Supersonic flow past bodies in dispersive hydrodynamics

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The problem of steady two-dimensional supersonic flow about slender pointed bodies is studied in dispersive hydrodynamics. The equivalence of this problem to the Gurevich–Pitaevskiĭ evolutionary problem of dissipationless shock waves in Korteweg–de Vries hydrodynamics is shown. The Whitham technique is used to derive a number of exact solutions describing different cases of flow around objects in dispersive hydrodynamics. © *1995 American Institute of Physics.*

1. INTRODUCTION

The theory of supersonic flow around objects is one of the classical problems of hydrodynamics.[1–4] The nonlinear system of Euler equations describing two-dimensional time-independent supersonic flows in ideal hydrodynamics is hyperbolic. When supersonic flow occurs around objects it is accompanied by wave-breaking singularities. In ordinary dissipative hydrodynamics this gives rise to a shock wave, consisting of a strong discontinuity on which the flow changes its characteristics discontinuously (Fig. 1). The flow ahead of the shock wave is unperturbed. Behind the shock wave the flow is described as before by the Euler equations, but now the gas density is higher than in the incident flow, and it has a higher temperature. Heating takes place due to the dissipation of some fraction of the energy of the flow into the shock wave.

In dispersive hydrodynamics, where dissipation is completely absent, the flow behind the point where singularities develop has a fundamentally different character. A broad region develops here, filled with small-scale oscillations; this is a dissipationless shock wave.[5–12] The distribution of the oscillations has a substantial effect on the flow pattern, so that in addition to the average properties of the flow in dispersive hydrodynamics it is necessary to also determine the structure of these oscillations.

We propose to study the supersonic flow around objects in dispersive hydrodynamics in a series of papers. The solution of such problems is not only important for a number of problems in the dynamics of low-density plasmas (e.g., in the flow of the solar wind around the magnetospheres of the earth and planets), wave hydrodynamics on water, etc., but is also of independent theoretical interest.

The principal difference between the present work and a large number of treatments devoted to both numerical and analytical studies of plasma flowing around objects is the purely dissipationless nonlinear self-consistent formulation of the problem, which does not require the imposition of any supplementary conditions (such as the Rankine–Hugoniot adiabat) on a shock wave.

In the present work we treat the problem of supersonic flow around slender pointed bodies, when the general equation of two-dimensional time-independent dispersive hydrodynamics can be reduced to the Korteweg–de Vries (KdV) equation.[13] Since the perturbations introduced into the flow by the slender body are small and do not significantly change the average properties of the flow, we can devote most of our attention to the structure of the waves that are excited. As an elementary example we consider the system of equations describing two-dimensional time-independent flow in a highly nonisothermal ($T_i \ll T_e$) plasma: nonlinear dispersive ion-acoustic waves. In Sec. 4 a formal derivation of the KdV equation is given for the problem of ion-acoustic flow around a slender body, and it is shown that the two-dimensional time-independent problem of flow around an object corresponds to a one-dimensional Gurevich–Pitaevskiĭ evolutionary problem for the dispersive analog of a shock wave.[5] The initial data for the KdV equation are obtained by differentiating the function specifying the shape of the object around which the flow takes place.

To date the Gurevich–Pitaevskii problem has been effectively integrated for a broad class of initial data.[5,8–12] In the Gurevich–Pitaevskii formulation the dissipationless shock wave region is described by a modulated quasisteady solution of the KdV equation in which the modulation parameters satisfy the averaged Whitham equations[14] with special boundary conditions resulting from matching the average flow in the wave zone with the external continuous flow at the boundaries of the dissipationless shock wave. The Whitham modulation system for the KdV equation consists of three equations and has a diagonal (Riemann) form. The Gurevich–Pitaevskii problem simplifies considerably in a number of cases of flow around objects when it is possible to fix one of the Riemann invariants and treat the so-called quasisimple dissipationless shock wave. The modulation sys-

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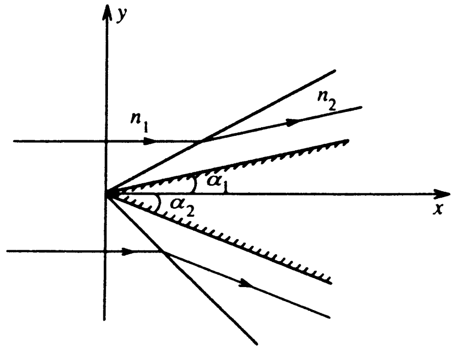

FIG. 1. Flow around a wedge in dissipative hydrodynamics.

tem describing the quasisimple dissipationless shock wave consists of two equations and can be integrated by the hodograph method. This problem was solved by Gurevich *et al.*[11] using a generalized form of the hodograph transformation (the Tsarev[15] method) and the "scalar potential" technique, which reduces the complicated system of hodograph equations to the well-known Euler–Poisson equation. In Sec. 5 the procedure for integrating the Gurevich–Pitaevskii problem is briefly summarized for arbitrary monotonic initial data.

The behavior associated with the problem of flow around an object is revealed when the (x,t) formulas are recalculated and interpreted in terms of two-dimensional steady flows. A number of specific examples of flow around monotonic shapes are studied in Sec. 6. The self-similar problem of flow around a wedge and the problem of flow around bodies whose shape is given by the power-law function $y=ax^q$ are treated. For $1<q<2$ the solution of the modulation equations possesses a generalized self-similarity. The associated explicit formulas for the solution and the dissipationless shock wave boundaries are given. For $q\geqslant 2$ the solution cannot be described using the KdV equation, since at the point where breaking occurs the perturbation amplitude formally becomes infinitely large. But for $q\geqslant 2$ this happens only in a small neighborhood of the second boundary of the body, so this problem can be treated using the KdV equation and is of considerable interest, because in this case the dissipationless shock wave associated with the flow separates from the object. An important difference between this and the analogous effect in dissipative hydrodynamics (Ref. 1, Sec. 115) is the absence of a weak discontinuity propagating from the object to the point where the shock wave forms. In the case $q>2$ (close to the boundary) it becomes necessary to integrate the complete modulation system consisting of three equations.

