

Soliton (vortex) thermodynamics of a quasi-2D easy-plane antiferromagnet

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The dynamics of a 2D topological soliton (magnetic vortex) in an easy-plane Heisenberg antiferromagnet in an external magnetic field \mathbf{H} is analyzed. The field is assumed to be perpendicular to the easy plane. Effective equations are written for the coordinates of the vortices in the vortex gas which exists in the 2D antiferromagnet above the point of the topological phase transition. The rms velocity of the vortex motion is calculated. The components of the dynamic structure factor of the 2D antiferromagnet in a magnetic field are calculated. It is predicted that characteristics of the soliton peaks (their positions and widths) in the dynamic structure factor will be strong functions of the magnetic field. © 1995 American Institute of Physics.

1. INTRODUCTION

Two-dimensional magnetism has attracted new interest in recent years. One reason is progress in the fabrication of high-quality samples of layered magnetic materials, including intercalated compounds¹ and artificial quasi-2D spin systems—magnetic lipid layers.² A second reason is that important experimental information has been acquired by the methods of inelastic neutron scattering, antiferromagnetic resonance, and NMR. A third reason is the existence of 2D spin subsystems in high- T_c superconductors. Two-dimensional magnetic materials are interesting from the theoretical standpoint because nonlinear excitations—topological solitons of the magnetic-vortex type—contribute substantially to their thermodynamic and high-frequency properties. The soliton approach to the thermodynamics of quasi-2D magnetic materials starts from the idea of a gas of vortices of finite density. This approach was apparently first taken in a paper by Kosterlitz and Thouless,³ but it is only in recent years that this approach has become particularly popular.^{4–12} (We recall that the familiar contribution of kink solitons to the thermodynamics of quasi-1D magnetic materials has been under study for nearly 20 years now; see Refs. 13–15 and some reviews.^{16,17})

The properties of solitons (magnetic vortices) are more complex than those of 1D solitons (kinks) and have received considerably less study. The presence of vortices in an easy-plane magnetic material determines the specific (Berezinskiĭ–) Kosterlitz–Thouless topological phase transition.^{18,3} It is assumed that bound vortex-antivortex pairs exist at sufficiently low temperatures. As the temperature is raised, the rms distance in the pair increases in proportion to $(T_c - T)^{-1/2}$, and at a certain T_c , the “temperature of the Kosterlitz–Thouless transition,” the pairs dissociate. A gas of quasifree vortices exists in the system at $T \geq T_c$. As these vortices move along with the hydrodynamic flows in the medium, they make a specific contribution to the re-

sponse function of the magnetic material, forming the so-called central peak in the dynamic structure factor. This contribution was calculated in Refs. 6–8 for the case of ferromagnets, in Ref. 11 for in-plane vortices in an antiferromagnet, and in Ref. 12 for both in-plane and extraplanar vortices in an antiferromagnet. An antisymmetric Dzyaloshinskiĭ interaction was also incorporated in the latter paper. Comparison with numerical simulations^{10,12} shows that the dynamical properties of extraplanar vortices in antiferromagnets and ferromagnets are quite different from those which have been discussed. These differences should be reflected in the mean velocity of the vortex gas and thus in the nature of the vortex contribution to the dynamic structure factor.

In the present paper we analyze the dynamics of magnetic vortices and ensembles thereof in a classical easy-plane, weakly anisotropy antiferromagnet, taking account of the Dzyaloshinskiĭ interaction and an external magnetic field \mathbf{H} oriented perpendicular to the easy plane. We calculate the rms velocity of the vortices and their contribution to the dynamic structure factor. It turns out that an external magnetic field is a factor of fundamental importance. It significantly changes the nature of the vortices and also the shape and position of the central peak. A calculation shows that the vortex velocity and the associated width of the central peak are significantly higher in an antiferromagnet than in a ferromagnet, for given parameter values, and they furthermore depend strongly on H . If the field is sufficiently weak, on the other hand, the dynamics of the gas of vortices is governed by the viscous slowing of the vortices. [The situation here stands in contrast with that in a ferromagnet. In that case, the so-called gyroforce plays a leading role (more on this below), and it is not a matter of fundamental importance to take the slowing of the vortices into account]. The dynamical characteristics of the vortex gas depend strongly on the nature of the relaxation term in the equation for the antiferromagnetism vector.

2. MODEL

We consider an ordinary two-sublattice model of an antiferromagnet. We replace the magnetic moments of the sublattices, \mathbf{M}_1 and \mathbf{M}_2 , $|\mathbf{M}_1|=|\mathbf{M}_2|=M_0$, with some normalized vectors: the magnetization vector \mathbf{m} and the antiferromagnetism vector \mathbf{l} :

$$\mathbf{m} = (\mathbf{M}_1 + \mathbf{M}_2)/2M_0, \quad \mathbf{l} = (\mathbf{M}_1 - \mathbf{M}_2)/2M_0.$$

These vectors are related by

$$\mathbf{m}^2 + \mathbf{l}^2 = 1, \quad (\mathbf{m}, \mathbf{l}) = 0. \quad (1)$$

Under the assumption $|\mathbf{m}| \ll |\mathbf{l}| \approx 1$ (which is valid for weak magnetic fields $H \ll H_e$ and weak Dzyaloshinskii interactions: $H_D \ll H_e$, where H_e and H_D are the exchange field and the field of the Dzyaloshinskii interaction), the energy density of an antiferromagnet is¹⁹

$$W = M_0^2 \left\{ \frac{\delta}{2} \mathbf{m}^2 + \frac{\alpha}{2} (\nabla \mathbf{l})^2 + \frac{\beta}{2} l_z^2 + d(\mathbf{e}_z, [\mathbf{m}\mathbf{l}]) - 2\mathbf{h}\mathbf{m} \right\}. \quad (2)$$

Here \mathbf{e}_z is a unit vector along the hard axis of the crystal; M_0 is the saturation magnetization; $\delta = H_e/2M_0$ and α are the constants of uniform and nonuniform exchange; respectively, $\beta > 0$ is the anisotropy constant; $h = H/M_0$, $d = 2H_D/M_0$ is the Dzyaloshinskii interaction constant; and a is the lattice constant.

The term with d is characteristic of the purely easy-plane case. For an anisotropic basal plane, however, there can be other invariants of the Dzyaloshinskii interaction, of the type $D_{ij}m_i l_j$ with a tensor $D_{ij} \neq d\epsilon_{ijk}(\mathbf{e}_z)_k$. We will discuss their role below. In writing the energy density in the form in (2), we omitted terms $\alpha\beta' m_z^2, \delta' \mathbf{l}^2$ by virtue of the conditions $|\mathbf{m}| \ll |\mathbf{l}|$, $\mathbf{m}^2 + \mathbf{l}^2 = 1$.

