Nonlinear dynamics of multiple-sublattice magnetic materials with modulated magnetic structure

A. L. Sukstanskiĭ

Donetsk Physicotechnical Institute, Ukrainian Academy of Sciences, 340114 Donetsk, Ukraine

B. A. Ivanov

Institute of Metal Physics, Ukrainian Academy of Sciences, 252142 Kiev, Ukraine

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The nonlinear dynamics of multiple-sublattice noncollinear antiferromagnets with modulated magnetic structures are studied using the effective Lagrangian method. An analysis is performed in the concrete example of a three-sublattice model of magnets of the $CsCuCl_3$ type, in which a helical structure exists along with the triangular ordering of the spins in the basal plane. One- and two-parameter solutions of the equations of motion, which describe different nonlinear excitations in the magnet, are found, and the fundamental influence of an external magnetic field on the character of the soliton solutions is demonstrated.

1. INTRODUCTION

Recently, there has been steady interest in theoretical and experimental investigations of noncollinear antiferromagnetic crystals, which include, in particular, many multiple-sublattice antiferromagnets, as well as numerous antiferromagnets with an incommensurate magnetic structure in the ground state. The investigations of such magnets have been concerned mainly with describing their static properties, studying phase transitions of various kinds, analyzing the spectrum of linear excitations (spin waves), etc.

Considerably less attention has been focused on the nonlinear dynamics of magnets, whose description requires the use of models with three or more sublattices [in contrast to one-sublattice ferromagnets and two-sublattice antiferromagnets, in which nonlinear excitations have been studied in great detail (see, for example, Refs. 1 and 2)]. Although some models of multiple-sublattice magnets make it possible to find very simple nonlinear solutions (for example, the domain boundaries in a four-sublattice antiferromagnet of the La_2CuO_4 type,[3]) in the general case the integration of the equations of motion (the Landau–Lifshitz equations) for the magnetization vectors $\mathbf{M}_n$ $(n=1,2,\ \ldots,\ N)$ in multiple-sublattice magnets runs into insurmountable mathematical difficulties, since it reduces (even taking into account the constancy of the magnitudes of the magnetization vectors) to a system of $2N$ differential equations in partial derivatives, which is fairly cumbersome and inconvenient for analysis.

There is, however, an alternative and very productive approach to the study of the dynamic properties of magnets, viz., the effective Lagrangian method, which was developed in Refs. 4–6. According to this method, any magnetic structure can be described in the exchange approximation by no more than three mutually perpendicular unit vectors $\mathbf{l}_\sigma(\mathbf{r},t)$ (where $\sigma=1,\ 2,\ 3$), whose relative orientations remain unchanged in different excited states, i.e., which form a "rigid" reference frame. Any excited state assigned by the vectors $\mathbf{l}_\sigma(\mathbf{r},t)$ can be obtained from the original uniform state $\mathbf{l}_\sigma^{(0)}$ by rotation through the angle $\phi(\mathbf{r},t)$, which depends on the coordinates and the time:

$$\mathbf{l}_\sigma(\mathbf{r},t)=\hat{D}(\phi)\mathbf{l}_s^{(0)}, \qquad (1)$$

where $\hat{D}(\phi)$ is a three-dimensional orthogonal matrix.

The dynamics of long-wavelength ("hydrodynamic") excitations, in which the characteristic scale of the spatial nonuniformity greatly exceeds the crystal-lattice constant and the frequencies are much smaller than the exchange frequencies, can be studied by the phenomenological Lagrangian method. Various relativistic interactions which fix the orientation of the magnetic vectors $\mathbf{l}_\sigma$ relative to the crystallographic axes can also be taken into account within the effective Lagrangian method,[6] if these interactions are much weaker than the exchange interactions.

The high degree of generality of effective Lagrangians permits their application to practically any multiple-sublattice magnet having either a uniform or an nonuniform ground state, including noncollinear antiferromagnets with triangular ordering of the spins, which have been studied intently (see, for example, the review in Ref. 7 and the literature cited therein), as well as magnets having a ground state in which a modulated magnetic structure is realized. Nonlinear excitations in some multiple-sublattice magnets were previously studied in Refs. 8 and 9 using the effective Lagrangian method. In Refs. 10 and 11 this method was used to analyze the nonlinear dynamics of spin glasses.

The present work is devoted to a theoretical analysis of the nonlinear dynamics in noncollinear antiferromagnets of the $CsCuCl_3$ type. This class of magnets is extremely interesting, since they exhibit both triangular ordering of the spins and an incommensurate periodic structure of the "simple helix" or "ferromagnetic helix" type (the SH or FH type in the notation adopted in accordance with Ref. 12) and represent a fairly rare case of a modulated magnetic structure of relativistic and exchange origin, i.e., which results from the competition between the exchange interaction and the Dzyaloshinskiĭ–Moriya interaction (in the overwhelming

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majority of magnetically ordered crystals the modulated structures are associated with competition between exchange interactions).

