetic field, and the force $\mathbf{f}$ is given by the following formula:

$$\mathbf{f}_l=\frac{\partial}{\partial x_l}\left\{-\rho^2\frac{\partial F}{\partial \rho}\delta_{lk}-\rho\frac{\partial F}{\partial(\partial\mu_i/\partial x_k)}\frac{\partial \mu_i}{\partial x_l}\right\}. \quad (4.3)$$

In contrast to the ordinary magnetohydrodynamic equation for a conducting medium, which contain the Lorentz force, our equations contain the force (4.3) caused by the nonuniformity of the magnetic field.

Equations (4.1)–(4.3) must be augmented by the equation of motion of the magnetic moment,

$$\frac{\partial \boldsymbol{\mu}}{\partial t}+v_i\frac{\partial \boldsymbol{\mu}}{\partial x_i}=g[\boldsymbol{\mu},\bar{\mathbf{H}}], \quad (4.4)$$

where $g=1.93e/m_pc$ is the gyromagnetic ratio and $\bar{\mathbf{H}}$ is the effective magnetic field,

$$\bar{\mathbf{H}}=\mathbf{H}-\frac{\partial F}{\partial \boldsymbol{\mu}}+\frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\rho\frac{\partial F}{\partial(\partial \boldsymbol{\mu}/\partial x_k)}\right), \quad (4.5)$$

and by the equations of magnetostatics,

$$\text{curl }\mathbf{H}=0, \quad \text{div}(\mathbf{H}+4\pi\rho\boldsymbol{\mu})=0. \quad (4.6)$$

(The magnetic moment density is $\mathbf{M}=\rho\boldsymbol{\mu}$.)

Equations (4.1)–(4.6) constitute the complete system of equations for describing the spectrum of the various vibrational modes of a magnetically ordered neutron liquid in the hydrodynamic approximation.

Below we assume that the energy density F has the form

$$F\left(\rho,\boldsymbol{\mu},\frac{\partial\mu_i}{\partial x_k}\right)=F_0(\rho,\boldsymbol{\mu})+\frac{\alpha_{ik}}{2}\frac{\partial \boldsymbol{\mu}}{\partial x_k}\cdot\frac{\partial \boldsymbol{\mu}}{\partial x_k},$$

where α is the exchange constant. To estimate the value of α in neutron matter we use the relationship between the exchange constant and the Curie temperature well-known from the theory of ferromagnetism. As is known (see, e.g., Ref. 5), in an ordinary ferromagnet

$$\frac{g_e\mu_B}{m_e}\alpha_s k^2=\frac{\theta}{\hbar}(ak)^2,$$

where $(g_e\mu_B/m_e)\alpha_s k^2=\omega_l$ is the spin wave spectrum, $\mu_B=e\hbar/2m_ec$ is the Bohr magneton, θ is the Curie temperature, and a is the electron separation. This relationship clearly implies that

$$\alpha_s=\theta a^2 m_e/\hbar g_e\mu_B.$$

Here, to estimate α, we replace θ with the energy U_2 of the nuclear interaction between neighboring neutrons, μ_B by μ_n, m_e by m_n, and g_e by g. We then have

$$\alpha=U_2a^2m_n/\hbar g\mu_n.$$

To linearize the system of equation (4.1)–(4.6) we first assume that the equilibrium state, the small deviation from which we are studying, is characterized by an equilibrium density ρ_0 and spontaneous magnetization $\boldsymbol{\mu}_0$, while the equilibrium velocity $\mathbf{v}_0$ and the magnetic field $\mathbf{H}$ are zero. Linearizing Eqs. (4.1)–(4.6) about this equilibrium state leads to the following expression for the frequencies of magnetohydrodynamic oscillations:

$$\omega_\pm^2=\frac{A\pm\sqrt{A^2-4B}}{2}, \quad (4.7)$$

where

$$A=\omega_s^2+\omega_0^2+\omega_1^2(\cos^2\vartheta+\alpha g^2\sin^2\vartheta)>0, \quad (4.8)$$

$$B=\omega_s^2(\omega_0^2+\omega_1^2\cos^2\vartheta)+\omega_1^2\omega_0^2\alpha g^2\sin^2\vartheta>0. \quad (4.9)$$

