Intermittency and turbulent diffusion

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New forms are proposed for logarithmically normal and multifractal models of turbulent intermittency that ensure consistency of the space and time representations of the scaling laws of turbulent diffusion. The corresponding corrections in the power exponents are determined on the basis of an exact solution of the balance equation for the mean square vorticity (enstrophy) for a system of strongly interacting point vortex dipoles (maximally small vortex rings).

1. INTRODUCTION

1. The investigations of vortex intermittency of turbulence begun in the experiments of Refs. 1–3 (see also Refs. 4 and 5 and the references given there) confirmed the conjecture of Landau and Lifshitz[6] regarding limitation in the applicability of the concept and theory of locally isotropic turbulence of Kolmogorov and Obukhov,[7–9] which does not take into account the unbounded growth in the fluctuations of the dissipation rate of the turbulent energy

$$\varepsilon=\frac{\nu}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2$$

when there is a corresponding increase of the Reynolds number $\mathrm{Re}=L\langle u^2\rangle^{1/2}/\nu$, where L is the external turbulence scale. Indeed, in the theory of locally isotropic turbulence,[7,8] the fluctuations of ε were not taken into account, and it was assumed that in the inertial interval of scales $r_\nu \ll r \ll L$ [$r_\nu=(\nu^3/\langle\varepsilon\rangle)^{1/4}$ is the internal turbulence scale, and the angular brackets denote averaging] just the one physical parameter $\langle\varepsilon\rangle$ is sufficient to describe turbulent regimes by means of scaling laws. Some of these laws have the form

$$\langle(\Delta u)^p\rangle \sim \langle\varepsilon\rangle^{p/3} r^{\gamma_0(p)}, \quad \gamma_0(p)=p/3, \tag{1}$$

$$K \simeq \frac{dr^2}{dt} \sim \langle\varepsilon\rangle^{1/3} r^{\gamma_1}, \quad \gamma_1=4/3, \tag{2}$$

$$r^2 \simeq \langle\varepsilon\rangle t^{\gamma_2}, \quad \gamma_2=3, \tag{3}$$

$$E_L(\omega) \simeq \langle\varepsilon\rangle \omega^{-\gamma_3}, \quad \gamma_3=2, \tag{4}$$

$$D_L \simeq \langle\varepsilon\rangle t^{\gamma_4}, \quad \gamma_4=1, \tag{5}$$

where K is the turbulent diffusion coefficient, E_L is the Lagrangian frequency spectrum of the turbulence energy, and D_L is the structure function of the velocities of the Lagrangian particles that corresponds to this spectrum. When allowance is made for the fluctuations of ε, which serve as a natural measure of the observed (see Refs. 1–5) intermittency of turbulence, it becomes necessary to introduce corrections to the exponents γ_0–γ_4 in (1)–(5) and other scaling laws in the theory of locally isotropic turbulence. By intermittency, we here understand nonuniformity of the space–time distribution of the energy of the small-scale motions and localized eddy regions.[1,10]

In accordance with the observational data, the corrections due to the fluctuations ε are quantitatively small, but their presence has fundamental importance, for example, in connection with the property of scale invariance of the Navier–Stokes hydrodynamic equations with respect to stretching (and compressing) scaling transformations:[11]

$$r'=\lambda r, \quad u'=\lambda^{\alpha/3}u, \quad t'=\lambda^{1-\alpha/3}t,$$
$$(p/\rho)'=\lambda^{2\alpha/3}p/\rho, \quad \nu'=\lambda^{1+\alpha/3}\nu, \quad K'=\lambda^{1+\alpha/3}K, \tag{6}$$

where α is any number. We also have for ε from (6)

$$\langle\varepsilon_r\rangle \sim \langle(\Delta u)^3\rangle/r \sim \varepsilon_L (r/L)^{\alpha-1}, \tag{7}$$