To study the nonlinear dynamics of an antiferromagnet we go over to effective equations for the vector \mathbf{l} on the basis of a generalized σ model of the \mathbf{n} field.¹⁹⁻²² It is convenient to use angular variables for the vector \mathbf{l} : $l_z = \cos \theta$, $l_x + il_y = \sin \theta \exp(i\phi)$. If we ignore dissipative processes, we can find equations of motion from the Lagrangian

$$L = \frac{\alpha M_0^2}{2} \int d^2x \left\{ \frac{1}{c^2} \left(\frac{\partial \theta}{\partial t} \right)^2 - (\nabla \theta)^2 - \frac{1}{l_H^2} \cos^2 \theta + \sin^2 \theta \left[\frac{1}{c^2} \left(\left(\frac{\partial \phi}{\partial t} \right)^2 - 2gH \frac{\partial \phi}{\partial t} \right) - (\nabla \phi)^2 \right] \right\}. \quad (3)$$

The magnetization vector \mathbf{m} can be expressed in terms of \mathbf{l} and $\partial \mathbf{l} / \partial t$. It is given by the expression

$$\mathbf{m} = \frac{d}{\delta} [\mathbf{e}_z] + \frac{2}{\delta} \{ \mathbf{h} - \mathbf{l}(\mathbf{h}, \mathbf{l}) \} + \frac{2}{g\delta M_0} \left[\frac{\partial \mathbf{l}}{\partial t} \right]. \quad (4)$$

In these equations, $c = gM_0\sqrt{\alpha\delta}/2$ is a characteristic velocity, which is the same as the minimum phase velocity of spin waves; g is the gyromagnetic ratio; $l_H = l_0(1 + H^2/H_0^2)^{-1/2}$ is a characteristic magnetic length;

$l_0 = (\alpha/\tilde{\beta})^{1/2}$ and $\tilde{\beta} = \beta(1 + d^2/\delta\beta^2)^{1/2}$ is an effective anisotropy constant, renormalized by the Dzyaloshinskii interaction. The characteristic field $H_0 = M_0(\beta\delta)^{1/2}/2$ is expressed in terms of the anisotropy and exchange constants, like the field of the spin-flop phase transition in an easy-axis antiferromagnet.

At $H=0$ the dynamics of the magnetization of an antiferromagnet is Lorentz-invariant with a characteristic velocity c . A magnetic field cause changes in not only the static energy but also the dynamic properties of the antiferromagnet. For a field \mathbf{H} directed perpendicular to the easy plane, this circumstance determines the onset of a gyroscopic term proportional to $gH(\partial\phi/\partial t)\sin^2\theta$ in the Lagrangian. This term destroys the Lorentz invariance [in a ferromagnet the situation is fundamentally different: the leading dynamic term in the Lagrangian, $\alpha(1 - \cos\theta)(\partial\phi/\partial t)$, is of a gyroscopic nature, and terms with $(\partial\theta/\partial t)^2$ and $(\partial\phi/\partial t)^2$ are unimportant]. According to the model in (2), the Dzyaloshinskii interaction is manifested only in static characteristics, namely, in the formula for \mathbf{m} and in the renormalization of β . However, gyroscopic terms can arise in the Lagrangian of an antiferromagnet not only because of an external magnetic field but also as the result of certain types of Dzyaloshinskii interaction.²³⁻²⁵ The energy of the Dzyaloshinskii interaction is determined by a term which is linear in the components of \mathbf{m} and which has an odd power of the components of \mathbf{l} . It can be written in general as

$$w_D = D_{ij}(\mathbf{l})m_i l_j,$$

where the tensor D_{ij} is determined by the symmetry of the antiferromagnet. The contribution to the equilibrium value of \mathbf{m} for this form of w_D is given by

$$(\mathbf{m}_D)_i = (D_{kl}l_k l_l - D_{ij}l_j) / \delta.$$

It is easy to see that for $D_{ij} = d\epsilon_{ijk}(\mathbf{e}_z)_k$ this expression becomes (4). When the Dzyaloshinskii interaction is taken into account, a dynamic increment $D_{ik}\epsilon_{ijp}l_j l_k (\partial l_p / \partial t)$ arises in the Lagrangian. In terms of angular variables, this increment can be written

$$\Delta L_d = \frac{M_0}{g\delta} \int d^2x \Delta(\theta, \phi) \left(\frac{\partial \phi}{\partial t} \right) \quad (5)$$

(see Ref. 25 for more details). For $D_{ij} = d\epsilon_{ijk}(\mathbf{e}_z)_k$, this expression has the form of a total time derivative, and it does not affect the form of the equations of motion for the angular variables. For several antiferromagnets with a nonantisymmetric Dzyaloshinskii interaction $D_{ij} \neq d\epsilon_{ijk}(\mathbf{e}_z)_k$, however, this term is important. For example, this term causes a pronounced change in the nature of the motion of 1D solitons (kinks) in an antiferromagnet.^{24,25} However, the effect of this term on the dynamics of vortices turns out to be unimportant, as we will see below. In particular, this effect is not manifested in the effective equation of motion of the vortices, (23).

In the dissipationless limit, the system has as integrals of the motion the energy of the magnetic material E and the

momentum of the magnetization field \mathbf{P} . From the expression for the Lagrangian, (3), we find $\mathbf{P}=\mathbf{P}_{LI}+\mathbf{P}_g$, where

$$\mathbf{P}_{LI}=-\frac{\alpha a M_0^2}{c^2} \int d^2 x \left[\nabla \theta \frac{\partial \theta}{\partial t} + \nabla \phi \frac{\partial \phi}{\partial t} \sin^2 \theta \right],$$

$$\mathbf{P}_g=\frac{\alpha a M_0^2}{c^2} \int d^2 x g H \nabla \phi \sin^2 \theta. \quad (6)$$

Here \mathbf{P}_{LI} is the ordinary Lorentz-invariant term, and the gyroscopic term \mathbf{P}_g is due to the magnetic field. This expression does not have any singularities associated with the fact that ϕ is not differentiable as $r \rightarrow 0$ and $\theta \rightarrow 0$ or π (the circumstance that ϕ is not differentiable was discussed in Ref. 26).