In addition, the quasi-one-dimensional nature of the nonlinear waves in these magnets, i.e., the smallness of the exchange parameters coupling the chains of spins compared with those within the chains, imparts special urgency to the analysis of these waves. In low-dimensional magnets, in particular, nonlinear excitations play an important role in shaping the low-temperature thermodynamics, since, although the density of the nonlinear excitations at low temperatures is small compared with the density of the magnons, their contribution to certain thermodynamic parameters can dominate in some cases.[13,14]

Although we shall consider the nonlinear dynamics of multiple-sublattice antiferromagnets in the specific example of the compound $CsCuCl_3$ below, the results obtained can be generalized to the case of other multiple-sublattice magnets with a similar magnetic symmetry, which can be described within the effective Lagrangian method.

2. STRUCTURE AND THERMODYNAMIC POTENTIAL OF THE $CsCuCl_3$ SYSTEM

The experimental and theoretical investigation of the magnetic properties of compounds of the $CsCuCl_3$ type has been the subject of numerous studies (see Refs. 15–20), in which various details of the magnetic structure of these magnets were discovered and various models were proposed to describe them. In particular, it was found that in the paramagnetic state the compound $CsCuCl_3$ has $P6_122$ spatial symmetry at temperatures below $T_t=423$ K and that six magnetically active Cu^{2+} ions ($s=1/2$, where s is the magnitude of the spin) are located in b positions, i.e., are displaced in the basal plane of the crystal relative to the hexagonal 6_1 axis by a small amount $\kappa\approx 0.06$. The Cu^{2+} ions form helical chains oriented along the 6_1 axis (the distance between neighboring ions in a chain is equal to $\sim c_0/6$ when the crystal-lattice parameters are $c_0=18.1777$ Å and $a_0=7.2157$ Å). The small value of κ and the nature of the exchange, indirect exchange, and relativistic-exchange interactions[19] (particularly, the weakness of the interchain interactions in comparison with the in-chain interactions) specify the nearly quasi-one-dimensional character of the magnetic behavior of the system under consideration.

Experimental neutron-diffraction data indicate that a triangular antiferromagnetic structure occurs in the $CsCuCl_3$ system in the basal plane of the crystal and that long-period modulation of this triangular structure occurs along the hexagonal 6_1 axis (henceforth the z Cartesian axis) in the magnetically ordered state below $T_N=10.7$ K. The rotation angle between the magnetizations of chain Cu^{2+} ions in neighboring basal planes averaged over a structural period is approximately equal to 5.1°. Therefore, the modulation period amounts to about 12 lattice constants.

The triangular antiferromagnetic structure in the basal plane requires tripling of the magnetic unit cell in comparison with the crystal-chemical unit cell in that plane. Therefore, in the general case 18 magnetic sublattices are needed to describe it theoretically. However, on the basis of the results in Ref. 20, the technique of extended translational symmetry can be used to derive a model with three sublattices (so that each spin chain of Cu^{2+} ions is described in the terms of a single magnetic sublattice), which makes it possible to simplify the problem significantly. Such a model was used in Ref. 21 in a theoretical analysis of linear magnetic excitations, whose results agree fairly well with the experimental data in Ref. 22.

Following Refs. 20 and 21, the thermodynamic potential of a system can be represented in the form

$$W=M_0^2\int dz\, w(z),$$

$$w(\mathbf{r})=\sum_{n=1}^{3}\left\{\frac{\alpha}{2}\left(\frac{\partial \mathbf{m}_n}{\partial z}\right)^2+\frac{\tilde{\beta}}{2}m_{nz}^2+\alpha_1\left(m_{nx}\frac{\partial m_{ny}}{\partial z}-m_{ny}\frac{\partial m_{nx}}{\partial z}\right)-\mathbf{h}\cdot\mathbf{m}_n\right\}+\delta(\mathbf{m}_1\cdot\mathbf{m}_2+\mathbf{m}_1\cdot\mathbf{m}_3+\mathbf{m}_2\cdot\mathbf{m}_3). \quad (2)$$

Here $\mathbf{m}_n=\mathbf{M}_n/M_0$; the $\mathbf{M}_n$ are the magnetizations of the sublattices, where $n=1, 2, 3$; $|\mathbf{M}_n|=M_0=\text{const}$; $\alpha>0$ is the nonuniform exchange coupling constant, which is determined by the exchange integral within the chains J ($\alpha\sim Jc_0^2$); α_1 is the in-chain nonuniform relativistic-exchange coupling constant,[23] whose sign is insignificant for the formation of a modulated structure, and $\alpha_1\sim\frac{\alpha}{c_0}\frac{v}{c}$, where v is the Fermi velocity of the electrons and c is the velocity of light; $\delta>0$ is the uniform intersublattice exchange coupling constant, which is determined by the interchain exchange integral I ($\delta\sim I$); $\tilde{\beta}>0$ is the crystallographic magnetic anisotropy constant, and $\mathbf{h}=\mathbf{H}/M_0$, where $\mathbf{H}$ is the external magnetic field (henceforth $\mathbf{h}=h\mathbf{e}_z$). The experimental values presented in Ref. 19 for J and I give the estimate $I/J\sim 0.1$, which implies a definite similarity between the magnetic $CsCuCl_3$ system and a quasi-one-dimensional system. We also note that $\delta\gg\tilde{\beta}$.