where ε_r is obtained from ε by averaging over a region with characteristic dimension r, and the angular brackets denote statistical averaging, including averaging over the fluctuations of ε. Only the case $\alpha=1$ corresponds to the scaling laws of the Kolmogorov–Obukhov theory of locally isotropic turbulence, for which the relation $\langle\varepsilon_r\rangle \simeq \langle\varepsilon\rangle \sim \varepsilon_L$ corresponding to the representation (1)–(5) must hold. Thus, in the theory of locally isotropic turbulence, it is assumed that not only the hydrodynamic equations are scale invariant but that also $\langle\varepsilon_r\rangle$ possesses the same properties. This implies that in (6) too the value of α cannot differ from unity even by a small amount, and at the same time it is effectively impossible to introduce corrections for γ_0–γ_4 in (1)–(5) without violating the scale invariance of the original hydrodynamic equations. On the other hand, in the theoretical modeling of intermittency effects, for example, in Refs. 12–20, the fluctuations of ε are taken into account by replacing $\langle\varepsilon\rangle$ with ε_r in (1)–(5) and the other scaling laws of the theory of locally isotropic turbulence. The corresponding expressions are then averaged over the probability distribution of ε_r under the assumption that in fact

$$\langle\varepsilon_r\rangle=\langle\varepsilon\rangle=\mathrm{const}, \tag{8}$$

which means that $\langle\varepsilon_r\rangle$ is independent of r and t. Then in the scaling laws proportional to the first power of ε_r [such as (3)–(5)], corrections do not appear in the power exponents that are due to the fluctuations of ε, while in the intimately related scaling laws (1), (2) such corrections do occur. This makes the corresponding modified scaling laws inconsistent and can violate the scale invariance of the hydrodynamic equations. Indeed, under the condition (8) the presence, for example, of corrections to (2) is already incompatible with (6) and (7) for the value $\alpha=1$ corresponding to (8). Thus, in the logarithmically normal (LN) model of turbulence intermittency (formulated by the pioneers in the theory of locally isotropic turbulence[12,13] and refined in[14,15]) it is assumed that $\ln \varepsilon(\mathbf{x},t)$ and $\ln \varepsilon_r$ have a normal probability distribution with variance of the form[9]

$$\sigma^2_{\ln \varepsilon_r} \simeq A(\mathbf{x},t)+\mu \ln \frac{L}{r}, \tag{9}$$

where for large Reynolds numbers μ is a universal constant, and $A(\mathbf{x},t)$ may depend on the details of the macroscopic structure of the turbulent flow. Asymptotically, the LN distribution corresponds to the distribution with respect to the diameters of particles obtained as a result of a series of successive independent fractionations, and therefore the LN model can serve as a natural model of a cascade process in which ever smaller eddies are generated successively in a turbulent flow. Below, in Sec. 3 of this paper, we shall demonstrate the compatibility of the LN distribution for ε with exact solutions of the hydrodynamic equations corresponding to a system of dynamically interacting point vortex dipoles (maximally small vortex rings).

For the LN model, the quantity $\langle \varepsilon_r^q \rangle$ (the angular brackets here denote averaging over the LN distribution of ε_r) has the form[9]

$$\frac{\langle \varepsilon_r^q \rangle}{\langle \varepsilon \rangle^q} = \exp\left(qm + \frac{\sigma^2 q^2}{2} \right). \tag{10}$$

where $m=\langle \ln \varepsilon_r \rangle$, and the form of σ^2 is determined in (9). From (8) we then obtain the relationship between m and σ^2:

$$m=-\sigma^2/2. \tag{11}$$

Thus, (8)–(10)yield the equation

$$\langle \varepsilon_r^q \rangle = \langle \varepsilon \rangle^q (L/r)^{\mu(q)}, \tag{12}$$

where for the considered formulation of the LN model the function $\mu(q)$ has the form

$$\mu(q)=\frac{\mu_0}{2} q(q-1), \tag{13}$$

where in accordance with Ref. 9 we have $\langle \varepsilon(\mathbf{x}+\mathbf{r})\varepsilon(\mathbf{x}) \rangle \sim (L/r)^{\mu_0}$.