The problems of flow around an object that give rise to a two-sheet hodograph transformation (flow around an infinite wing, flow around bounded objects, etc.) require further investigation.

2. GENERAL QUALITATIVE PICTURE OF FLOW AROUND OBJECTS IN DISPERSIVE HYDRODYNAMICS

In dispersive hydrodynamics there is a characteristic spatial scale, the dispersion parameter D. We will treat only hydrodynamic flows, i.e., flows whose characteristic scales satisfy $l\gg D$.

Note that the analogous spatial scale in ordinary dissipative hydrodynamics is $D_0\approx c_s/\nu$ (here c_s is the speed of sound and ν is the kinematic viscosity), and there the condition $l\gg D$ is assumed to hold.

Nonlinear time-dependent one-dimensional flows in dispersive hydrodynamics have been treated previously.[5,6,8–12,16] From these results we can deduce the following properties of dispersive hydrodynamics:

1. Before the development of singularities, hydrodynamic flows in both dispersive hydrodynamics and ordinary hydrodynamics can be described by the Euler equations. Singularities develop at a point where a simple Riemann wave breaks, or in the more general case at points where characteristics intersect.

2. Beyond a point where wave breaking occurs in dispersive hydrodynamics, small-scale waves develop. A region which spreads as a function of time can be distinguished where oscillations have a nonlinear character and their amplitude a does not decrease with time t. We call this region a dissipationless shock wave.

3. The oscillations in a dissipationless shock wave have a completely determined structure. For media with negative dispersion they begin with a chain of solitons whose amplitude gradually decreases, and they go over to a sinusoidal wave. The dissipationless shock wave has a head soliton at its leading edge, while on its trailing edge the oscillation amplitude goes to zero. In media with positive dispersion the order of motion is reversed: the sinusoidal oscillations are in front and the solitons behind.

4. In the case of negative dispersion the oscillations become weaker behind the dissipationless shock wave algebraically as a function of time: $a\propto t^{\alpha}$, $\alpha=-1/2,\ -1/3$. They vanish asymptotically in the limit $t\to\infty$. Thus, behind the trailing edge of a dissipationless shock wave a region can be distinguished which contains no oscillations and is described by the Euler equations. In front of the dissipationless shock wave the flow is unperturbed and continuous as before. The Euler equations therefore fail to hold only in the dissipationless shock wave region.

5. The change in the average quantities (density and flow velocity) associated with passage through the dissipationless shock wave region, if the latter is formed as a result of the breaking of a simple Riemann wave, is described by equations derived by Gurevich and Meshcherkin.[6] Thus, in dispersive hydrodynamics, as in ordinary hydrodynamics, a transition is possible through the shock-wave region; the Rankine–Hugoniot relations are replaced by the Gurevich–Meshcherkin relations.

6. In a number of cases an important phenomenon occurs in which the leading soliton breaks if its amplitude attains a certain value.[6] After breaking occurs the flow becomes multistreaming, and consequently is no longer described by the equations of dispersive hydrodynamics. The

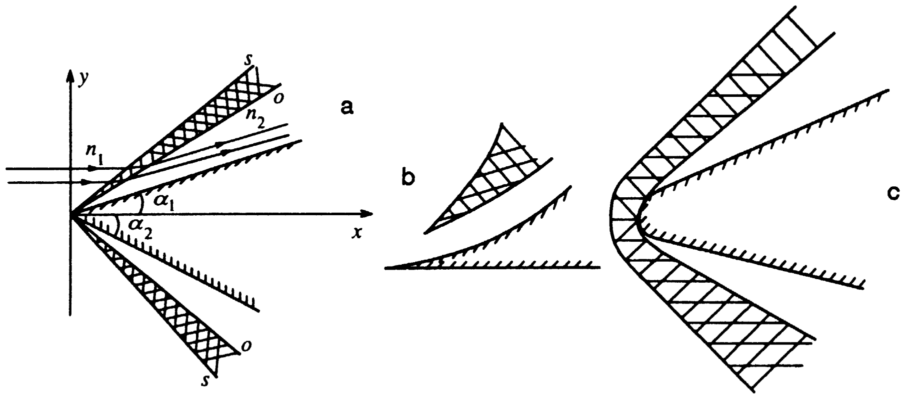

FIG. 2. Flow around bodies in dispersive hydrodynamics: a) flow around a straight wedge; b) separation of the dissipationless shock wave from the boundary of the object; c) flow around a blunt body.

shock wave then changes in structure. As the amplitude discontinuity increases it is gradually smeared out, and the flow assumes a multistreaming kinetic character.[17]

Taking this behavior into account, we can propose the following qualitative picture of two-dimensional flow around objects in dispersive hydrodynamics. Figure 2 shows the flow around the leading part of the object. Taking the flow velocity (in the x direction) to be relatively large, $u_x=u_0 \gg c_s$, in the first approximation we can disregard the variation of u_x. Then the perturbations propagate in the y direction and are advected in the x direction with constant velocity u_0: the propagation in x acts in a manner like a sweep in the time $t=x/u_0$. This allows one in part to transfer the well-known time-dependent solutions (Refs. 5, 6, 8–12, 16) to the theory of two-dimensional flow around an object. The qualitative picture of flow around a wedge thus constructed is shown in Fig. 2a. Here the region of the dissipationless shock wave is indicated by hatching; s is its leading edge, where the solitons are traveling (for negative dispersion); and o is the trailing edge. The amplitude of the oscillations in the dissipationless shock wave is constant along the lines $y/x=$const, while the oscillation amplitude drops off as one goes from the leading edge toward the trailing edge. Between the trailing edge and the surface of the object there are also oscillations, but their amplitude decreases monotonically with x as $x^{-1/2}$. The velocity of the incident flow undergoes a rotation inside the dissipationless shock wave; in the region behind the dissipationless shock wave it is directed parallel to the surface of the object. The picture of the deflected flow can be seen to resemble that in ordinary hydrodynamics (Fig. 1), but only in dispersive hydrodynamics is there a continuous rotation of the velocity within the dissipationless shock wave (Fig. 2a), rather than a sharp discontinuity. The average gas density increases in passing through the dissipationless shock wave; beyond the dissipationless shock wave in flowing around the wedge it is a constant, $n=n_2$ (Fig. 2a), as in ordinary hydrodynamics, but the value n_2 is different.