The presence of relativistic-exchange terms which are linear with respect to the first spatial derivative in (2) leads to the formation of a long-period modulated magnetic structure, which creates periodic modulation of the simple harmonic triangular antiferromagnetic structure in the three-sublattice magnet with a strong antiferromagnetic exchange interaction between the sublattices under consideration. In other words, three FH structures similar to the ones first described in Ref. 23, which are correlated in the triangular antiferromagnetic structure in the basal plane of the system, appear in the magnet.

The wave vector of an FH structure is $q=-\alpha_1/\alpha$, and the angle χ by which the magnetization vectors $\mathbf{M}_n$ of the sublattices depart from the basal plane is determined by the magnitude of the external magnetic field: $\sin\chi=h/3\delta$. In the absence of an external magnetic field, the harmonic FH structure transforms into a structure of the SH type ($\chi=0$), and the sublattices form angles equal to $2\pi/3$ between one another.

3. EFFECTIVE LAGRANGIAN AND EQUATIONS OF MOTION

In the absence of an external magnetic field $\mathbf{h}$, the magnetization vectors $\mathbf{m}_n$ of the sublattices in the system lie in the basal (xy) plane and form a triangular antiferromagnetic structure. Therefore, the following vectors can be chosen as the mutually orthogonal unit vectors $\mathbf{l}_\sigma$:

$$\mathbf{l}_1^{(0)}=\frac{1}{3}(2\mathbf{m}_3^{(0)}-\mathbf{m}_1^{(0)}-\mathbf{m}_2^{(0)}),\quad \mathbf{l}_2^{(0)}=\frac{1}{\sqrt{3}}(\mathbf{m}_1^{(0)}-\mathbf{m}_3^{(0)}),$$

$$\mathbf{l}_3^{(0)}=\mathbf{l}_1^{(0)}\times\mathbf{l}_2^{(0)}. \tag{3}$$

The orientations of $\mathbf{l}_1^{(0)}$ and $\mathbf{l}_2^{(0)}$ relative to the Cartesian x and y axes were not fixed; therefore, without any loss of generality, we can set $\mathbf{l}_1^{(0)}=\mathbf{e}_x$ and $\mathbf{l}_2^{(0)}=\mathbf{e}_y$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the corresponding unit vectors. Then

$$l_{1i}(\mathbf{r},t)=D_{xi}(\mathbf{r},t),\quad l_{2i}(\mathbf{r},t)=D_{yi}(\mathbf{r},t). \tag{4}$$

The effective Lagrangian L which describes the noncollinear antiferromagnet under consideration has the form[6]

$$L=\int dz\left\{\frac{\chi_\perp}{2g^2}[\omega_1^2(\phi,\dot\phi)+\omega_2^2(\phi,\dot\phi)]+\frac{\chi_\parallel}{2g^2}\,\omega_3^2(\phi,\dot\phi)-U(\hat D)\right\}. \tag{5}$$

Here $\omega_i(\phi,\dot\phi)$ are differential Cartan forms, which are related to the rotation matrix $\hat D(\phi)$ by the expression

$$\omega_i(\phi,\dot\phi)=\tfrac{1}{2}\epsilon_{ikl}D_{kj}\dot D_{lj}; \tag{6}$$

g is the gyromagnetic ratio; the dot signifies differentiation with respect to time; $\chi_\perp$ and $\chi_\parallel$ are the transverse and longitudinal (with respect to the vector $\mathbf{l}_3^{(0)}=\mathbf{l}_1^{(0)}\times\mathbf{l}_2^{(0)}$) susceptibilities of the antiferromagnet, and $\chi_\perp,\chi_\parallel\sim\delta^{-1}$; ϵ_{ikl} is a totally antisymmetric third-rank tensor, and U is the "potential" energy of the magnet, whose form can easily be obtained from the expression for the density of the thermodynamic potential (2) with consideration of the relations (3) and (4):

$$U(\hat D)=\frac{\alpha}{4}(D_{1i}'^2+D_{2i}'^2)+\frac{\tilde\beta}{4}(D_{13}^2+D_{23}^2)+\frac{\alpha_1}{2}(D_{11}D_{12}'-D_{12}D_{11}'+D_{21}D_{22}'-D_{22}D_{21}'), \tag{7}$$

where the prime denotes a derivative with respect to the coordinate z.

The magnetization vector $\mathbf{M}$ in the system described by the effective Lagrangian (5) is specified by the relation[6]

$$M_k=\frac{1}{g}[\chi_\perp(\omega_1D_{1k}+\omega_2D_{2k})+\chi_\parallel\omega_3D_{3k}].$$

We note that the density of the "kinetic" part of the Lagrangian (5) is analogous to the kinetic energy of a symmetric top and that the entire system under consideration can be interpreted as the continuum limit of a system of distributed symmetric tops.

If the external magnetic field is sufficiently weak in comparison with the field of the intersublattice exchange interaction ($h\ll\delta$), it can also be taken into account in the analysis of the dynamics of a magnet using an effective Lagrangian.[6] The new effective Lagrangian can be obtained from (5) by the replacement $\omega_i\to\omega_i+gH_i$.