In Ref. 4 it is noted that there is better agreement between the conclusions of the LN model and the experimental data [including the data for large q ($q \leqslant 16$)] than for forms of the β model[16–21] with linear dependence $\mu(q) \sim q$ (for all q) in (12). In Ref. 17, it is shown that quite generally all models of intermittency using the conditions of scaling of the cascade process of vortex fractionation can correspond to the representation (12) [with different expressions for $\mu(q)$]. In particular, for the Novikov–Styuart model,[16] which is the basis of various fractal and multifractal forms of the β model,[18–21] the function $\mu(q)$ has the form

$$\mu(q)=\mu_0(q-1), \tag{13'}$$

where we write $\mu_0=\log_2(1/\beta)>0$, and β does not depend on the step n of the cascade fractionation of the vortices and is equal to the ratio of the volume occupied by the vortices with scale $L/2^{n+1}$ to the volume of the original (decaying) vortex with scale $L/2^n$. In the multifractal models of Refs. 20 and 21, β is assumed to be dependent on the step n and may be a random variable. The LN model in the interpretation of Ref. 9 can be regarded as a special case of a multifractal model but with continuous distribution of the vorticity in space and LN distribution for $1/\beta_n$ that is mutually independent for each n.

It is obvious that both for $\mu(q)$ in (13) and for $\mu(q)$ in (13′) [and the modifications of it corresponding to the various multifractal models,[18,20] [see, e.g., (A9) in the Appendix] $\mu(1)=0$ by virtue of the condition (8) of scale invariance of $\langle \varepsilon_r \rangle$ assumed in these models. Therefore, in these models corrections to the exponents γ_2–γ_4 cannot arise in (3)–(5), but for (1) and (2) such corrections to γ_0, γ_1 do occur,[22] and this leads to inconsistent representations of interrelated scaling laws such as (2) and (3) [(2) and (4)].

In this paper, we consider some ways of obtaining mutually consistent corrections to the scaling laws (1)–(5) corresponding to preservation of the property of scale invariance of the Navier–Stokes equations for $\alpha \neq 1$ in (6), (7). In Refs. 21 and 23, attempts were also made to obtain such consistent representations for the modifications (2), (3) on the basis of approaches that differ from that of the present paper.

2. The observation of Landau and Lifshitz[6] noted above actually already indicates that in the theory of locally isotropic turbulence it is necessary to introduce, in addition to the parameter $\langle \varepsilon \rangle$, additional physical quantities (one or several) that carry information about the large-scale properties of the turbulent flow. In this section, we propose to use, as the parameter in addition to $\langle \varepsilon \rangle$, the mean energy $e=\langle \mathbf{u}^2 \rangle$ of the turbulent flow. We note that Kraichnan's attempt[24] to describe turbulence by means of the single parameter e established the relative acceptability of such a description only for Lagrangian statistical characteristics of the turbulence, since neither the 5/3 law nor the other associated scaling laws of the theory of locally isotropic turbulence can be reproduced by such a description. On the other hand, not only the possibility but also the necessity of describing turbulence (in the asymptotic limit of short evolution times t) by means of two parameters—the mean energy e and the mean square vorticity (enstrophy) $\Omega^2=\langle (\mathrm{curl}\ \mathbf{u})^2 \rangle$—was established in Ref. 25. Then the role of ε is played by $\tilde{\varepsilon}=e\Omega$, and the mutually consistent scaling laws corresponding to (2)–(5) have in the inertial interval the form (obtained from dimensional and scaling considerations)[25]

$$K(r) \simeq \frac{dr^2}{dt} \sim \tilde{\varepsilon}^{1/(3+z)} \Omega^{z/(z+3)} r^{4/3+2z/3(3+z)}, \tag{14}$$

$$r^2 \simeq \tilde{\varepsilon} \Omega^z t^{3+z}, \tag{15}$$

$$E_L(\omega) \simeq \tilde{\varepsilon} \Omega^z \omega^{-(2+z)}, \tag{16}$$

$$D_L(t) \simeq \tilde{\varepsilon} \Omega^z t^{1+z}, \tag{17}$$

which is identical to (2)–(5) if $z=0$ and $\tilde{\varepsilon}$ is replaced by $\langle \varepsilon \rangle$.