Figure 2b shows flow around a body with a sharp forward portion. Under some conditions the dissipationless shock wave can begin at a point outside the body, as in ordinary hydrodynamics, and not on its surface. A dissipationless bow shock wave develops in flow around a blunt body. In this case the flow is two-valued immediately behind the front dissipationless shock wave. We can assume that the entire region from the leading edge of the dissipationless bow shock wave to the surface of the body is filled with oscillations (Fig. 2c). In the transition to the supersonic lateral region a shock-wave structure develops similar to that shown in Fig. 2a; the oscillation amplitude in the Euler zone between the surface of the object and the trailing edge of the lateral wave gradually decays.

In the region in back of the object a rarefaction wave develops, causing a gradual drop in density. The average density and velocity profile here is described by Eulerian dynamics, since qualitatively it should be completely similar to that observed in ordinary hydrodynamics. In particular, a second shock wave should develop as a result of the flows converging on the axis behind the body (Fig. 3). In dispersive hydrodynamics this is a dissipationless shock wave, i.e., it has an oscillatory structure and expands as a function of distance from the body. Thus, two dissipationless shock waves propagate outward behind the body in either direction. Note that for large x the structure of the nose and tail dissipationless shock waves differ fundamentally. The nose dissipationless shock wave decays into a large number of noninteracting constant-amplitude solitons, the distance between which increases with distance in the x direction (a soliton wave).[8,11,12] This differs substantially from the asymptotic behavior of a shock wave in ordinary hydrodynamics. The trailing dissipationless shock wave is converted asymptotically into a linear wave packet whose amplitude decreases as $\propto x^{-1/2}$ (Ref. 18).

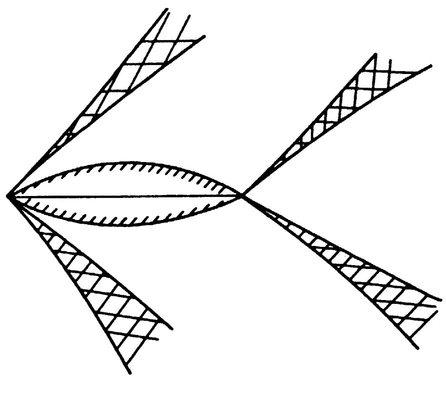

FIG. 3. Flow around a finite body.

The qualitative picture presented here will be confirmed in the following section of this paper, where we present corresponding exact solutions of the equations of dispersive hydrodynamics.

3. STARTING EQUATIONS; DISPERSION RELATIONS

To be specific let us consider a system describing nonlinear flows in an unmagnetized two-temperature ($T_1 \ll T_e$) plasma:[6,16]

$$\partial_t n + \nabla(n\mathbf{u}) = 0, \quad \partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} + \nabla\phi = 0,$$
$$D^2 \Delta\phi = e^{\phi} - n. \tag{1}$$

Here n and $\mathbf{u} = (u, v, w)$ are the ion density and velocity, ϕ is the electric potential, and $D = (T_e/4\pi e^2 n_0)^{1/2}$ is the Debye radius, which determines the scale of the dispersive effects; the dependent variables are nondimensionalized by scaling them with their characteristic values: $n/n_0 \to n$, $u_x/c_s \to u$, $u_y/c_s \to v$, $u_z/c_s \to w$, $e\phi/T_e \to \phi$. Here n_0 is the ion density at infinity, $c_s = (T_e/M_0)^{1/2}$ is the sound speed, M_0 is the ion mass, and we have taken $\phi|_{(x,y)=\infty} = 0$. For now it is convenient to leave the independent variables dimensional.

Equations (1) possess properties which are general for different dispersive hydrodynamic systems. In particular, when the dispersive scaling D goes to zero dispersive hydrodynamics goes over to ideal Eulerian hydrodynamics with some adiabatic equation state $P \propto n^{\gamma}$ [for case (1) this is ideal isothermal hydrodynamics with $\gamma = 1$].

For two-dimensional steady flows in the (x,y) plane, Eqs. (1) assume the form

$$\partial_x(nu) + \partial_y(nv) = 0, \quad u\partial_x u + v\partial_y u + \partial_x\phi = 0,$$
$$u\partial_x v + v\partial_y v + \partial_y\phi = 0, \quad D^2(\partial^2_{xx}\phi + \partial^2_{yy}\phi) = e^{\phi} - n. \tag{2}$$

The linear equations in the system (2) are characterized by a "dispersion relation" relating the components k_y and k_x of the wave vector. If we linearize Eqs. (2) about uniform flow with $n=1$, $\phi=0$, $v=0$, $u=M$ (where M is the Mach number), we find the dispersion relation for two-dimensional steady ion-acoustic flows:

$$\left(\frac{\kappa_y}{\kappa_x}\right)^2 = (M^2-1)\,\frac{1 + \dfrac{\kappa_x^2}{M^2-1}}{1-\kappa_x^2}, \tag{3}$$

where

$$\kappa_x = MDk_x, \quad \kappa_y = MDk_y. \tag{4}$$

For nonnegative right-hand side Eq. (3) determines the region where the linear modulations are hyperbolic:

$$1 - M^2 \leqslant \kappa_x^2 \leqslant 1. \tag{5}$$

For supersonic flows the first inequality of Eq. (5) is satisfied automatically. In this case for long waves such that

$$\kappa_x \equiv MDk_x \ll 1 \tag{6}$$

we have in place of (3) in the upper half-plane

$$\kappa_y = \sqrt{M^2-1}\,\kappa_x\left(1 + \frac{1}{2}\frac{M^2}{M^2-1}\kappa_x^2\right). \tag{7}$$

The form of the dispersion relations (7) indicates that under weakly nonlinear conditions the long waves (6) in Eqs. (2) are described by the KdV equation.