To study the dynamic properties of the model with a modulated ground state that we are considering, it is convenient to represent the rotation matrix $\hat D$ in the form of a product of two matrices:

$$\hat D(z,t)=\hat G(z,t)\cdot\hat D_0(z), \tag{8}$$

where $\hat D_0(z)$ describes the rotation of the basal reference frame through the angle $\zeta_0=qz$ around the z axis, which corresponds to the ground state of the magnet (a helical structure with a wave vector $q=-\alpha_1/\alpha$),

$$\hat D_0=\begin{pmatrix}\cos\zeta_0 & -\sin\zeta_0 & 0\\ \sin\zeta_0 & \cos\zeta_0 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{9}$$

Substituting (8)–(9) into the expression (7) for the potential energy of the system, we can easily see that the latter is dependent only on the elements of the matrix $\hat G$:

$$U(\hat G)=\frac{\alpha}{4}(G_{1i}'^2+G_{2i}'^2)+\frac{\beta}{4}(G_{13}^2+G_{23}^2), \tag{10}$$

where $\beta=\tilde\beta+\alpha_1^2/\alpha$ is the effective uniaxial anisotropy constant.

In fact, the transition from $\hat D$ to $\hat G$ is a transition to a coordinate system which rotates in space and in which, unlike (7), the effective potential energy (10) does not contain terms that are linear with respect to the spatial derivatives. This situation greatly simplifies the analysis of the dynamics of the system.

The rotation matrix $\hat G(\phi)$ can have different parametrizations. The parametrization $\boldsymbol{\phi}=\mathbf{n}\tan(\psi/2)$, $\mathbf{n}^2=1$ has been used for it in numerous papers. It has a simple physical meaning: rotation of the reference frame through the angle ψ around the axis assigned by the unit vector $\mathbf{n}$. A parametrization of the rotation matrix defined by the components of the four-dimensional unit vector ν_μ, where $\mu=1-4$, was found to be more convenient in the analysis of the spectrum of linear spin waves in a multiple-sublattice antiferromagnet in Ref. 21.

Analysis reveals that a parametrization of the matrix $\hat G$ based on Euler angles is most convenient for studying nonlinear excitations in the model under consideration here:

$$\hat G=\begin{pmatrix}\cos\psi\cos\varphi-\cos\theta\sin\psi\sin\varphi & \cos\theta\sin\psi\cos\varphi+\cos\psi\sin\varphi & \sin\theta\sin\psi\\ -\sin\psi\cos\varphi-\cos\theta\cos\psi\sin\varphi & \cos\theta\cos\psi\cos\varphi-\sin\psi\sin\varphi & \sin\theta\cos\psi\\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta\end{pmatrix}. \tag{11}$$

The differential forms ω_i, which are defined in (6) and have the meaning of components of the angular velocity, like the potential energy (10), do not contain elements of $\hat{D}_0$ and are equal to

$$\omega_1=-\dot{\theta}\cos\psi-\dot{\varphi}\sin\theta\sin\psi,$$
$$\omega_2=\dot{\theta}\sin\psi-\dot{\varphi}\sin\theta\cos\psi, \tag{12}$$
$$\omega_3=-\dot{\psi}-\dot{\varphi}\cos\theta.$$

Plugging (10)–(12) into (5) and taking into account the external magnetic field H, which is aligned parallel to the z axis, we obtain the effective Lagrangian of the system in the form

$$L=M_0^2\int dz\left\{\frac{\chi_\perp}{2(gM_0)^2}(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+\frac{\chi_\parallel}{2(gM_0)^2}\right.$$
$$\times(\dot{\psi}+\dot{\varphi}\cos\theta)^2-\frac{\chi_\parallel}{gM_0}h(\dot{\psi}+\dot{\varphi}\cos\theta)-\frac{\beta}{4}\sin^2\theta$$
$$\left.-\frac{\alpha}{4}[\theta'^2+\varphi'^2\sin^2\theta+2(\psi'+\varphi'\cos\theta)^2]\right\}. \tag{13}$$

We note that in a spin glass, which has a higher symmetry than a noncollinear antiferromagnet, $\chi_\perp=\chi_\parallel$ holds (Ref. 6), and in the absence of an external magnetic field a spin glass can be described by a "Lorentz-invariant" model.[10,11]

The Lagrangian (13) contains two cyclic variables, viz., ψ and φ, which correspond to two different integrals of the motion:

$$N_\varphi=M_0^2\int dz\left\{\frac{\chi_\perp-\chi_\parallel}{(gM_0)^2}\dot{\varphi}\sin^2\theta+\frac{2\chi_\parallel}{(gM_0)^2}\right.$$
$$\left.\times(gM_0h-\dot{\psi})\sin^2\frac{\theta}{2}\right\}, \tag{14}$$

$$N_\psi=M_0^2\int dz\,\frac{\chi_\parallel}{(gM_0)^2}(\dot{\psi}+\dot{\varphi}\cos\theta). \tag{15}$$