In the case of a steady nonuniform state of the antiferromagnet such as $\theta=\pi/2$, $\phi=\mathbf{k}\mathbf{r}$, the term \mathbf{P}_g gives the antiferromagnet a nonzero momentum: $\mathbf{P}=\mathbf{k}gH\alpha a M_0^2 S/c^2$, where S is the area of the 2D antiferromagnet. This behavior is characteristic of superfluids which can be described by a complex order parameter $\Psi=|\psi|\exp(i\phi)$. The momentum density of the superfluid motion is given by the familiar expression $\mathbf{p}=|\psi|^2 \nabla \phi = \rho_s \mathbf{v}_s$, where ρ_s is the density of the superfluid component and \mathbf{v}_s is its velocity. By virtue of the similarity of these expressions, one can say that there is a fundamental analogy between superfluid systems and easy-plane magnetic materials (this analogy has been discussed for a ferromagnet^{27,28}). Using this analogy \mathbf{P}_g , we would naturally associate the quantity $\rho_s \mathbf{v}_s$ with the magnetization density, and $\rho_s \mathbf{v}_s^2/2$ with the energy density. Consequently, the quantity $\rho_s^{\text{AFM}} = 2\alpha a M_0^2 (gH/c^2)^2 = 8aH^2/c^2 \delta$ represents the density of the superfluid component, ρ_s , for the dynamics of an antiferromagnet. This analogy between an antiferromagnet in an external field and superfluid systems is manifested in an analysis of the dynamics of vortices (more on this below). For an antiferromagnet, in contrast with a ferromagnet, the quantity ρ_s^{AFM} is proportional to H^2 and vanishes if there is no magnetic field.

Incorporating dissipation disrupts the integrals of the motion E and \mathbf{P} . When dissipation is taken into account, the Lagrange equations $\delta L/\delta \theta=0$, $\delta L/\delta \phi=0$ are replaced by $\delta L/\delta \theta - \delta Q/\delta \dot{\theta}=0$ and $\delta L/\delta \phi - \delta Q/\delta \dot{\phi}=0$, where Q is the dissipation function of the magnetic material. The rate of energy dissipation, dE/dt , of a soliton is equal to twice the dissipation function Q calculated from the soliton solution. In the limit of a low vortex velocity we can use immobile solutions, replacing \mathbf{r} by $\mathbf{r}-\mathbf{v}t$ in them, where \mathbf{v} is the vortex velocity [see Eq. (10) below]. The viscous drag force acting on a vortex moving at a constant velocity \mathbf{v} is given in the low-velocity limit by

$$\mathbf{F}_{\text{dis}}=-\eta \mathbf{v}, \quad \eta=\frac{2Q}{v^2}, \quad (7)$$

where η is the coefficient of viscous drag. Its value is independent of v in the limit $v \rightarrow 0$.

A Hilbert dissipation function, whose density is proportional to $(\partial \mathbf{l}/\partial t)^2$, is ordinarily used to describe magnetic relaxation. In some recent papers by Bar'yakhtar,²⁹⁻³² however, it was shown through an analysis of the dynamical

symmetry of the magnetization field that the dissipation function of magnetic materials is more complicated. It should contain terms differing in nature, specifically an exchange term Q_e and a relativistic term Q_r :

$$Q=Q_e+Q_r=\alpha a M_0^2 \int d^2 x (q_r+q_e).$$

An exchange dissipation function is chosen in order to achieve the dynamical symmetry of the exchange interaction, specifically, to conserve the resultant magnetization of the magnetic material. For an antiferromagnet this approach has the consequence that the density of the exchange dissipation function contains two terms,³¹

$$\lambda_e \left(\nabla \left[\frac{\partial \mathbf{l}}{\partial t} \right] \right)^2 + \tilde{\lambda}_e \left(\frac{\partial^2 \mathbf{l}}{\partial t^2} \right)^2.$$

At low soliton velocities, only the first term is important, and q_e is given by

$$q_e=\frac{\lambda_e}{2\omega} \{ (\nabla \dot{\theta} - \dot{\phi} \nabla \phi \sin \theta \cos \theta)^2 + [\nabla \dot{\phi} \sin \theta + \cos \theta (\dot{\phi} \nabla \theta + \dot{\theta} \nabla \phi)]^2 + \sin^2 \theta (\dot{\phi} \nabla \theta - \dot{\theta} \nabla \phi)^2 \}, \quad (8)$$

where λ_e and $\tilde{\lambda}_e$ are exchange relaxation constants, and the dot denotes the time derivative, here and below. The fairly high symmetry of the exchange interaction has the consequence that gradients of the variables θ , ϕ , $\dot{\theta}$, and $\dot{\phi}$ appear in the dissipation function, and there is no dissipation in the case of a uniform rotation of \mathbf{l} (Refs. 29-31).

Relaxation processes of a relativistic nature (due to spin-orbit and dipole-dipole interactions) are incorporated in the dissipation function Q_r . In principle, the symmetry of these interactions is lower than that of the exchange interactions, and it allows us to write Q_r as a quadratic form in $\dot{\theta}$ and $\dot{\phi}$ which does not contain gradients. However, in the case of a purely uniaxial magnetic material (this approximation corresponds to the assumption that there is no anisotropy in the basal plane; it is the one ordinarily used in an analysis of vortices), this function must conserve the z projection of the magnetization. Bar'yakhtar showed^{30,32} that this function should accordingly contain no terms with $\dot{\phi}^2$. For the linear approximation, this assertion was made even earlier, by Halperin and Hoenberg,³³ who constructed dynamic equations for the spin density of planar magnetic materials from symmetry considerations on the basis of the Goldstone theorem and the Adler principle. The form of Q_e and Q_r agrees with calculations of the magnon damping coefficients in an easy-plane antiferromagnet. (A Hilbert dissipation function leads to a different dependence of the magnon damping coefficient on the magnon quasimomentum.²⁹⁻³²)

We also note that relaxation terms containing mixed derivatives of the magnetization vector with respect to the coordinates and the time have been used by Pokrovskii and Khveshenko⁵ to describe the dynamics of an ensemble of vortices in a ferromagnet. The approximation of a purely uniaxial magnetic material may be an unjustified idealization

in a description of dissipation processes, since it ignores (for example) the anisotropy in the basal plane, which always exists in reality, or magnetoelastic relaxation processes. In addition, it is interesting to make a comparison with the results of papers which incorporate the Hilbert terms. We accordingly choose a relativistic dissipation function in the form

$$q_r = \frac{\lambda}{2\omega_0 l_0^2} \dot{\theta}^2 + \frac{\lambda_z}{2\omega_0 l_0^2} \sin^2 \theta \dot{\phi}^2. \quad (9)$$

When we set $\lambda_z = \lambda$, this expression becomes the ordinary Landau–Lifshitz or Hilbert dissipation function. According to Refs. 30 and 32, the case $\lambda_z = 0$ corresponds to the model of a purely uniaxial antiferromagnet. As we will see, the results of the analysis depend strongly on the form of the dissipation functions. In particular, the case $\lambda_z = 0$ differs in a fundamental way from a case with any $\lambda_z \neq 0$. The quantity η_r has a finite value only in the case $\lambda_z = 0$; for $\lambda_z \neq 0$, a logarithmically divergent term arises in η_r . This term was ultimately dropped in a paper by Huber.⁴ In the ferromagnet studied in Refs. 4–7, the viscosity coefficient appears in an expression for the vortex velocity in the combination $G^2 + \eta^2$, where G is the gyrotropy constant (more on this below). Since η contains the small dissipation constant λ_r , while G does not contain small parameters, this is not a meaningless assumption for a ferromagnet, even if there is a weak (logarithmic) divergence in η . For an antiferromagnet we have $G \propto H$ (more on this below), and the ratio λ_r/G may not be small at small values of H . Whether there is a divergence (even a logarithmic one) in either the effective mass of the vortex or the viscous drag coefficients helps determine the nature of the time-varying motion of the vortex (see the following section of this paper).