The relations (14)–(17) are consistent for all values of the free parameter z, the value of which can be determined, for example, in an experimental investigation of turbulence intermittency, in particular, from the rate of "separation" of Lagrangian particles. The parameters e and Ω (especially e, since the low harmonics make the largest contribution to its value) contain information about the components of the motion in the turbulent flow with the largest scales, and the parameter z actually regulates their influence on the turbulent regime in the inertial interval of scales. Scale invariance of the hydrodynamic equations is then preserved for α in (6), (7) of the form

$$\alpha=1+\frac{2z}{3+z}. \tag{18}$$

In contrast to the invariant parameter e, the enstrophy Ω^2 in a three-dimensional turbulent flow can have singularities (regularized only by dissipative effects) that develop in the process of evolution in time as a result of the stretching of vortex filaments.[9] Therefore, the use of Ω^2 in (14)–(17) is limited to time intervals determined, for example, by the time of explosive growth of Ω^2 (Refs. 26 and 27).

At the same time, in the theory of locally isotropic turbulence there is implicitly assumed to exist a finite limit

$$\langle\varepsilon\rangle\sim\lim_{\nu\to 0}\nu\Omega^2,$$

which, as it happens, can be realized at the explosive singularities in the evolution of Ω^2 (Refs. 26 and 27). In this respect, the use of the parameter $\langle\varepsilon\rangle$ instead of Ω^2 is more convenient. Then in place of (14)–(17) we obtain

$$K\simeq\langle\varepsilon\rangle^{(1+\gamma)/(3+\gamma)}e^{-\gamma/(3+\gamma)}r^{2(2+\gamma)/(3+\gamma)}, \tag{14'}$$

$$r^2\simeq\langle\varepsilon\rangle^{1+\gamma}e^{-\gamma}t^{3+\gamma}, \tag{15'}$$

$$E_L(\omega)\simeq\langle\varepsilon\rangle^{1+\gamma}e^{-\gamma}\omega^{-2-\gamma}, \tag{16'}$$

$$D_L(t)\simeq\langle\varepsilon\rangle^{1+\gamma}e^{-\gamma}t^{1+\gamma}. \tag{17'}$$

From comparison of (14′)–(17′) with (14)–(17) it is obvious that $\gamma=z$ holds, and the relation (18) keeps its form for the representation (14′)–(17′).

3. We now consider the possibility of determining the corrections to the exponents γ_0–γ_4 in (1)–(5) by analyzing the dynamics of strongly interacting vortex charges. For this, we use the evolution equation obtained in Ref. 27 for the balance of the enstrophy Ω^2 corresponding to the dynamical interactions of pairs of point vortex dipoles (i.e., the smallest possible vortex rings or spherical Hill's vortices). Indeed, in Ref. 27 the following equation was obtained for the dimensionless quantity $u=\Omega^2/\Omega^2(0)$:

$$\frac{du}{dt}=A(t)u-\nu B(t)u^{7/5}, \tag{19}$$

where the functions $A(t)$ and $B(t)$ have an explicit form determined by the exact solutions obtained in Ref. 27 for the dynamics of two interacting vortex dipoles. Here we define $\Omega^2=\int d^3x\omega^2(\mathbf{x},t)$, where ω is the distribution of the vorticity corresponding to this vortex system.

It is obvious from (19) that in the limit $\nu\to 0$ the quantity u can indeed have an LN distribution if the function $A(t)$ corresponds to a Gaussian random process.

With allowance for the equations of the vortex dipole dynamics,[27] Eq. (19) can be written in a form corresponding to a certain new conservation law (for $\nu=0$):

$$\frac{d}{dt}\ln\left[\frac{u(l/l_0)^{3(p-1)/2}}{(\gamma/\gamma_0)^p}\right]=-\frac{6\nu\alpha^2}{b^2}, \tag{20}$$

where $l(t)$ is the separation between two vortex dipoles $[l_0\equiv l(0)]$, $\gamma\equiv|\gamma|$ is the absolute value of their Lamb momentum (these momenta are equal in absolute magnitude but have opposite directions, corresponding to zero invariant total momentum of the vortex pair). In (20), the geometrical factors α ($\alpha\gg 1$), b ($b\to 0$), and p ($p=$const) correspond to a regularizing "smearing" of the Dirac δ function in the definition of the vortex field of a point vortex dipole. At the same time, α and b can be functions of t, so that, for example, (19) corresponds to (20) for $\alpha=$const and $b=b(t)$. Such a regularization, made in Ref. 27, is analogous to the method of regularization proposed by Landau[28] in quantum field theory. In Ref. 27 the δ function is replaced by a regular smooth normalized function $\tilde{\delta}$ whose support has radius b:

$$\tilde{\delta}(\mathbf{x})=a_0r^\alpha(b-r)^2\theta(b-r)\left[1+y\,\frac{3\cos^3\vartheta-1}{2}\right], \tag{21}$$

where θ is the Heaviside function; $\int d^3x\tilde{\delta}(\mathbf{x})=1$ (a_0 is determined from this normalization condition); $y=$const; and r and ϑ are spherical coordinates. For $\tilde{\delta}(\mathbf{x})$ in (21), the value of p in (20) can, in particular, have the form

$$p\simeq\begin{cases}1+\frac{3}{5}y+\ldots, & \alpha\gg 1,\quad y\ll 1,\\ \frac{11}{8}\left(1-\frac{13}{248y}+\ldots\right), & \alpha\gg 1,\quad y\gg 1.\end{cases} \tag{22}$$

In accordance with Ref. 27,

$$\Omega^2\simeq\frac{\gamma^2(t)\alpha^3}{48\pi b^5}\left(1+\frac{1}{5}y+\frac{8}{35}y^2\right),\quad \alpha\geq 137,$$

so that the values of α and b in (20) can be functions of the time t.

In contrast to the two-dimensional case, when for $\nu=0$ the enstrophy Ω^2 is invariant (i.e., $u=1$), in three-dimensional turbulence the enstrophy Ω^2 does depend on t due to the effect of stretching of the vortex filaments.[9] However, as can be seen from (20), for $\nu=0$ and for the three-dimensional case there exists a similar but new invariant J for a turbulent regime formed from an ensemble of realizations of pairs of dynamically interacting vortex dipoles:

$$J=\frac{\Omega^2(t)}{\Omega^2(0)}\,\frac{[l(t)/l(0)]^{3(p-1)/2}}{[\gamma(t)/\gamma(0)]^p}=\text{const}. \tag{23}$$

In the case when the "smearing" of the δ function is spherically symmetric [in particular, for $y=0$ in (21)], i.e., $\delta\to\tilde{\delta}=\delta_0(r)$, we have $p=1$ in (20) and (23) for all normalized functions $\delta_0(r)$ of compact support.

Let us consider, for example, the case corresponding to asymptotic separation of the dipoles in the limit $t \to \infty$, when the invariant interaction energy

$$H = \frac{\gamma^2}{4\pi l^3}\left[1 - \frac{3(\gamma \mathbf{l})^2}{\gamma^2 l^2}\right]$$

of the vortices is negative, so that[27]

$$\frac{\gamma^2}{l^3} \to 2\pi|H| = \text{const}, \quad l(t) \simeq t^{2/5}\left(\frac{25|H|}{2\pi}\right)^{1/5}, \quad t \to \infty. \tag{23'}$$

From (23′) and (20) for all p, we obtain

$$\frac{d}{dt}\ln\left[u\left(\frac{l_0}{l(t)}\right)^{3/2}\right] = -\frac{6\nu\alpha^2}{b^2}. \tag{24}$$

For $\nu=0$, the invariant J from (23) corresponding to (24) has the form

$$J = \frac{\Omega^2(t)}{\Omega^2(0)}\left(\frac{l_0}{l(t)}\right)^{3/2} = \text{const}, \tag{25}$$

which corresponds to unbounded growth of the enstrophy Ω^2 with increasing intervortex separation $l(t)$ in the limit $t \to \infty$.