The equations of motion corresponding to the Lagrangian (13) have the form

$$\alpha(\theta''+\varphi'^2\sin\theta\cos\theta+2\psi'\varphi'\sin\theta)-\frac{2\chi_\perp}{(gM_0)^2}$$
$$\times(\ddot{\theta}-\dot{\varphi}^2\sin\theta\cos\theta)-\frac{2\chi_\parallel}{(gM_0)^2}\dot{\varphi}\sin\theta(\dot{\psi}+\dot{\varphi}\cos\theta)$$
$$+\frac{2\chi_\parallel}{gM_0}h\dot{\varphi}\sin\theta-\beta\sin\theta\cos\theta=0, \tag{16}$$

$$\alpha\frac{\partial}{\partial z}\left[\psi'\cos\theta+\frac{1+\cos^2\theta}{2}\varphi'\right]-\frac{\chi_\perp}{(gM_0)^2}\frac{\partial}{\partial t}(\dot{\varphi}\sin^2\theta)$$
$$-\frac{\chi_\parallel}{(gM_0)^2}\frac{\partial}{\partial t}[\cos\theta(\dot{\psi}+\dot{\varphi}\cos\theta)]-\frac{\chi_\parallel}{gM_0}h\dot{\theta}\sin\theta=0, \tag{17}$$

$$\alpha\frac{\partial}{\partial z}(\psi'+\varphi'\cos\theta)-\frac{\chi_\parallel}{(gM_0)^2}\frac{\partial}{\partial t}(\dot{\psi}+\dot{\varphi}\cos\theta)=0. \tag{18}$$

Since the total rotation angle of the reference frame ϕ is related to the Euler angles by the known expression

$$\cos\frac{\phi}{2}=\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, \tag{19}$$

the localized solutions of the system of equations (16)–(18) of interest to us have the boundary conditions

$$\theta\to0,\quad(\psi+\varphi)\to0,\quad\theta'\to0,\quad(\psi+\varphi)'\to0$$
$$\text{for } z\to\pm\infty. \tag{20}$$

Let us move on to an analysis of some particular solutions of the equations of motion (16)–(18) which satisfy the boundary conditions (20).

4. SIMPLE WAVES

First we consider the simple-wave solutions, for which $\theta=\theta(\xi)$, $\psi=\psi(\xi)$, and $\varphi=\varphi(\xi)$, where $\xi=z-Vt$ and V is the velocity of the wave.

From Eqs. (17) and (18) we find

$$\varphi'=\frac{2(R_1-\tilde{R}_2\cos\theta)}{\alpha\sin^2\theta(1-V^2/c^2)}, \tag{21}$$

$$\psi'=\frac{R_2(gM_0)^2}{\alpha(gM_0)^2-\chi_\parallel V^2}-\frac{2(R_1-\tilde{R}_2\cos\theta)\cos\theta}{\alpha\sin^2\theta(1-V^2/c^2)}, \tag{22}$$

where $\tilde{R}_2=R_2-\chi_\parallel V/gM_0$, R_1 and R_2 are the first integrals of Eqs. (17) and (18), and $c^2=\alpha(gM_0)^2/2\chi_\perp$ is the characteristic velocity, which coincides with the minimal phase velocity of the linear spin waves (see Ref. 21). In an antiferromagnet this velocity has an order of magnitude of 10^6 cm/s.

Substituting the expressions (21) and (22) into Eq. (16), after some simple manipulations we obtain

$$\alpha\left(1-\frac{V^2}{c^2}\right)\theta''-\beta\sin\theta\cos\theta$$
$$+\frac{4(R_1-\tilde{R}_2\cos\theta)(\tilde{R}_2-R_1\cos\theta)}{\alpha(1-V^2/c^2)\sin^3\theta}=0. \tag{23}$$

Note that Eq. (23) has the same structure as the equation describing a heavy symmetric top and can be integrated in general form, i.e., with arbitrary values of R_1 and $\tilde{R}_2$ (see, for example, Ref. 24). Then the solutions can be expressed explicitly in terms of elliptic functions.

However, we are interested only in localized solutions which satisfy the boundary conditions (20). It is easy to see that such solutions are obtained with the following definite values of the parameters:

$$R_2=0,\quad R_1=\tilde{R}_2=-\frac{\chi_\parallel}{gM_0}V. \tag{24}$$

When the parameters have such values, Eq. (23) can be brought into the form

$$l_0^2\theta''-\sin\theta\cos\theta+\frac{B^2}{2}\frac{\sin(\theta/2)}{\cos^3(\theta/2)}=0, \tag{25}$$

where $l_0=z_0\sqrt{1-V^2/c^2}$, $B=2\chi_\parallel hV/\beta gM_0l_0$, and $z_0=\sqrt{\alpha/\beta}$.