3. STRUCTURE AND DYNAMICS OF VORTICES

The structure of a vortex is determined by the equations for θ and ϕ which follow from (3). For an immobile vortex the solution is

$$\phi^{(0)} = \phi_0 + \nu\chi, \quad \theta^{(0)} = \theta^{(0)}(\xi), \quad \xi \equiv \frac{r}{l_H}, \quad (10)$$

where $\phi_0 = \text{const}$, χ and r are polar coordinates in the plane of the magnetic material, xy , and the quantity $\nu = \pm 1, \pm 2, \dots$ determines the topological charge of the vortex [more precisely, one of its topological charges—the so-called circulation of the vortex—since vortices in Heisenberg magnetic materials are described by elements of the group of relative homotopies $\pi_2(S^2, S^1) = Z \times Z$; Ref. 28]. The function $\theta^{(0)}(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2 \theta^{(0)}}{d\xi^2} + \frac{1}{\xi} \frac{d\theta^{(0)}}{d\xi} = \sin \theta^{(0)} \cos \theta^{(0)} \left(1 - \frac{\nu^2}{\xi^2} \right) \quad (11)$$

with the natural boundary conditions $\theta^{(0)}(0) = \pi(1-p)/2$ and $\theta^{(0)}(\infty) = \pi/2$. This equation can be solved numerically by the shooting method.¹⁹ Here the quantity $p = \pm 1$, the so-called polarization of the vortex, determines the second topological

charge of a vortex. The energy of an immobile soliton, E_0 , diverges as the logarithm of the area of the antiferromagnet, S , per vortex. (We recall that for vortices there is a logarithmic divergence of the energy of a soliton as a function of the area.) For a vortex with $|\nu| = 1$, the calculation carried out in Ref. 19 yields $E_0 = (1/2)\pi\alpha M_0^2 \ln(5.67S/l_H^2)$. The properties of immobile vortices were described in detail in Refs. 19 and 28.

Constructing a solution to describe a moving vortex is a considerably more complicated problem. For a ferromagnet, no exact solution of this problem is known even in the case of a low vortex velocity. It has simply been pointed out⁴ that the motion of a vortex in a ferromagnet is necessarily accompanied by the onset of a nonzero gradient of ϕ (or by a deformation of the spin flux in the terminology of Ref. 27) far from the vortex: $\nabla\phi \rightarrow \mathbf{k} = \text{const} \cdot \mathbf{v}$ as $|\mathbf{r}| \rightarrow \infty$. The unique relationship between $\nabla\phi$ and \mathbf{v} leads to motion of the vortex when ϕ is nonuniform for any reason. In particular, this situation determines the nature of the motion of an ensemble of vortices: the resultant value of $\nabla\phi$ created by a system of vortices near some selected vortex determines the velocity of this vortex. It is this mechanism which determines the rms velocity of an ensemble of vortices⁴ and thus the width of the central peak in the correlation functions in an easy-plane ferromagnet.⁶

There is no problem in constructing a moving solution for a vortex for a Lorentz invariant model of an antiferromagnet. It is sufficient to carry out a Lorentz transformation of the immobile solution, (10). A Lorentz contraction of the vortex core arises in the process, as does a distortion of the field $\phi(\mathbf{r})$ far from the vortex:

$$\phi = \nu \arctan \left[\frac{y \sqrt{1 - v^2/c^2}}{x} \right]$$

(we are assuming that the vortex is moving along the x axis). The energy of a moving vortex in the case $H = 0$ is given by $E = E_0/\sqrt{1 - v^2/c^2}$ or $E = \sqrt{E_0^2 + c^2 \mathbf{P}^2}$. In other words, the effective mass of the vortex, $m_* = E_0/c^2$, has the same logarithmic divergence as the energy of an immobile vortex, E_0 . For the case of interest, with $H \neq 0$, the question is considerably more complicated. In order to construct a vortex thermodynamics of an antiferromagnet in the spirit of Refs. 4–6, we need to write an effective equation of motion for the vortex coordinates \mathbf{X}_α , where α specifies the vortices making up the ensemble or gas of interacting vortices. Equations of this type for vortices in a ferromagnet have been found by various methods: through the use of equations of motion for θ and ϕ (Ref. 5), from a force balance condition,⁴ and through the use of a soliton perturbation theory²⁸ in the spirit of Ref. 34. These methods can also be applied to antiferromagnets. Using the same assumptions as in Refs. 4, 28 and 5, one finds that they lead to a system of first-order equations for \mathbf{X}_α , which describe the viscous motion of vortex α under the influence of the other vortices. To construct this equation we follow Ref. 5.

Using the equations of the preceding section, we write the dynamical equations for the angular variables $\theta(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ of the antiferromagnet:

$$\square \theta - \sin \theta \cos \theta \left[(\nabla \phi)^2 - \frac{1}{c^2} \left(\left(\frac{\partial \phi}{\partial t} \right)^2 - 2gH \frac{\partial \phi}{\partial t} \right) - \frac{1}{l_H^2} \right] = \frac{\lambda}{c l_0} \left(\frac{\partial \theta}{\partial t} \right) + \frac{\lambda_e l_0}{c} \left[\nabla \phi \nabla \left(\frac{\partial \phi}{\partial t} \right) \sin 2\theta + \frac{\partial \theta}{\partial t} (\nabla \phi)^2 - \Delta \frac{\partial \theta}{\partial t} \right], \quad (12)$$

$$\begin{aligned} & \nabla (\nabla \phi \sin^2 \theta) - \frac{1}{c^2} \frac{\partial}{\partial t} \left[\left(\frac{\partial \phi}{\partial t} - gH \right) \sin^2 \theta \right] \\ &= \frac{\lambda_z}{c l_0} \left(\frac{\partial \phi}{\partial t} \right) \sin^2 \theta + \frac{\lambda_e l_0}{c} \left[\frac{\partial \phi}{\partial t} (\nabla \phi)^2 \cos 2\phi \sin^2 \theta \right. \\ & \quad \left. - \nabla \left(\nabla \frac{\partial \phi}{\partial t} \sin \theta \right) + \frac{\partial \phi}{\partial t} (\nabla \phi)^2 \right. \\ & \quad \left. - r \nabla \phi \nabla \frac{\partial \phi}{\partial t} \sin 2\theta - \frac{1}{2} \frac{\partial \phi}{\partial t} \Delta \sin^2 \theta \right]. \quad (13) \end{aligned}$$