It is obvious from (25) that u and $\varepsilon \sim \nu\, \Omega^2$ can have the LN probability distribution if $\ln|H|$ is a random Gaussian variable. Since ε_r is also proportional to Ω^2, we have for $\langle \varepsilon_r^q \rangle$ the relation (10), in which m and σ^2 have the form

$$m = \frac{3}{5}\ln t + \frac{3}{10}\langle \ln|H| \rangle,$$

$$\sigma^2 = 0.09(\langle(\ln|H|)^2\rangle - \langle \ln|H| \rangle^2). \tag{26}$$

It is clear from (26) that the conditions (8) and (11) [which are usually assumed in the LN model (see Ref. 9)] are not satisfied in the limit $t \to \infty$, since

$$m \simeq \frac{3}{5}\ln t, \quad \sigma^2 = \text{const}.$$

For $\nu \neq 0$, the representation (26) is not changed qualitatively. In particular, for the case $b=\text{const}$ and an unknown function $\alpha = \alpha\ (t)$ the solution of Eq. (24) has the form

$$u = \frac{(t/t_0)^{3/5}}{[1 + (4/\mathrm{Re}_b)(t/t_0)^{3/5}]^{3/2}}, \tag{27}$$

since $\Omega^2 \simeq (\gamma^2\alpha^3/48\pi b^5)(1 + O(y))$, where $t_0 = (2\pi l_0^5/25|H|)^{1/2}$ is the characteristic time of approach (collapse) of the vortex dipoles, after which they separate unboundedly, $\mathrm{Re}_b = b_0^2/\nu t_0 \alpha_0^2$ is the internal Reynolds number determined by the ratio of the time of viscous spreading of the internal structure of the vortex to $t_0[b_0 \equiv b(0), \alpha_0 \equiv \alpha(0))$. Note that we have $\mathrm{Re}_b = (b_0^2/l_0^2\alpha_0^2)\mathrm{Re}$, where $b_0 \ll l_0$, $\mathrm{Re} = v_0 l_0/\nu$, and $v_0 = l_0/t_0$. Thus, for $t \ll t_0\, \mathrm{Re}_b^{5/3}$ the representation (26) is unchanged for finite $\nu \neq 0$ as well, while for $t \gg t_0\, \mathrm{Re}_b^{5/3}$ we have $u \sim (t/t_0)^{-0.3}\, \mathrm{Re}_b^{3/2}$ and the exponent 3/5 in (26) must be replaced by $-3/10$.

We now consider the case when, conversely, $\alpha=\text{const}$, and the unknown function determined from (24) is the "smearing" radius $b(t)$. Then the solution of (24) has the form

$$u(t) = \frac{(t/t_0)^{3/5}}{[1 + (60/19\ \mathrm{Re}_b)(t/t_0)^{19/25}]^{5/2}}. \tag{28}$$

Therefore, as for (27), the exponent 3/5 in (26) is changed (here to $-13/10$) only for $t \gg t_0\, \mathrm{Re}_b^{25/19}$.

Thus, in the limit $\nu \to 0$ ($t \ll t_0\, \mathrm{Re}_b^{5/3}$) the turbulent diffusion coefficient K and the variance r^2 of the cloud of impurity for the modification of the LN model of intermittency considered in this section have a mutually consistent form:

$$K \simeq \frac{dr^2}{dt} \simeq \langle \varepsilon \rangle^{1/3} r^{4/3 + 1/9}, \tag{29}$$

$$r^2 \simeq \langle \varepsilon \rangle t^{3 + 3/5}, \tag{30}$$

where (29) and (30) are obtained from (10) and (26) for $q=1/3$ and $q=1$ in the limit of large t ($t \gg t_0$). For finite ν and $t \to \infty$ ($t \gg t_0\, \mathrm{Re}_b^{5/3}$), we obtain from (27) and (28) in place of (30)

$$r^2 \simeq \langle \varepsilon \rangle t^3 (t/t_0)^{-0.3}\, \mathrm{Re}_b^{3/2} \quad (\alpha = \alpha(t), \quad b = \text{const}), \tag{31}$$

$$r^2 \simeq \langle \varepsilon \rangle t^3 (t/t_0)^{-13/10}\, \mathrm{Re}_b^{5/2} \quad (b = b(t), \quad \alpha = \text{const}). \tag{32}$$

Thus, without the conditions (8) and (11) we can obtain consistent corrections to (2) and (3) even in the LN model due to the fluctuation effects of the turbulence intermittency. At the same time, scale invariance of the hydrodynamic equations is preserved for $\alpha \neq 1$ in (6) and (7), since for (29) and (30) we have $\alpha = 4/3$.