The localized solution of Eq. (25) can be represented in the form

$$\tan^2\frac{\theta}{2}=\frac{V_*}{|V|}\frac{1-V^2/V_s^2}{\sqrt{1-V^2/c^2}\cosh(2\kappa\xi)+|V|/V_*},\tag{26}$$

where we have introduced the notation

$$V_s=\frac{c}{\sqrt{1+(\chi_\parallel h)^2/2\beta\chi_\perp}},\quad V_*=\frac{V_s}{\sqrt{1-V_s^2/c^2}}$$

$$\kappa=\frac{1}{l_0}\sqrt{\frac{1-V^2/V_s^2}{1-V^2/c^2}}=\frac{1}{z_0}\frac{\sqrt{1-V^2/V_s^2}}{\sqrt{1-V^2/c^2}}.$$

Solution (26) describes a one-parameter simple-wave soliton propagating along the axis of a helical structure. This solution exists for $V<V_s$, where V_s is the limiting velocity of the soliton. Note that the value of the limiting velocity is smaller than the characteristic velocity c and decreases as the external magnetic field h increases.

The amplitude $\theta_{\max}=\theta(0)$ of a soliton is determined by the parameter B and equals

$$\theta_{\max}=2\arctan\sqrt{\frac{2}{B}-1}$$

$$=2\arctan\sqrt{\frac{V_s}{|V|}\sqrt{\frac{1-V^2/c^2}{1-V_s^2/c^2}}-1}.\tag{27}$$

It is not difficult to see that $\theta_{\max}$, as in the case of a ferromagnetic soliton,[1,25] decreases when the velocity of the soliton V increases and that as $V\rightarrow V_s$ it tends to zero, $\theta_{\max}\sim\sqrt{V_s-V}$, and the soliton degenerates. We also note that when the velocity of the soliton is fixed, its amplitude decreases with increasing magnitude of the external magnetic field.

In the absence of an external field or when $V=0$, the coefficient B in Eq. (25) vanishes, a soliton solution like (26) does not exist, and Eq. (25) has the nontrivial solution of a moving (when $h=0$) or resting (when $V=0$) "180° domain boundary:"

$$\sin\theta=\frac{1}{\cosh(\xi/l_0)}.\tag{28}$$

But if $B\ll 1$, i.e., if the velocity of the soliton is sufficiently small ($V\ll V_s$) or if the value of the external field is small ($h\ll\sqrt{2\beta\chi_\perp}/\chi_\parallel$ the amplitude of the soliton is close to π, and it can be regarded as a bound state of two 180° domain boundaries separated by a distance proportional to $|\ln B|$.

The energy E of the soliton excitation (26) equals

$$E=E_0\frac{\sqrt{1-V^2/V_s^2}}{1-V^2/c^2},\quad E_0=2M_0^2\sqrt{\alpha\beta}.\tag{29}$$

The constant of motion N_ψ in Eq. (22) corresponding to the soliton solution (26) is equal to zero, and the integral N_φ (21) can be represented in the form

$$N_\varphi=\frac{\chi_\parallel H z_0}{g\sqrt{1-V^2/c^2}}\ln\left[\frac{\sqrt{1-V^2/c^2}+\sqrt{1-V^2/V_s^2}}{\sqrt{1-V^2/c^2}-\sqrt{1-V^2/c^2}}\right].\tag{30}$$

The energy E of the soliton and the motion integral N_φ increase without bound as $V\rightarrow 0$ and tend to zero as $V\rightarrow V_s$.

To conclude this section we once again mention the fundamental significance of the influence of the external magnetic field on the character of the solution of the equations of motion under consideration: if there is no field, the localized soliton solution (26) does not exist, and only the nonlocalized solution (28) of the domain-boundary type is realized.

5. PRECESSION SOLITONS

Let us consider one more example of a one-parameter soliton solution of Eqs. (16)–(18), in which $\psi=-\varphi=\omega t$ and $\theta=\theta(z)$. It is not difficult to see that in this case the equalities (17) and (18) hold identically and that the function $\theta(z)$ satisfies the equation

$$\Lambda_0^2\theta''-\sin\theta\cos\theta+\Omega\sin\theta=0,\tag{31}$$

where we have introduced the notation

$$\Lambda_0^2=\frac{\alpha}{\beta+2(\chi_\parallel-\chi_\perp)\omega^2/(gM_0)^2},$$

$$\Omega=\frac{2\chi_\parallel\omega(\omega-gH)}{\beta+2(\chi_\parallel-\chi_\perp)\omega^2/(gM_0)^2}.$$

A localized solution of Eq. (31) exists for $\Omega<1$ or $\omega_-<\omega<\omega_+$, where

$$\omega_\pm=\frac{g}{2\chi_\perp}[\chi_\parallel H\pm\sqrt{2\chi_\perp\beta M_0^2+(\chi_\parallel H^2)}].\tag{32}$$

The functional dependence of the solution of Eq. (31) is determined by the sign of the parameter Ω:

$$\tan\frac{\theta}{2}=\begin{cases}A/\cosh(\tilde\kappa z), & \Omega>0,\\ A/\sinh(\tilde\kappa z), & \Omega<0,\end{cases}\tag{33}$$

where

$$A=\sqrt{\frac{1-\Omega}{|\Omega|}},\quad \tilde\kappa=\frac{\sqrt{1-\Omega}}{\Lambda_0}.$$