4. FLOW AROUND SLENDER BODIES: FORMULATION OF THE PROBLEM

For the problem of an ideal flow around a body the boundary conditions are obtained by setting the normal component of the velocity equal to zero on the surface of the object, which is at rest in the coordinate frame being used, together with the requirement that the flow be uniform at infinity. Let us consider the analogous problem for the dispersive hydrodynamic system (2). We assume that the body has a shape of a "sharp" cylinder with a generator parallel to the z axis, specified in the upper half-plane by the function

$$\Phi(x,y) = y - bF(x/l) = 0,$$
$$x \geqslant 0, \quad y \geqslant 0, \quad F(0) = 0. \tag{8}$$

Here b and l are the effective thickness and length, respectively of the object. Let the uniform flow be incident on the object on the left with velocity $M\mathbf{e}_x$; here $M > 1$. We restrict ourselves to the case of zero angle of attack. By virtue of the supersonic nature of the flow the motions in the upper and lower half-planes are independent (Fig. 2), so in what follows we will treat only the flow with $y \geqslant 0$. The flow in the lower half-plane can be treated completely analogously.

The "impenetrability" condition takes the form

$$(\mathbf{u}\ \mathrm{grad}\ \Phi)_{\mathrm{body}} = 0. \tag{9}$$

Let the object be slender, so that

$$\delta = b/l \ll 1, \quad F(x/l) = O(1). \tag{10}$$

The introduction of the characteristic length l implies that an additional dimensionless parameter

$$\beta = D/l \ll 1 \tag{11}$$

appears in the problem, which determines the typical value of the component κ_x of the scaled wave vector (4).

Assuming that the perturbations introduced into the flow by the slender body are weak, we represent the dependent variables in Eq. (2) as an expansion in the small parameter δ about uniform supersonic flow with $M - 1 \ll 1$:

$$n = 1 + \delta n_1 + \delta^2 n_2 + \ldots,$$
$$u = M + \delta u_1 + \delta^2 u_2 + \ldots,$$
$$v = \delta v_1 + \delta^2 v_2 + \ldots,$$
$$\phi = \delta\phi_1 + \delta^2\phi_2 + \ldots . \tag{12}$$

The requirement that the flow be uniform at infinity imposes the conditions

$$n_i \to 0, \quad u_i \to 0, \quad v_i \to 0, \quad \phi_i \to 0 \quad \text{for } x \to \infty,$$
$$y \to \infty. \tag{13}$$

Consistent with the expansions (12) and the nature of the nonlinearity in Eqs. (2), the transformation of the independent variable consisting of dilation and rotation takes the form

$$\xi=\frac{x-\sqrt{M^2-1}y}{l}, \quad \tau=\delta\frac{y}{l}. \tag{14}$$

Substituting Eqs. (12) and (14) into the original system (2) under the condition

$$\frac{\beta^2}{\delta}\lesssim 1 \tag{15}$$

we obtain the KdV equation for the lowest-order corrections in the expansion (12):[1)]

$$\partial_\tau u_1+\frac{M^3}{\sqrt{M^2-1}}u_1\partial_\xi u_1-\frac{M^4\beta^2}{2\sqrt{M^2-1}\delta}\partial^3_{\xi\xi\xi}u_1=0. \tag{16}$$

The inequality (15) implies that perturbations introduced into the flow by the slender pointed body and described by the KdV equation have long wavelengths and satisfy condition (6). The line $\xi=0$, i.e., $y=x/(M^2-1)^{1/2}$, is the Mach line (the characteristic of the linearized equations of ideal hydrodynamics).

The corrections n_1, v_1, ϕ_1 are related to u_1 as follows:

$$n_1=\phi_1=-Mu_1, \quad v_1=-\sqrt{M^2-1}\,u_1. \tag{17}$$

The KdV equation for steady weakly nonlinear flows in a dispersive medium was derived by Karpman[13] in a less formal manner from the so-called *Boussinesq* system. This approach has certain advantages: e.g., it enables one to calculate the coefficient associated with the nonlinear term in the KdV equation in general form (i.e., to express this coefficient in terms of the adiabatic index γ of the corresponding ideal hydrodynamics. The term in question takes the form

$$\frac{\gamma+1}{2}\frac{M^3}{\sqrt{M^2-1}}u_1\partial_\xi u_1. \tag{18}$$

Now let us consider the boundary condition (9). Using Eqs. (12) and (17) we find that to first order in δ it takes the form

$$u_1|_{y=0}=-\frac{M}{(M^2-1)^{1/2}}F'(x/l), \quad x\geqslant 0, \tag{19}$$

i.e., information about the shape of the body is "advected" along the x axis. Condition (19) ensures an accuracy $O(\delta)$ in this solution and thus corresponds to the accuracy of the KdV approximation [provided, of course, that $F'(x/l)=O(1)$].

Since perturbations cannot propagate upstream in a supersonic flow, we have

$$u_1=0 \quad \text{for } x<0. \tag{20}$$

Using the more general form of the nonlinear term (18) we perform some "cosmetic" changes of variables in (16):

$$X=\frac{\sqrt{M^2-1}y-x}{l}, \quad T=\frac{M^4(\gamma+1)}{2(M^2-1)}\delta\frac{y}{l},$$
$$\eta=-\frac{\sqrt{M^2-1}}{M}u_1, \quad \varepsilon^2=\frac{2\sqrt{M^2-1}\beta^2}{(\gamma+1)\delta}, \tag{21}$$

which yield the standard form of the KdV equation,

$$\partial_T\eta+\eta\partial_X\eta+\varepsilon^2\partial^3_{XXX}\eta=0. \tag{22}$$

The boundary conditions (19) and (20) are transformed into initial data for Eq. (22) of the form

$$\eta(X,0)=\begin{cases} r_0(X)\equiv F'(-X) & \text{for } X\leqslant 0,\\ 0, & \text{for } X>0. \end{cases} \tag{23}$$

The behavior of the solution of Eq. (22) with the initial conditions (23) having unit scale depends strongly on the magnitude of the effective dispersion parameter ε given by (21) and (15). For $\varepsilon\sim 1$ the initial localized perturbation decays into a finite number of solitons and an oscillatory wave "tail." The case

$$\varepsilon\ll 1, \tag{24}$$

which is more realistic and has more content from the hydrodynamic standpoint, has initial data which "contain" a very large number of solitons in a finite interval. The evolution of such a perturbation gives rise to a dissipationless shock wave. There are several approaches to solving such problems. One of these is associated with the study of the exact multisoliton solution of the KdV equation in the limit $\varepsilon\to 0$ and was developed in a series of papers.[19] The other is based on the application of the Whitham method[14] and is the one used in the present work.