Note that the numerical value 3/5 of the correction to γ_2 in (3) corresponding to (30) agrees with the analogous value of the correction obtained in Ref. 23 (when $t \gg t_R$) for $\mu_0 \approx 0.17$. In turn, as recent experiments[5] have shown, we have $\mu_0 \approx 0.2$. In Ref. 29, the experimentally observed dependence $r^2 \simeq t^{3+z}$, where z lies in the interval $0.15 \leq z \leq 0.45$, was noted, while for the multifractal model of Ref. 21 the value of z (for $p_1 = 0.5$, $p_2 = 1$, $x = 0.125$) was found to be negative [in Ref. 21, we have $z \simeq -0.28$ in the same way as happens in (31) when allowance is made for finite $\nu \neq 0$].[1)]

In the Appendix, we give a modification of the multifractal model (described in Ref. 18) in which it is possible to reconcile the corrections to (2) and (3) even when the condition (8) of scale invariance of $\langle \varepsilon_r \rangle$ is maintained.

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APPENDIX

It will be shown below that in the usual multifractal model of turbulence intermittency it is possible to obtain agreement of the time and space representations of the scaling laws of turbulent diffusion if we bear in mind $\langle l^2 \rangle_\varepsilon \neq \langle l \rangle_\varepsilon^2$ and recall that the coefficient of turbulent diffusion K is expressed in terms of the derivative of $\langle l \rangle_\varepsilon^2$ (and not of $\langle l^2 \rangle_\varepsilon$, as is usual), i.e.,

$$\tilde{K} = \frac{1}{2}\frac{d}{dt}\langle l \rangle_\varepsilon^2, \quad l = \langle \mathbf{r}^2 \rangle_L^{1/2},$$

where $\langle ... \rangle$ denotes averaging over the ensemble of realizations of the turbulent flow, $\mathbf{r}$ is the radius vector connecting two fluid particles in the flow, and $\langle ... \rangle_\varepsilon$ denotes statistical averaging over the distribution of the probabilities of the fluctuations of the dissipation rate of the turbulence energy ε.

We consider the representation for l in terms of ε and t (we shall then omit the subscript ε of $\langle ... \rangle_\varepsilon$):

$$l \sim \varepsilon_{\langle l \rangle}^{1/2} t^{3/2}, \tag{A1}$$

where the time of relative diffusion is assumed to be independent of ε, and $\varepsilon_{\langle l \rangle}$, in contrast to ε_l [or ε_r in (7) and (8)] is the value of ε averaged over a sphere of radius $\langle l \rangle$ at a fixed instant of time. The value of $\langle l \rangle$ itself can be determined by averaging the left- and right-hand sides of (A1) over the probability distribution of the fluctuations of ε, which depends parametrically on $\langle l \rangle$. For this, by analogy with (12), we use the relation

$$\langle \varepsilon_{\langle l \rangle}^q \rangle = \langle \varepsilon \rangle^q \left(\frac{L}{\langle l \rangle} \right)^{\mu(q)}, \tag{A2}$$

which differs from (12) only by the replacement of l by $\langle l \rangle$. From (A1) and (A2), we readily obtain the representation

$$\langle l \rangle = \langle \varepsilon \rangle^{1/2[1+\mu(1/2)]} t^{3/2[1+\mu(1/2)]}, \tag{A3}$$

from which it is obvious that $\langle l \rangle^2 \neq \langle l^2 \rangle$ and the law of time evolution for $\tilde{K}$ has the form

$$\tilde{K} \equiv \frac{1}{2} \frac{d\langle l \rangle^2}{dt} \sim \langle \varepsilon \rangle^{1/[1+\mu(1/2)]} t^{[2-\mu(1/2)]/[1+\mu(1/2)]}. \tag{A4}$$

To obtain the space representation of $\tilde{K}$, it is necessary to average the corresponding scaling law $\tilde{K} \simeq \varepsilon_{\langle l \rangle}^{1/3} \langle l \rangle^{4/3}$ over the fluctuations of $\varepsilon_{\langle l \rangle}$ with allowance for (A2). Then, taking into account the fact that $\langle l \rangle$ does not depend on $\varepsilon_{\langle l \rangle}$ [i.e., $\langle f(\langle l \rangle) \rangle = f(\langle l \rangle)$ for any function f], we obtain

$$\tilde{K} \sim \langle \varepsilon \rangle^{1/3} \langle l \rangle^{4/3-\mu(1/3)}. \tag{A5}$$