At positive values of the parameter $\Omega>0$ (to which the frequency ranges $\omega_-<\omega<0$ and $gH<\omega<\omega_+$ correspond) the amplitude of the soliton equals $\theta_{\max}=\theta(0)=2\arctan A<\pi$; but if $\Omega<0$ holds (i.e., in the frequency range $0<\omega<gH$) the amplitude equals $\theta_{\max}=\pi$. In the latter case the solution (33) corresponds to a nonzero topological charge.[1]

It is important to note that solitons with a nonzero topological charge (kinks) are of great importance in the solution of so-called soliton thermodynamics in one-dimensional magnetic systems (see, for example, Refs. 13, 14, and 26). As was shown in those papers, a kink density in thermodynamic equilibrium exists at finite temperatures, causing significant changes in the structure of the spin–spin correlation functions of the system, which determine the character of the neutron scattering. In particular, just such excitations form the experimentally observed central peak in the neutron-scattering spectrum.[14,26]

The value $\Omega=0$, to which two frequency values, viz., $\omega=0$ and $\omega=gH$, correspond, is a special point of Eq. (31). Here, as in the case of Eq. (25) considered in the preceding section, the nontrivial solution of Eq. (31) defines a solution of the domain-boundary type and has a form similar to (28).

We stress that in the absence of an external magnetic field, $\Omega\geqslant 0$ holds for all values of the frequency ω and that there is, therefore, no solution with a nonzero topological charge.

If the external magnetic field is sufficiently small, the frequency range (ω_-, ω_+), in which there are no precession solitons, is determined by the antiferromagnetic resonance frequency $\omega_0=c/z_0\sim 10^{11}-10^{12}\ \mathrm{s}^{-1}$: $\omega_\pm\simeq\pm\omega_0$. Here the range for the existence of solitons with a nonzero topological charge $(0,gH)$ is small, if the magnitude of the external magnetic field is small. As the field increases, the width of the range for the existence of soliton excitations like (33) increases.

On the basis of the structure of the rotation matrix $\tilde{G}$ (11), it is not difficult to see that the solution under consideration corresponds to the precession of the vector $\mathbf{l}_3$ in a spatially nonuniform reference system with an angular velocity ω around the z axis, which forms an angle θ with that axis. At the same time, the vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ precess in the plane perpendicular to $\mathbf{l}_3$.

The energy of a precession soliton and the integrals N_ψ and N_φ for either sign of Ω are specified by the common formulas:

$$E=2\beta M_0^2\Lambda_0\left[\sqrt{1-\Omega}+\left(\Omega+\frac{2\chi_\parallel h\omega}{gM_0}\right)\ln\frac{1+\sqrt{1-\Omega}}{|\Omega|}\right], \tag{34}$$

$$N_\psi=\frac{4\Lambda_0\chi_\parallel}{g^2}\,\omega\,\ln\frac{1+\sqrt{1-\Omega}}{|\Omega|}, \tag{35}$$

$$N_\varphi=\frac{4\Lambda_0\chi_\parallel}{g^2}\,\omega\left[\rho\sqrt{1-\Omega}+\left(\frac{gH}{\omega}-1+\rho\Omega\right)\right.$$
$$\left.\times\ln\frac{1+\sqrt{1-\Omega}}{|\Omega|}\right], \tag{36}$$

where $\rho=1-\chi_\parallel/\chi_\perp$.

When $\omega\to\omega_\pm$ $(\Omega\to 1)$ holds, both the energy and the motion integrals N_ψ and N_φ tend to zero; in the limit $\omega\to 0$, the values of E and N_φ diverge, and $N_\psi\to 0$. In the vicinity of the second special point $\omega=gH$ all three motion integrals diverge.

We present a simple relation, which holds for the motion integrals E, N_φ, and N_ψ:

$$2\omega\chi_\parallel\rho(E-gHN_\psi)=\beta(gM_0)^2\left[N_\varphi+\left(1-\frac{gH}{\omega}\right)N_\psi\right]. \tag{37}$$

6. TWO-PARAMETER SOLITONS

In the preceding sections we considered one-parameter solutions of the equations of motion (16)–(18). More complicated localized solutions of these equations can be obtained for $H=0$, if ψ is set equal to $-\varphi=\mathrm{const}$. In that case (17)–(18) hold identically, and (16) reduces to the familiar sine–Gordon equation:

$$z_0^2\theta''-\frac{1}{c^2}\ddot{\theta}-\sin\theta\cos\theta=0, \tag{38}$$

where the parameters z_0 and c were defined above.

The sine–Gordon equation is exactly integrable,[27] making it possible to construct solutions describing multisoliton excitations of a multiple-sublattice quasi-one-dimensional magnet. These solutions have been adequately studied, and there is, therefore, no sense in dwelling on them in detail here. We only note that, by virtue of the condition $\psi=-\varphi=\mathrm{const}$, in the system under consideration all such solutions correspond to oscillations of the rigid reference frame around an axis lying in the xy plane, which in turn rotates around the z axis owing to the nonuniformity of the ground state. For example, when $\psi=\varphi=0$ holds, the oscillations occur around the vector $\mathbf{l}_1$, and for $\psi=-\varphi=\pi/2$ they occur around $\mathbf{l}_2$.