5. QUASISIMPLE DISSIPATIONLESS SHOCK WAVES IN FLOWS AROUND OBJECTS

In Secs. 1 and 2 we showed that if the higher-order corrections to the time-independent two-dimensional equations of ideal hydrodynamics are dispersive rather than dissipative, then in flow past a sharp body with $F'(0)>0$ instead of a sharp jump in the density a dissipationless shock wave develops, i.e., a stationary wedge-shaped region filled with small-scale nonlinear oscillations. In terms of the evolutionary problem (22), (23) the formation of the dissipationless shock wave is associated with the tendency of the initial profile to break if it has a decaying section near zero. This breaking is prevented by dispersive effects, which is what gives rise to the oscillations. The Whitham method[14] has been found to be extremely effective in describing dissipationless shock waves (see for example Review[7] and the references given there); the corresponding problem for the KdV equation was first considered by Gurevich and Pitaevskii.[5]

In the GP treatment the (x,t) plane is broken up into three regions (Fig. 2a). In the external regions the flow is described by the solution of the Hopf equation for $\eta\simeq r(X,T)$:

$$\partial_T r+r\partial_X r=0 \tag{25}$$

with the initial data (23), while in the interior region it is a dissipationless shock wave, a quasisteady modulated solution of the KdV equation in the form of a traveling conoidal wave,

$$\eta(X,T)=a\,\mathrm{cn}^2\left[\left(\frac{a}{12m}\right)^{1/2}\frac{X-UT}{\varepsilon}\Big|m\right]+d, \tag{26}$$

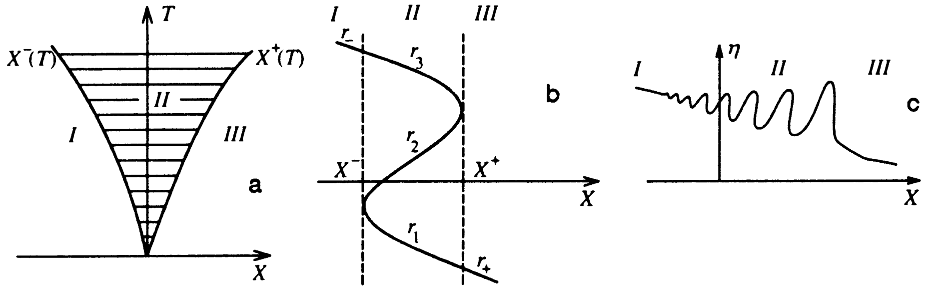

FIG. 4. Dissipationless shock wave in dispersive hydrodynamics: a) $X-T$ plane in the Gurevich–Pitaevskiĭ problem; b) behavior of the Riemann invariants in a dissipationless shock wave; c) oscillatory structure of the dissipationless shock wave.

where $en(y,m)$ is a Jacobi elliptic function and the modulation parameters a, m, U, and d are expressed in terms of three functions $r_1(X,T)\leqslant r_2(X,T)\leqslant r_3(X,T)$:

$$a=2(r_2-r_1),\quad m=\frac{r_2-r_1}{r_3-r_1},\quad d=r_3-r_2+r_1,$$
$$U=\frac{1}{3}(r_1+r_2+r_3). \tag{27}$$

The desired functions $r_i(X,T)$ are Riemann invariants of the Whitham modulation system:[14,5]

$$\partial_T r_i+V_i(\mathbf{r})\partial_X r_i=0,\quad i=1,2,3, \tag{28}$$

where there is no summation over repeated indices, and the characteristic velocities $v_i(r)$ can be represented in the form[9–11]

$$V_i(\mathbf{r})=U-\partial_i U(\partial_i \ln L)^{-1}, \tag{29}$$

where

$$\partial_i\equiv\frac{\partial}{\partial r_i},\quad L=\int_{r_1}^{r_2}\frac{d\tau}{[(\tau-r_1)(\tau-r_2)(\tau-r_3)]^{1/2}}$$
$$=\frac{2K(m)}{(r_3-r_1)^{1/2}}. \tag{30}$$

Here $K(m)$ is the complete elliptic integral of the first kind. The Gurevich–Pitaevskii boundary conditions which ensure continuity of the average flow at the boundaries $X^{\pm}(T)$ of the dissipationless shock wave (in accordance with the definition) take the form (Fig. 2b)

$$r_3(X^-,T)=r(X^-,T),\quad r_2(X^-,T)=r_1(X^-,T),$$
$$r_1(X^+,T)=r(X^+,T),\quad r_2(X^+,T)=r_3(X^+,T). \tag{31}$$

The system (28) is hyperbolic and at each point has three characteristic directions $dX_i/dT=V_i(\mathbf{r})$ ($i=1,2,3$). The matching of (31) with the exterior solution (25) is shown in Fig. 4b. The problem of integrating Eqs. (28) with the boundary condition (31) is nonlinear and its boundary is not known; nonetheless, as will be shown below, it can be completely linearized by going to the hodograph space in which the boundary conditions go over to very simple linear conditions on the coordinate axes. This is a consequence of the Riemann form of Eq. (28) and the special "natural" boundary conditions (31).