For the function $\mu(q)$, we have in accordance with Ref. 18 the conditions

$$\mu(1) = \mu(0) = 0, \quad 0 < \mu(2) \equiv \mu_0 < 1,$$
$$\mu(q+\delta) - \mu(q) \leq \delta, \quad \delta \geq 0,$$
$$\mu(q) \leq \mu + q - 2, \quad q \geq 2. \tag{A6}$$

If the space, and time, representations of $\tilde{K}$ [(A5) and (A4)] are to be mutually consistent, we must in addition to (A6) require fulfillment of a new condition on the form of the function $\mu(q)$:

$$3\mu(1/3) = 2\mu(1/2), \tag{A7}$$

which has not previously been considered in intermittency models. For analogous consistency of the representations for $d^{m-1}\tilde{K}/dt^{m-1}$, we obtain instead of (A7)

$$2\mu(1/2) = (3/m)\mu(m/3), \tag{A8}$$

where $m=1,2,...$. For $m=1$, (A8) is identical to (A7). The case $m=2$, which corresponds to the correlation function of the Lagrangian velocities, also has physical meaning.

For $m=1$ and $m=2$, the condition (A5) is satisfied, for example, by a function $\mu(q)$ of the form

$$\mu(q) = \begin{cases} q \sum_{k=0}^{\infty} b_k \cos(12\pi q k), & 1/3 \leq q \leq 2/3, \\ C_1[\log_2(\bar{g}_1^q + \bar{g}_2^q) - 1], & q < 1/3, \\ \log_2(g_1^q + g_2^q) - 1,, & q \geq 2/3, \end{cases} \tag{A9}$$

where

$$C_1 = \frac{1}{2} \frac{\log_2(g_1^{2/3} + g_2^{2/3}) - 1}{\log_2(\bar{g}_1^{1/3} + \bar{g}_2^{1/3}) - 1}, \quad \sum_{k=0}^{\infty} b_k = \frac{3}{2} [\log_2(g_1^{2/3} + g_2^{2/3}) - 1],$$

and $g_1 + g_2 = \bar{g}_1 + \bar{g}_2 = 2$. In (A9), $\mu(q)$ satisfies the conditions (A6), since for $q \geq 2/3$ the function (A9) is identical to the representation for $\mu(q)$ in the multifractal model of intermittency [see the expression (30) in Ref. 18]. Of course, (A9) is not the only possible representation of $\mu(q)$ that satisfies (A6) and (A7), i.e., ultimately the form of $\mu(q)$ depends on the particular form of the model of turbulence intermittency.

Thus, we give one further example for which it is possible to obtain consistency of the space and time representations of the scaling laws of turbulent diffusion. In particular, for the $g_1 = 0.6$ and $g_2 = 1.4$ adopted in Ref. 18, we obtain from (A9) and (A3) the estimate

$$\langle l \rangle^2 \sim t^{3+0.154}, \tag{A10}$$

which for $\langle l \rangle \sim r$ already agrees quantitatively with the observational data of Ref. 30 (see above $r^2 \sim t^{3+z}$, $0.15 < z < 0.45$ in Ref. 30). Experimental observations of passive Lagrangian particles (in Ref. 31, these were balloons of neutral buoyancy) can be used to determine more accurately the corrections in the scaling laws (1)–(5) and their mutual consistency.

Since regions with pronounced enhancement of dissipation can be related to collapse effects and coalescence of localized vortex singularities, it will be of interest to determine the parameters that occur in the scaling laws of turbulent diffusion [as was done above, see (29)–(32)]by analyzing the separation of particles of real admixtures (having different coefficients of molecular diffusion, see Ref. 32) in a field of vortex singularities of different multipolarities.[27,32,33]

[1)]In Ref. 29, $z = -4/3$, and this differs little from $z = -13/10$ in (32). In Ref. 29, this estimate was obtained from dimensional and scaling considerations from the idea of the realization of a local cascade transformation of angular momentum that is invariant for the system of point vortex dipoles considered in Ref. 27.

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