To study dynamical multivortex solutions of system (12) and (13), we work from the approximation of a low-density vortex gas. We seek N -soliton solutions approximately in the form⁵

$$\phi(\mathbf{r}, t) = \sum_{\alpha=1}^N \phi^{(0)}(\mathbf{r} - \mathbf{X}_\alpha(t)), \quad (14)$$

$$\cos \theta(\mathbf{r}, t) = \sum_{\alpha=1}^N \cos \theta^{(0)}(\mathbf{r} - \mathbf{X}_\alpha(t)), \quad (15)$$

where $\phi^{(0)}$ and $\theta^{(0)}$ are static one-vortex solutions of (10), and $\mathbf{X}_\alpha(t)$ determines the coordinates of the centers of the vortices. We multiply Eq. (12) by $\partial_\alpha \theta$, multiply Eq. (13) by $\partial_\alpha \phi$, add the results term by term, and integrate over the area occupied by the vortices. As a result we find the effective equations of motion which we need for the ensemble of vortices. In the approximation used in Refs. 4 and 5, this equation can be written

$$\begin{aligned} \frac{E_0}{c^2} \frac{\partial^2 \mathbf{X}_\alpha}{\partial t^2} &= G \left[\frac{\partial \mathbf{X}_\alpha}{\partial t} \mathbf{e}_z \right] + 2\pi\alpha a M_0^2 [\mathbf{e}_z \nabla \phi(\mathbf{X}_\alpha)] \\ & \quad - \eta \frac{\partial \mathbf{X}_\alpha}{\partial t}. \quad (16) \end{aligned}$$

Formally, the term on the left side of Eq. (16) is an inertial term with a Lorentz-invariant mass E_0/c^2 . Terms of this sort arise in (16) because of terms of the type $\partial^2 \theta / \partial t^2$ and $\partial^2 \phi / \partial t^2$ in Eqs. (12) and (13). There are no such terms for a ferromagnet.^{4,5} We will discuss the role played by inertial terms below; at this point we move on to an analysis of the terms on the right side of (16), which will be used to describe the dynamics of the ensemble of vortices.

The first term on the right side of Eq. (16) is a gyroscopic force acting on a given vortex. The gyroscopic constant for a vortex in an antiferromagnet is

$$G_{\text{AFM}} = -2\pi\nu\alpha a M_0^2 gH/c^2 = -16\pi\nu a M_0 H/gH_e. \quad (17)$$

This expression is quite different from G_{FM} . In the first place, the gyrotropy constant is proportional to only one of the two topological charges of a vortex (not to the product $\nu\rho$, as in a ferromagnet). Second, the gyroscopic force in an antiferromagnet exists only in a magnetic field. The value of G_{AFM} is, for all reasonable field values, smaller than that in a ferromagnet: $G_{\text{AFM}} = G_{\text{FM}}(4H/H_e) \ll G_{\text{FM}}$. (We recall that all the calculations in the present paper are being carried out under the approximation $H \ll H_e$; furthermore, the antiferromagnetic order is disrupted at a value of the field H above the exchange field H_e .) Since the constant G_{AFM} is small and vanishes as $H \rightarrow 0$, there are features of the dynamics of an ensemble of vortices in an antiferromagnet which differ from those in a ferromagnet (more on this below³⁷).

The second term on the right side of (16) contains the value of $\nabla \phi$ at the position of vortex α . This value is determined by the positions of the other vortices: $\nabla \phi(\mathbf{X}_\alpha) = \sum_{\beta \neq \alpha} \nabla \phi^{(0)}(\mathbf{X}_\alpha - \mathbf{X}_\beta)$. Taking the explicit form of the function $\phi^{(0)}$ into account [see (10)], we easily see that this term has the meaning of a potential force \mathbf{F}_α^e which is exerted on vortex α by all the other vortices. This term can be written

$$\begin{aligned} \mathbf{F}_\alpha^e &= -(\partial/\partial \mathbf{X}_\alpha) \mathcal{H}_{\text{int}}, \quad \mathcal{H}_{\text{int}} = - \sum_{\alpha \neq \beta} e_\alpha e_\beta \ln |\mathbf{X}_\alpha - \mathbf{X}_\beta|, \\ e_\alpha &= \nu_\alpha M_0 \sqrt{\pi \alpha a}, \quad (18) \end{aligned}$$

where the vortex interaction Hamiltonian \mathcal{H}_{int} has the form of the interaction Hamiltonian of a 2D Coulomb gas. Finally, we write the force \mathbf{F}_α^e , acting on vortex α , as follows:

$$\mathbf{F}_\alpha^e = \sum_{\beta \neq \alpha} e_\alpha e_\beta \frac{\mathbf{X}_\alpha - \mathbf{X}_\beta}{|\mathbf{X}_\alpha - \mathbf{X}_\beta|^2} = e_\alpha \mathbf{E}_\alpha, \quad (19)$$

where the quantity \mathbf{E}_α is analogous to an electric field which is acting on a given charge (vortex) and which is produced by the other vortices.

The last term in Eq. (16) is the viscous drag force in (7). The viscosity coefficient

$$\eta = \frac{2\pi\alpha a M_0^2}{\omega_0 l_0^2} (\lambda \gamma_1 + \lambda_e \gamma_2) \quad (20)$$

consists of two terms, η_r and η_e , which are of relativistic and exchange origin, respectively, and we have

$$\begin{aligned} \gamma_1 &= \frac{1}{2} \int \xi d\xi \theta^{(0) \prime 2}, \\ \gamma_2 &= \frac{1}{2} \int \xi d\xi \left[\frac{4\nu^2}{\xi^2} \theta^{(0) \prime 2} \sin^2 \theta^{(0)} - \frac{\nu^2}{4\xi^2} \sin^2 2\theta^{(0)} \right. \\ & \quad \left. - \theta^{(0) \prime 2} \cos 2\theta^{(0)} \right]. \quad (21) \end{aligned}$$

Values have been found for the coefficients γ_1 and γ_2 by evaluating the integrals, using the specific expression for $\theta(r)$ found through a numerical integration of Eq. (11). The results are $\gamma_1 \approx 3.003$ and $\gamma_2 = 0.751$.

The choice of the exchange dissipation function in the particular form in (8) is a point of fundamental importance.

Terms of the type $\dot{\phi}^2(\nabla\phi)^2$, $(\nabla\dot{\phi})^2$, etc., may lead, in a calculation of the vortex viscosity coefficient, to integrals which diverge as $r \rightarrow 0$. Only when they are present in the combination in q_e in (8) do we obtain a finite expression for η_e .