If breaking takes place on the boundary with the uniform regime, i.e., at the point $X=0$, then the solution of the Whitham equation with the condition (31), (23) constitutes a so-called quasisimple wave[8,11] with $r_1\equiv 0$ and variable r_2, r_3. The modulation system for these variables can be integrated using the ordinary hodograph transformation;[8] here we introduce a convenient modern version of it, based on the generalized hodograph method[15,7] and the "scalar potential" technique.[9–12]

Thus, Eqs. (28) become a system of two equations in the variables r_2 and r_3. Accordingly, the boundary conditions (31) simplify:

$$r_3(X^-,T)=r(X^-,T),\quad r_2(X^-,T)=0,$$
$$r_2(X^+,T)=r_3(X^+,T). \tag{31a}$$

We will look for solutions $r_i(X,T)$ implicitly:

$$X-V_i(\mathbf{r})T=W_i(\mathbf{r}),\quad i=2,3, \tag{32}$$

where $W_i(\mathbf{r})$ are new unknown functions satisfying the system[15] by substituting (32) in initial system (25):

$$\frac{\partial_i W_j}{W_i-W_j}=\frac{\partial_i V_j}{V_i-V_j},\quad i,j=2,3,\quad i\neq j. \tag{33}$$

From (33) and (29) it follows that we can introduce for W_i the scalar (potential) representation

$$W_i(r_2,r_3)=f-\partial_i f[\partial_i \ln L]^{-1}. \tag{34}$$

Then (33) goes over to the well-known scalar Euler–Poisson equation for the potential f,

$$2(r_3-r_2)\partial_{32}^2 f=\partial_3 f-\partial_2 f. \tag{35}$$

The boundary condition for (35) takes the form

$$f(0,r_3)=\frac{1}{2}r_3^{-1/2}\int_0^{r_3}z^{-1/2}W(z)dz. \tag{36}$$

Here $W(r)$ is the function which is the inverse of the initial distribution $r_0(X)$. Furthermore, for monotonic initial data the function f must be bounded at $r_2=r_3$ (for further details see Refs. 9–11).

The solution of Eqs. (35) and (36) is (11)

$$f=\int_{r_2}^{r_3}\frac{\phi(\tau)d\tau}{[\tau(r_3-\tau)(\tau-r_2)]^{1/2}}, \tag{37}$$

where the function $\phi(\tau)$ is related to the initial data by the Abel transformation

$$\phi(\tau)=\frac{1}{2\pi}\int_0^{\tau}\frac{W(x)dx}{(\tau-x)^{1/2}}. \tag{38}$$

This solution can also be represented as a single integral: substituting (38) in (37) and interchanging the orders of integration we find

$$f(r_2,r_3)=\frac{1}{\pi(r_3-r_2)^{1/2}}\int_{r_2}^{r_3}\frac{W(x)}{x^{1/2}}K(z)dx$$

$$+\frac{1}{\pi r_2^{1/2}}\int_0^{r_2}\frac{W(x)}{(r_3-x)^{1/2}}K\left(\frac{1}{z}\right)dx, \quad (39)$$

where

$$z=\frac{r_2(r_3-x)}{x(r_3-r_2)}.$$

To determine the boundaries $X^{\pm}(T)$ of the dissipationless shock wave we must consider the solution of (32) for $r_2=r_3$ (the leading or "+" edge) and $r_2=0$ (the trailing or "−" edge). Then the boundaries are defined as multiple characteristics (envelopes) of the modulation system for this solution, i.e.,.

$$\frac{dX^{\pm}}{dT}=V^{\pm}, \quad (40)$$

where

$$V^{+}\equiv V_2(0,r_3^+,r_3^+)=V_3(0,r_3^+,r_3^+)=\tfrac{2}{3}r_3^+,$$
$$V^{-}\equiv V_2(0,0,r_3^-)=V_3(0,0,r_3^-)=-r_3^-. \quad (41)$$

Here $r_3^{\pm}$ are r_3 invarient values on boundaries of the nondissipative shock wave.

For the trailing edge we can easily find the parametric form[8]

$$X^{-}=W(r)-\frac{r^{1/2}}{2}\int_0^r\frac{W'(x)}{x^{1/2}}dx,$$
$$T^{-}=-\frac{1}{2r^{1/2}}\int_0^r\frac{W'(x)}{x^{1/2}}dx. \quad (42)$$

For the leading edge we have

$$X^{+}=\frac{1}{2r^{1/2}}\int_0^r\frac{W(x)-xW'(x)}{(r-x)^{1/2}}dx,$$
$$T^{+}=-\frac{3}{4r^{3/2}}\int_0^r\frac{xW'(x)}{(r-x)^{1/2}}dx. \quad (43)$$

Note that the determination of the boundaries $X^{\pm}(T)$ in the problem of flow around an object is of particular interest, since the curves $y^{\pm}(x)$ which are found using (42), (43), and (21) are the geometric boundaries of a stationary dissipationless shock wave in the (x,y) plane.

The oscillatory structure of the dissipationless shock wave (Fig. 4c) is described by Eqs. (26) and (27), where we must substitute the functions $r_i(X,T)$ that have been found. The resulting solution is sinusoidal and has a vanishingly small amplitude close to the boundary $y^{-}(x)$ and takes the form of individual solitons near the boundary $y^{+}(x)$ directed toward the incident flow. The average variations of the hydrodynamic variables in the dissipationless shock wave are found from the relations

$$\bar{u}_1=-\frac{M}{(M^2-1)^{1/2}}\bar{\eta}, \quad \bar{n}_1=\bar{\phi}_1=-M\bar{u}_1,$$

$$\bar{v}_1=-(M^2-1)^{1/2}\bar{u}_1, \quad (44)$$

where

$$\bar{\eta}=2(r_3-r_1)\mu(m)+r_1+r_2-r_3. \quad (45)$$

Here we have written $\mu(m)=E(m)/K(m)$, where $E(m)$ is the complete elliptic integral of the second kind.

6. EXAMPLES

In the vicinity of the point $x=0$, where the shape of the body is given by the function $y(x)\propto x^q$, Eqs. (37)–(43) can be evaluated in terms of quadratures. It turns out that the cases $q=1$, $1<q<2$, $q=2$, and $q>2$ are qualitatively different. Let us consider them in succession.

1. $q=1$. Flow around an infinitely sharp wedge (Fig. 2a).