We present two two-parameter solutions of Eq. (38). One of them is the two-soliton Perring–Skyrme solution:

$$\tan\frac{\theta}{2}=\frac{V}{c}\,\frac{\sinh(z/l_0)}{\cosh(Vt/l_0)}, \tag{39}$$

where $l_0=z_0\sqrt{1-V^2/c^2}$. This solution describes the interaction of two nonlinear waves moving in opposite directions to one another with a velocity V.

The second example is a solution which describes a bound state of two solitons (a breather):

$$\tan\frac{\theta}{2}=\frac{\sqrt{z_0^{-2}-\omega^2}}{\omega}\,\frac{\cos[\omega(t-zV/c^2)/\sqrt{1-V^2/c^2}]}{\cosh[(z-Vt)\sqrt{z_0^{-2}-\omega^2}/\sqrt{1-V^2/c^2}]}. \tag{40}$$

The solution (40) corresponds to an excitation moving with a velocity V, in which the vector $\mathbf{l}_3$ in the coordinate system associated with the wave precesses with a frequency $\omega'=\omega\sqrt{1-V^2/c^2}$.

In addition, we note that the sine–Gordon equation has various multisoliton solutions, including solutions with a nonzero topological charge (kinks) [in particular, a one-soliton solution like (28)]. As we have already noted in the preceding section, such solitons form the central peak in neutron-scattering experiments, which is presently the main evidence of their existence.

7. CONCLUSIONS

In the present work the nonlinear dynamics in multiple-sublattice magnets with a modulated structure have been considered in the concrete example of the three-sublattice model of $CsCuCl_3$; however, within the effective Lagrangian method used the number of sublattices is of no importance (for $N\geqslant 3$), since the structure of the effective Lagrangian used to construct the dynamic equations of motion is specified by the symmetry of the crystal and is not dependent on the number of sublattices. This refers primarily to the entire class of ABX_3-type magnets (A and B are cations, and X is a halogen), which are structurally similar to $CsCuCl_3$ considered here, since the effective Lagrangian (5) is characteristic of all such crystals. An effective Lagrangian like (5) can also be constructed for multiple-sublattice magnets having a different symmetry, regardless of the number of sublattices. In particular, as was noted in Ref. 6, the kinetic term in the Lagrangian for an arbitrary antiferromagnet has the form

$$T=\frac{1}{2g^2}\chi_{\alpha\beta}\omega_\alpha\omega_\beta,$$

where $\chi_{\alpha\beta}$ is the magnetic susceptibility tensor. In uniaxial antiferromagnets (including $CsCuCl_3$, which we considered) this tensor has two independent components, $\chi_\perp$ and $\chi_\|$. In an antiferromagnet of the UO_2 type, which has cubic symmetry, there is only one independent component ($\chi_\perp=\chi_\|$). In noncollinear antiferromagnets with orthorhombic or lower symmetry the magnetic vectors transform only according to one-dimensional representations, and the tensor $\chi_{\alpha\beta}$ will therefore have three independent components. The structure of the potential part of the Lagrangian can also be derived on the basis of symmetry arguments (as was done, for example, in Ref. 6 for magnets of the $YMnO_3$ and UO_2 types) or (as was done in the present work) on the basis of the form of the ordinary magnetic energy written in terms of the vectors $\mathbf{M}_n$ of the sublattices, which is known for this magnet. Moreover, in the general case the choice of the three mutually perpendicular vectors $\mathbf{l}_\sigma^{(0)}$, which are related in definite manners to the vectors $\mathbf{M}_n$, is ambiguous and is determined by considerations of convenience. In particular, any three (nonzero in the ground state) irreducible vectors which are characteristic of a magnet with a specific symmetry can always (but need not) be chosen as such vectors. It seems most convenient to us to choose two irreducible vectors ($\mathbf{l}_1^{(0)}$ and $\mathbf{l}_2^{(0)}$) and their vector product [see (3)] as such vectors. The latter procedure was used in Ref. 9 to analyze the magnetic dynamics in six-sublattice perovskites, where the irreducible vectors $\mathbf{l}_1^{(0)}$ and $\mathbf{l}_2^{(0)}$ are related to the magnetization vectors of the sublattices by the expressions $\mathbf{l}_1^{(0)}\sim\mathbf{M}_1^{(0)}-\mathbf{M}_2^{(0)}+\mathbf{M}_4^{(0)}-\mathbf{M}_5^{(0)}$ and $\mathbf{l}_2^{(0)}\sim\mathbf{M}_1^{(0)}+\mathbf{M}_2^{(0)}-2\mathbf{M}_3^{(0)}+\mathbf{M}_4^{(0)}+\mathbf{M}_5^{(0)}-2\mathbf{M}_6^{(0)}$. The effective Lagrangians and thus the equations of motion obtained for all noncollinear antiferromagnets have a similar structure, and therefore the results obtained above can be extended to other multiple-sublattice noncollinear antiferromagnets with a predominant exchange interaction between the sublattices.

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