The terms on the right side of Eq. (16) thus have a clear physical meaning: They are forces acting on a vortex: the gyroscopic force \mathbf{F}_α^g , the external force \mathbf{F}_α^e , and the viscous drag force $-\eta\dot{\mathbf{X}}_\alpha$.

The condition

$$\mathbf{F}_\alpha^g + \mathbf{F}_\alpha^e - \eta\dot{\mathbf{X}}_\alpha = 0 \quad (22)$$

determines the relationship between \mathbf{v} and \mathbf{F}^e . It describes the viscous (overdamped) motion of a vortex. This is the condition used in all papers of which we are aware that describe the dynamics of a vortex gas and calculate the rms velocity of vortices.

For an antiferromagnet, the equations of motion for θ and ϕ contain second derivatives with respect to the time. Consequently, an inertial term arises in Eq. (16). Actually, however, the inertial properties of vortices in a ferromagnet or antiferromagnet at $H \neq 0$ cannot be described with this term. As Ivanov and Stephanovich have shown,³⁵ the ansatz in (14) and (15) is insufficient for describing the inertial properties of 2D solitons in magnetic materials whose Lagrangian has gyroscopic terms. The ansatz must contain increments in θ and ϕ proportional to the soliton velocity. These increments, ϑ and φ , are determined by the solution of a system of linear equations. The condition under which this system has a solution consists of equations for $\mathbf{X}_\alpha(t)$. These equations determine the mass of a soliton in a ferromagnet or the increment in the Lorentz-invariant value of the mass, E_0/c^2 , in an antiferromagnet. An equivalent approach was developed by Wysin *et al.* (see their paper in the collection of Ref. 36). This approach was taken in Ref. 35 to calculate the effective mass of a localized 2D soliton in a ferromagnet. An attempt to use this method to calculate the mass of a vortex was made in Ref. 37. It turned out that a slow (power-law) decrease in the perturbations of the magnetization field in a vortex leads to a stronger than logarithmic divergence in the formula for the mass. This result agrees qualitatively with the results found by Mertens *et al.*³⁶ They carried out a numerical simulation of the rotational dynamics of a pair of vortices in circular geometry. Working from an analysis of data on the frequency and radius of the motion, they found a value of the effective mass for an extraplanar vortex in a Heisenberg ferromagnet at $T=0$. This mass was a linear function of the dimensions of the system: $m_* \propto L$. A numerical simulation was carried out only for the case in which the size of the pair was comparable to the size of the system, L . In our opinion, those results cannot be used in effective equations such as (16). We are thus left with the conclusion that there has not yet been a systematic description of inertial effects in the dynamics of magnetic vortices. However, the paper by Mertens *et al.*³⁶ contains a result which is of fundamental importance to the problem of incorporating the inertia of vortices in 2D magnetic materials in a description of the response functions of a vortex gas at $T > T_c$. According to that paper, inertial effects are manifested by the onset of a fairly fast motion (of the nature of motion along a cycloid)

against the background of a slow motion described by force balance condition (22). In a numerical simulation for non-zero temperatures $T \approx T_c$, this "fast" motion is never seen. This fact is an argument in favor of the proposition that the force balance equation which is ordinarily used,

$$G \left[\frac{\partial \mathbf{X}_\alpha}{\partial t} \mathbf{e}_z \right] - \eta \frac{\partial \mathbf{X}_\alpha}{\partial t} = e_\alpha \mathbf{E}_\alpha, \quad (23)$$

$$\mathbf{E}_\alpha = \sum_{\beta \neq \alpha}^i e_\beta \frac{\mathbf{X}_\alpha - \mathbf{X}_\beta}{|\mathbf{X}_\alpha - \mathbf{X}_\beta|^2}, \quad (24)$$

successfully describes the dynamics of a vortex gas at temperatures $T > T_c$, at which this gas exists in a 2D magnetic material.

4. MEAN VELOCITY OF A VORTEX GAS

An effective equation of motion for vortices like that in (23) was used by Huber⁴ for a thermodynamic calculation of the velocity of a vortex gas in a ferromagnet. Mertens *et al.*⁶ proposed a method for calculating the contribution of a vortex gas to the response function of a magnetic material at $T > T_c$ (so-called vortex phenomenology). A calculation carried out for a ferromagnet in that paper showed that vortices contribute substantially in the region of the central peak. Vortex phenomenology operates with two parameters: the vortex density n_v and the rms velocity \bar{u} , which also determines the width of the central peak. A numerical simulation revealed that the observed value of \bar{u} in a ferromagnet agrees fairly well with the Huber formula,¹¹ except for a slight decrease in \bar{u} with increasing temperature in the immediate vicinity of T_c (more on this below).

For an antiferromagnet, a singularity associated with the possible vanishing of G as $H \rightarrow 0$ arises in the thermodynamic characteristics of a vortex gas. A numerical simulation^{11,12} has demonstrated a significant discrepancy between the observed values of \bar{u} for an antiferromagnet and those for a ferromagnet.

To calculate \bar{u} for a vortex gas in an antiferromagnet we use the method developed in Ref. 4. In the continuum approximation we introduce a self-consistent effective "electric field" \mathbf{E} , which describes the interaction with other vortices. There is an analog between dynamical equations (23) for vortices and the equations for particles in a plasma in a transverse magnetic field.⁴ We can accordingly use results found in Ref. 38 to estimate $\langle E^2 \rangle$: $\langle E^2 \rangle = n_v \pi e^2 \ln \Lambda$, where n_v is the equilibrium vortex density, and Λ is a measure of the proximity of the vortex gas to a state of thermodynamic equilibrium. In the limits of random and equilibrium distributions, Λ is given by $\Lambda_R = S/\pi a^2$ and $\Lambda_T = 4\pi^2 T/n_v e^2 a^2$, respectively, where S is the area of the system, and T the temperature.

Using the expression for $\langle E^2 \rangle$ along with (23), we find an expression for the rms velocity of the vortices:

$$\bar{u} = \frac{e \sqrt{\langle E^2 \rangle}}{(G^2 + \eta^2)^{1/2}} = \frac{c}{2} \sqrt{\frac{\pi n_v l_0^2 H_0^2}{H^2 + H_T^2}} \ln \Lambda. \quad (25)$$

For convenience here we have introduced a characteristic field H_0 (as discussed above) and $H_r = \eta(g\delta/8\pi a)$. The field H_r is proportional to the relaxation constant η . We use Eqs. (21) for η and estimates of the relaxation constants according to Ref. 5 (some results which agree with those data in order of magnitude were found for an antiferromagnet in Ref. 39). We find $H_r \approx 0.05H_0$ at $T \approx T_c$. For $H \gg H_r$, the quantity $\bar{u}/\bar{u}_H \approx H_e/4H$ is thus seen to be inversely proportional to the magnetic field H ; at $H \ll H_r$, \bar{u}/\bar{u}_H is independent of the field, having a value $\bar{u}/\bar{u}_H \approx H_e/4H_r$, where

$$\bar{u}_H = \frac{gM_0}{2} \sqrt{\alpha\beta} \sqrt{\pi n_v l_0^2 \ln \Lambda}$$

is the vortex velocity found for a ferromagnetic by Huber.⁴ We see that for any reasonable values of the field $H \ll H_e$ the velocity in an antiferromagnet is higher than that in a ferromagnet. These results agree with the numerical simulation of Ref. 12. The maximum values of \bar{u} (in dimensionless units) are on the order of 0.5 for a ferromagnet¹⁰ and 2.3 for an antiferromagnet.¹² We would like to test the dependence $\bar{u} \propto 1/H$, but no numerical simulations have been carried out for the case $H \neq 0$, to the best of our knowledge.