For a wedge with opening angle α about the x axis the small nonlinearity parameter defined in Sec. 4 satisfies $\delta\approx\alpha$, i.e., the condition for the applicability of the KdV approximation is $\alpha\ll 1$. It can readily be shown that the corresponding initial data for the evolution problem (22), (23) assume the form of a unit step:

$$\eta(X,0)=\begin{cases}1, & X\leq 0\\ 0, & X>0.\end{cases} \quad (46)$$

The problem of the decay in the initial discontinuity (46) in KdV hydrodynamics was solved by Gurevich and Pitaevskii.[5] The solution of the modulation equations (28), (29) for the functions $r_i(X,T)$ in the cnoidal wave (26) is self-similar:

$$r_1=0, \quad r_3=1, \quad V_2(r_2)=\frac{X}{T} \quad (47)$$

or explicitly (for more details see Refs. 5 and 17)

$$r_1=0, \quad r_3=1, \quad r_2=m,$$
$$\frac{1+m}{3}-\frac{2}{3}\frac{m(1-m)}{\mu(m)-(1-m)}=\frac{X}{T}. \quad (48)$$

The equations of the boundaries which follow from (48) for $m=0$ and $m=1$ take the form

$$X^{-}(T)=-T, \quad X^{+}(T)=\tfrac{2}{3}T. \quad (49)$$

Using (21) to transform to the variables x, y we find the boundary equations in the physical plane:

$$y^{\pm}(x)=a^{\pm}x,$$
$$a^{-}=\frac{1}{(M^2-1)^{1/2}}-\frac{M^4(\gamma+1)}{2(M^2-1)^2}\alpha, \quad (50)$$
$$a^{+}=\frac{1}{(M^2-1)^{1/2}}+\frac{M^4(\gamma+1)}{3(M^2-1)^2}\alpha.$$

Correspondingly the angular width of the centered self-similar steady dissipationless shock wave is

$$\Delta\varphi=\arctan a^{+}-\arctan a^{-}\approx\frac{5}{6}\frac{M^2(\gamma+1)}{(M^2-1)}\alpha, \quad (51)$$

proportional to the opening angle of the deflecting wedge.

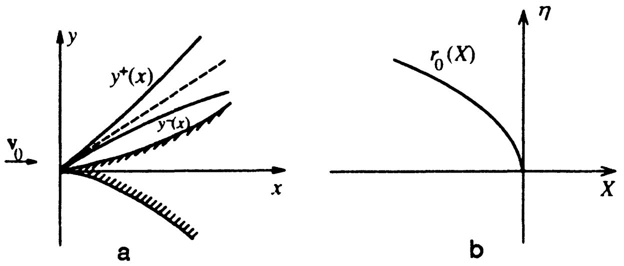

FIG. 5. Flow around a concave wedge defined by $y=\nu x^2$, $1<q<2$: a) qualitative flow pattern: the broken trace is the Mach line; b) initial data in the Gurevich–Pitaevskiĭ problem.

2. Flow around an infinite concave wedge.

Assume

$$y(x)=\nu x^q, \quad \text{where} \quad 1<q<2. \tag{52}$$

From (52) it follows that $y'(0)=0$, $y''(0) = -r'(0) = \infty$, and breaking occurs in the flow at the coordinate origin (Fig. 5). The initial data for the breaking problem corresponding to (52) take the form

$$\eta(X,0)=\begin{cases} r_0(X)=q(-X)^{q-1}, & X\leqslant 0, \\ 0, & X>0. \end{cases} \tag{53}$$

Since breaking occurs at the boundary with the uniform region, the solution of the modulation system is a quasisimple wave, described in Sec. 5. The solution of the Euler–Poisson equations (37)–(39) corresponding to the initial data (53) can be expressed in terms of the Gauss hypergeometric function[22] (see also Ref. 9):

$$f(r_2,r_3)=-r_3^{1/(q-1)}\,\frac{\pi^{1/2}q^{-1/(q-1)}\Gamma\left(\dfrac{1}{q-1}\right)}{2\Gamma\left(\dfrac{3q-2}{2(q-1)}\right)} \times F\left(\frac{1}{2},-\frac{1}{q-1};1,1-\frac{r_2}{r_3}\right). \tag{54}$$

For integer $1/(q-1)=N$ the hypergeometric series terminates and the solution assumes a symmetric polynomial form:

$$f(r_2,r_3)=-\left(\frac{2N}{N+1}\right)^{-N}\frac{N!}{(2N+1)!!}\sum_{n+m=N} \times\frac{\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_n}{m!n!}\,r_2^m r_3^n, \tag{55}$$

where

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}.$$

It is not hard to see that the desired solution (32), (34) for the functions $r_{2,3}(X,T)$ given by the potentials (54), (55) is a generalized self-similar form[5,9,10]:

$$r_i(X,T)=\frac{1}{T^\mu}\,l_i\left(\frac{X}{T^{\mu+1}}\right), \quad \mu=\frac{q-1}{2-q}. \tag{56}$$

The boundaries of the dissipationless shock wave (42), (43), like the solution (54), can be evaluated in terms of quadratures:

$$X^\pm=Z^\pm(q)T^{1/(2-q)},$$

where

$$Z^+(q)=\frac{2-q}{q-1}\left(\frac{1}{2}\int_0^1\frac{x^{1/(q-1)}}{(1-x)^{1/2}}\,dx\right)^{-(q-1)/(2-q)} \times\left(\frac{2}{3}\,q(q-1)\right)^{1/(2-q)},$$

$$Z^-(q)=-(2-q)q^{1/(2-q)}(3-q)^{(q-1)/(2-q)}. \tag{57}$$

The equation for the boundaries in the physical plane is

$$x^\pm=(M^2-1)^{1/2}y-Z^\pm(q)\left(\frac{M^4(\gamma+1)}{2(M^2-1)l^{q-1}}\,\delta y\right)^{1/(2-q)}. \tag{58}$$

It is not difficult to see that at some point the trailing edge of the dissipationless shock wave must intersect the body (see Fig. 5a), which is physically meaningless, since the solution then extends into the region occupied by the body. In one-dimensional time-independent dispersive hydrodynamics, as is well known,[5] such complications do not arise: the dissipationless shock wave expands monotonically and exists for all $T>0$. It is easy to resolve the contradiction if we recall that in the present slender-body approximation the shape about which the flow moves is advected along the x axis [cf. Eq. (19)] and cannot intersect the monotonically increasing function $y^-(x)$. It is clear, however, that the KdV description actually ceases to be satisfactory much earlier (the condition $\delta\ll 1$ is violated). The accuracy of the resulting solution decreases monotonically as it moves along the x axis to the right, and at each point it is determined by the slope of the curve $y(x)$. We note in passing that in consequence of the supersonic nature of the flow Eqs. (54)–(58) have a finite "region of influence": if a real body is prescribed by the equation

$$y(x)=\begin{cases} \nu x^q, & 0\leqslant x\leqslant l, \\ \zeta(x), & x>l, \end{cases} \tag{59}$$

then for (x,y) bounded by the characteristic of the modulation equations leaving the body at the point $(l,\nu l^q)$ these expressions continue to be valid.