Here ϑ is the angle between the wave vector $\mathbf{k}$ and the magnetization vector $\boldsymbol{\mu}_0$, $\omega_s=g\mu_0\alpha k^2$ is the spin wave frequency, $\omega_0=k\sqrt{\partial P/\partial\rho}$ is the acoustic wave frequency, $P=\rho^2\partial F/\partial\rho$ is the pressure of neutron matter, and $\omega_1^2=4\pi\rho_0\mu_0^2k^2$.

Clearly, the conditions (4.8) and (4.9) imply that

$$A^2-4B=\{\omega_s^2-\omega_0^2+\omega_1^2(\alpha g^2\sin^2\vartheta-\cos^2\vartheta)\}$$
$$+\omega_1^4\alpha g^2\sin^2 2\vartheta>0,$$

with the result that $\omega_\pm^2>0$.

Let us analyze the dispersion relationship (4.7) in the two limiting cases of small and large wave vectors k.

When k is small, the term $\omega_s^2\sim k^4$ in the expressions for A^2 and A^2-4B can be ignored. Then for the case of weak magnetoelastic coupling $\omega_1^2\ll\omega_0^2$, or $4\pi M_0^2/\rho_0\ll\partial P/\partial\rho$, we have

$$\omega_\pm^2=\begin{cases}\omega_0^2+\omega_1^2\cos^2\vartheta,\\ \omega_1^2\alpha g^2\sin^2\vartheta.\end{cases} \quad (4.10)$$

We see that the branch ω_+ is a slightly modified ordinary acoustic branch, while ω_- coincides with the spin branch caused by the dipole–dipole interaction at small values of k. For the case of strong magnetoelastic coupling $\omega_1^2\gg\omega_0^2$, or $4\pi M_0^2/\rho_0\gg\partial P/\partial\rho$, we have

$$\omega_\pm^2=\begin{cases}\omega_1^2(\cos^2\vartheta+\alpha g^2\sin^2\vartheta)+\omega_0^2\dfrac{\cos^2\vartheta}{\cos^2\vartheta+\alpha g^2\sin^2\vartheta},\\ \omega_0^2\dfrac{\alpha g^2\sin^2\vartheta}{\cos^2\vartheta+\alpha g^2\sin^2\vartheta}.\end{cases}$$

In the limit of large wave vectors $k\to\infty$, when

$$A^2-4B=\omega_s^4-2\omega_s^2(\omega_0^2-\omega_1^2(\alpha g^2\sin^2\vartheta-\cos^2\vartheta)),$$

we have

$$\omega_\pm^2=\begin{cases}\omega_s^2+\omega_1^2\alpha g^2\sin^2\vartheta,\\ \omega_0^2+\omega_1^2\cos^2\vartheta.\end{cases}$$

The first branch, ω_+, coincides with the well-known expression for the frequency of a spin wave modified by the dipole–dipole interaction.

It appears that in the case of neutron matter we have weak magnetoelastic coupling. To verify this we estimate the value of $\sqrt{\partial P/\partial\rho}$ by using the following expression for the pressure of an ideal degenerate Fermi gas:[1] $P=(3\pi^2)^{2/3}\hbar^2n^{5/3}/5m_n$. Then for $n\sim10^{36}$ cm^{-3} we have $\sqrt{\partial P/\partial\rho}=10^9$ cm/s. For a neutron gas of this density we have $\sqrt{4\pi\rho_0\mu_0^2}\sim2\sqrt{\pi}\,10^7$ cm/s. Thus, we believe that the spectra of magnetoelastic oscillations (4.10) realize themselves in neutron stars.

Assuming that $k\sim1/R$, where $R\sim10^6$ cm is the neutron star radius, we find that $\omega_+\sim10^4$ s and $\omega_-\sim10^3$ s.

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