For weak magnetic fields $H < H_r$, in particular, for $H=0$, the velocity \bar{u} is independent of H but inversely proportional to the relaxation constant. Since we have $\eta \propto T^n$, where $n=2$ for a ferromagnet⁵ and $n=3$ for an antiferromagnet,³⁹ this formula describes a decrease in \bar{u} with increasing T . Data found from the numerical simulation demonstrate a decrease in \bar{u} in a small neighborhood of T_c as T increases, for both ferromagnets¹⁰ and antiferromagnets.¹² For an antiferromagnet, however, the decrease is much more pronounced: $\bar{u}=2.27$ and 0.97 for $T=0.83$ and 0.90 , respectively.¹² This result can be explained on the basis of the theoretical result $\bar{u} \propto 1/\eta \propto T^{-n}$ found above. For a ferromagnet, the decrease is far weaker ($\bar{u}=0.49$ and 0.38 at $T=0.82$ and 0.87 , respectively). This result can be explained by assuming, for example, that at lower temperatures a vortex gas having a low density is in a state which is farther from thermodynamic equilibrium than that of a dense vortex gas at a higher temperature (in this case the logarithmic factor $\ln \Lambda$ in \bar{u} becomes important). Testing this hypothesis, however, will require a more detailed analysis of the simulation conditions.

5. VORTEX CONTRIBUTION TO THE DYNAMIC STRUCTURE FACTOR

Let us calculate this contribution. By virtue of relation (1), \mathbf{m} and \mathbf{l} make independent and additive contributions to spin correlation functions. Furthermore, they determine components of the dynamic structure factor which are centered at different points in \mathbf{q} space: \mathbf{l} determines a central peak near the Bragg peak of an antiferromagnet at the point $\mathbf{K}^0 = (\pi/a, \pi/a)$ (Ref. 10), while \mathbf{m} contributes near the point $\mathbf{q}=0$.

Extrplanar components of the dynamic structure factor are locally sensitive to the presence of vortices, since the component $l_z^{(0)}(\mathbf{r})$ is localized near the center of a vortex. The central peak for extrplanar correlations has a Gaussian

shape. The height of this peak increases with the number of vortices, and it depends strongly on the vortex form factor.

In-plane components of the dynamic structure factor are globally sensitive to the presence of vortices, which destroy the long-range order in the plane of a 2D magnetic material. A Lorentzian central peak for in-plane correlations does not depend on the vortex form factor, i.e., on the nature of the deviation of the antiferromagnetism vector from its equilibrium value at the center of the vortex. The height of the central peak decreases with increasing number of vortices, since the magnetic material becomes uniform, on the average, with increasing n_v .

A concrete calculation of the vortex contribution to the dynamic structure factor is carried out by analogy with the calculation for a ferromagnet (Ref. 6; see also Refs. 10–12), and we will not go into it in detail here. For both extrplanar and in-plane components, the widths of the central peak, $\Delta\Gamma_z = q\bar{u}$ and $\Delta\Gamma_x \approx 1.14\bar{u}n_v^{1/2}$, are proportional to the rms velocity of the vortices, and they increase substantially with decreasing field [see (25)].

It is important to note that a magnetic field and the Dzyaloshinskii interaction independently affect different components of the dynamic structure factor. The form of the in-plane components does not depend on the field; it is given by expressions analogous to those found in Refs. 10 and 12 (the Dzyaloshinskii interaction was taken into account in the latter paper):

$$S^{xx}(\mathbf{q}, \omega) = F_L(\mathbf{K}^0 - \mathbf{q}, \omega) + \frac{d^2}{\delta^2} F_L(\mathbf{q}, \omega),$$

where

$$F_L(\mathbf{q}, \omega) = \frac{\kappa^3}{8\pi n_v (\omega^2 + \kappa^2 [1 + (q/2\sqrt{n_v})^2]^2)}$$

determines the known expression for a Lorentzian central peak; $\kappa = \bar{u}(\pi n_v)^{1/2}$, and $q = |\mathbf{q}|$.

For the extrplanar components, the Dzyaloshinskii interaction has no effect, but the shape of the dynamic structure factor depends strongly on the magnetic field:

$$S^{zz}(\mathbf{q}, \omega) = |f_1(\mathbf{q})|^2 F_G(\mathbf{K}^0 - \mathbf{q}, \omega) + \frac{4h^2}{\delta^2} n_v \delta(\mathbf{q}) \delta(\omega) + \frac{4h^2}{\delta^2} |f_2(\mathbf{q})|^2 F_G(\mathbf{q}, \omega).$$

Here $F_G(\mathbf{q}, \omega)$ determines the expression for a Gaussian central peak, which is characteristic of extrplanar components of dynamic structure factors of various magnetic materials:^{6,7}

$$F_G(\mathbf{q}, \omega) = \frac{n_v}{2\pi^{3/2} q\bar{u}} \exp[-(\omega/q\bar{u})^2],$$

where $f_k(\mathbf{q}) = \int d^2x \cos^k \theta(\mathbf{r}) \exp^{i\mathbf{q}\mathbf{r}}$, and $k=1,2$, are two different vortex form factors, which determine the distributions of \mathbf{l} and \mathbf{m} , respectively. We would like to point out that the intensity of the third term is proportional to H^2 . An analysis of this functional dependence might be a good test for a comparison of experimental data with theory.

6. CONCLUSION

The analysis of this paper demonstrates that there is substantial difference between the dynamics of vortices in an antiferromagnet and in a ferromagnet. It also demonstrates that an external magnetic field perpendicular to the easy plane has a strong influence. Even in weak fields $H \leq H_0 \ll H_e$, which do not destroy the antiferromagnetic order, this circumstance is manifested in the following circumstances.

The dynamics of vortices in an antiferromagnet in the absence of a transverse magnetic field is described by Lorentz-invariant equations, and the rms velocity of the vortices is determined by the viscosity: $\bar{u} \propto 1/\eta$. As a result, \bar{u} is a strong function of the temperature.