3. Flow around a rectangular profile (Fig. 6).

Now consider the case $q=2$, or more precisely, let the function $y(x)$ grow near the origin quadratically and then more slowly, so that it behaves as νx^q, where $1<q<2$. The meaning of the latter restriction will become clear shortly.

For $y=\nu x^2$ we have

$$r_0'(0)=-2. \tag{60}$$

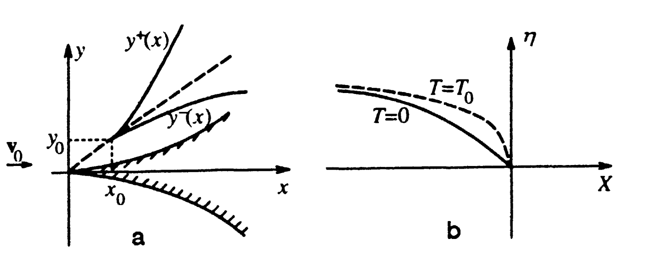

FIG. 6. Flow around a body with a quadratic ($q=2$) profile: a) separation of the dissipationless shock wave from the boundary of the object; b) evolution of the initial data up to the instant when breaking occurs.

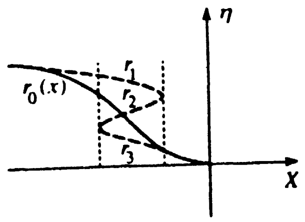

FIG. 7. Breaking of the profile $r_0(X) \propto (-X)^{q-1}$ having a section with $q>2$ near $X=0$.

In terms of the time-dependent problem (60) this means that breaking occurs at the point $x_0=0$, not immediately but at some point in time (Fig. 6b)

$$T_0=\frac{1}{2}. \tag{61}$$

Substituting X_0, T_0 in (21) we find the coordinates of the corresponding point in the (x,y) plane. This point is the peak of the dissipationless shock wave, which of course lies on the Mach line:

$$x_0=\frac{(M^2-1)^{3/2}}{\nu M^4(\gamma+1)}, \quad y_0=\frac{(M^2-1)}{\nu M^4(\gamma+1)}. \tag{62}$$

Thus, when flow takes place around a body with a quadratic profile a dissipationless shock wave separates from the boundary of the object and begins in the flow itself (Fig. 6a). In this case the solution is identical with that described in the previous section to within the substitutions $\tilde{X}=X$, $\tilde{T}=T-T_0$, $\tilde{W}(r)=W(r)-rW'(0)$ (respectively, $\tilde{x}=x-x_0$, $\tilde{y}=y-y_0$).

The phenomenon of the separation of the shock wave from the boundary of the body is also well known in ordinary hydrodynamics (Ref. 1, Sec. 115). In this case the separation is accompanied by the appearance of two weak discontinuities. One of these passes through the peak of the shock wave and coincides with the corresponding characteristic proceeding toward the object. Its occurrence is associated with a discontinuity in the entropy in the passage through the shock wave. The second discontinuity is a weak tangential discontinuity and coincides with the current line passing through the vertex of the shock wave. The weak tangential discontinuity arises because the flow ceases to be irrotational at the point where the shock wave forms. The flow in the region adjacent to the body and bounded by these discontinuities is no longer a simple wave.

The steady dissipationless shock wave that occurs in dispersive hydrodynamics does not prevent the flow from remaining irrotational and isentropic, so its formation is not accompanied by the occurrence of the discontinuities mentioned above, and the flow outside the dissipationless shock wave is a simple wave everywhere.

If the profile of the body is quadratic over an extended region, not only in a small neighborhood of the origin, then the conditions for the applicability of the KdV approximation may fail near the breaking point, since the amplitude of the initial perturbation satisfies $r_0(X) \propto -x$ (Fig. 6b) and is determined by the length of the region in which quadratic behavior persists. Thus, if we have $y \sim x^2$ for all x then the initial data at the breaking point formally diverge, which contradicts the assumptions made in deriving the boundary condition (19).

A similar effect also occurs for $q>2$ (Fig. 7). But if for some x_0 the behavior of the function $y(x)$ changes so that it grows more slowly than x^2, then the treatment of this problem yields a dissipationless shock wave in which all three Riemann invariants change, and the corresponding solution of the modulation equations is no longer a quasisimple wave. The generalized hodograph method nevertheless permits the full modulation system (28) to be integrated with the boundary conditions (31) (see Ref. 10).

In the initial profile $r_0(X)$ this point x_0 corresponds to (X_0, η_0) at which $r_0'(X)$ is a minimum i.e. $y'''(x_0)=0$. Breaking occurs at the time $T^*=-W'(\eta_0)$ at the point $X^*=X_0-\eta_0 W'(\eta_0)$. Then the solution takes the form

$$\begin{aligned} f(r_1,r_2,r_3)=&\frac{1}{\pi(r_3-r_2)^{1/2}}\int_{r_2}^{r_3}\frac{W_1(x)}{(x-r_1)^{1/2}}K(z)dx \\ &+\frac{1}{\pi(r_2-r_1)^{1/2}}\int_{r_1}^{r_2}\frac{W_1(x)}{(r_3-x)^{1/2}}K\left(\frac{1}{z}\right)dx \\ &+X^*-UT^*, \end{aligned} \tag{63}$$

where

$$W_1(x)=W(x)+xT^*-X^*, \quad z=\frac{(r_2-r_1)(r_3-x)}{(r_3-r_2)(x-r_1)}.$$

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[1)]A more general transformation of the independent variables leading to the KdV equation takes the form $\xi=\delta^p(x-\sqrt{M^2-1}y/l)$, $\tau=\delta^{p+1}(y/l)$, $p\geqslant 0$; where instead of (15) we have $\beta^2\delta^{2p-1}\lesssim 1$. However, specifying the characteristic scale l parallel to the x axis in the boundary conditions implies $p=0$.

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