A field causes a special contribution to extraplanar components of the dynamic structure factor, whose intensities depend strongly on the magnitude of H . In a field, a gyromotion of vortices arises. As the field is increased, there is a transition from viscous motion to gyroscopic motion, which is manifested in the behavior of \bar{u} and which causes characteristics of the peaks to depend on the field.

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- ¹L. J. De Jongh, *Magnetic Properties of Layered Transition Metal Compounds*, Kluwer Academic, London (1990).
- ²M. Pomerantz, *Surf. Sci.* **142**, 556 (1984).
- ³J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**, 1181 (1973).
- ⁴D. L. Huber, *Phys. Rev. B* **26**, 3758 (1982).
- ⁵V. L. Pokrovskii and D. V. Khveshchenko, *Fiz. Nizk. Temp.* **14**, 385 (1988) [*Sov. J. Low Temp. Phys.* **14**, 213 (1988)].
- ⁶F. G. Mertens, A. R. Bishop, G. M. Wysin, and C. Kawabata, *Phys. Rev. B* **39**, 591 (1989).
- ⁷M. Gouvêa, G. M. Wysin, A. R. Bishop, and F. G. Mertens, *Phys. Rev. B* **39**, 11840 (1989).
- ⁸M. Gouvêa, G. M. Wysin, A. R. Bishop, and F. G. Mertens, *J. Phys. C* **1**, 4387 (1989).
- ⁹A. R. Völkel, F. G. Mertens, A. R. Bishop, and G. M. Wysin, *Phys. Rev. B* **43**, 5992 (1991).
- ¹⁰A. R. Völkel, G. M. Wysin, A. R. Bishop, and F. G. Mertens, *Phys. Rev. B* **44**, 10066 (1991).
- ¹¹A. R. Völkel, A. R. Bishop, F. G. Mertens, and G. M. Wysin, *J. Phys. C* **4**, 9411 (1992).

- ¹²A. R. Völkel, F. G. Mertens, A. R. Bishop, and G. M. Wysin, *Ann. Phys. (Leipzig)* **2**, 308 (1993).
- ¹³J. A. Krumhansl and J. R. Schriber, *Phys. Rev. B* **11**, 3535 (1975).
- ¹⁴H. J. Mikeska, *J. Phys. C* **11**, L29 (1978).
- ¹⁵J. F. Currie, J. A. Krumhansl, A. R. Bishop, and S. E. Trullinger, *Phys. Rev. B* **22**, 477 (1980).
- ¹⁶Yu. A. Izyumov, *Usp. Fiz. Nauk* **155**, 553 (1988) [*Sov. Phys. Usp.* **31**, 689 (1988)].
- ¹⁷H. J. Mikeska and M. Steiner, *Adv. Phys.* **40**, 191 (1991).
- ¹⁸V. L. Berezinskiĭ, *Zh. Éksp. Teor. Fiz.* **61**, 1144 (1971) [*Sov. Phys. JETP* **34**, 610 (1972)].
- ¹⁹A. M. Kosevich, V. A. Ivanov, and A. S. Kovalev, *Phys. Rep.* **194**, 117 (1990).
- ²⁰U. V. Bar'yakhtar and B. A. Ivanov, *Fiz. Nizk. Temp.* **5**, 759 (1979) [*Sov. J. Low Temp. Phys.* **5**, 361 (1979)].
- ²¹A. F. Andreev and V. I. Marchenko, *Usp. Fiz. Nauk* **130**, 39 (1980) [*Sov. Phys. Usp.* **23**, 21 (1980)].
- ²²H. J. Mikeska, *J. Phys. C* **13**, 2913 (1980).
- ²³A. K. Zvezdin and A. A. Mukin, *Kratk. Soobshch. Fiz.* **126**, 10 (1981).
- ²⁴B. A. Ivanov and G. K. Oksyuk, Preprint 84-8R, Institute of Theoretical Physics, Ukrainian Academy of Sciences, Kiev (1984).
- ²⁵E. V. Gomonaĭ, B. A. Ivanov, Zh. A. L'vov, and G. K. Oksyuk, *Zh. Éksp. Teor. Fiz.* **97**, 307 (1990) [*Sov. Phys. JETP* **70**, 174 (1990)].
- ²⁶N. Papanicolaou and T. N. Tomaras, *Nucl. Phys. B* **300**, 425 (1991).
- ²⁷É. B. Sonin, *Usp. Fiz. Nauk* **137**, 264 (1982).
- ²⁸A. V. Nikiforov, É. B. Sonin, *Zh. Éksp. Teor. Fiz.* **85**, 642 (1983) [*Sov. Phys. JETP* **58**, 373 (1983)].
- ²⁹V. G. Bar'yakhtar, *Zh. Éksp. Teor. Fiz.* **87**, 1501 (1984) [*Sov. Phys. JETP* **60**, 863 (1984)].
- ³⁰V. G. Bar'yakhtar, *Physica B* **159**, 20 (1989).
- ³¹V. G. Bar'yakhtar, *Fiz. Nizk. Temp.* **11**, 1198 (1985) [*Sov. J. Low Temp. Phys.* **11**, 662 (1985)].
- ³²V. G. Bar'yakhtar, *Fiz. Tverd. Tela (Leningrad)* **29**, 1317 (1987) [*Sov. Phys. Solid State* **29**, 754 (1987)].
- ³³B. I. Halperin and P. C. Hohenberg, *Phys. Rev.* **188**, 898 (1969).
- ³⁴D. W. McLaughlin and A. C. Scott, in *Solitons in Action*, K. Lonngren and A. Scott (eds.), Academic, New York (1978).
- ³⁵B. A. Ivanov and V. A. Stephanovich, *Phys. Lett. A* **141**, 89 (1989).
- ³⁶G. M. Wysin, F. G. Mertens, A. R. Völkel, and A. R. Bishop, in *Nonlinear Coherent Structures in Physics and Biology*, F. G. Mertens and K. H. Spatschek (eds.), Plenum, New York (1993), p. 177; F. G. Mertens, G. M. Wysin, A. R. Völkel *et al.*, *ibid.*, p. 191.
- ³⁷B. A. Ivanov, A. K. Kolezhuk, and D. D. Sheka, "Two-dimensional magnetic soliton gas dynamics and its contribution to the response functions of layered magnets," Preprint ITP-93-61E, Bogolyubov Institute for Theoretical Physics, Kiev (1994); B. A. Ivanov, D. D. Sheka, *Phys. Rev. Lett.* **72**, 404 (1994).
- ³⁸J. B. Taylor and B. McNamara, *Phys. Fluids* **14**, 1492 (1971).
- ³⁹B. A. Ivanov and A. I. Sukstanskiĭ, *J. Magn. Magn. Mater.* **117**, 102 (1992).

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