# Velocity selection and instability spectrum in 3*D* dendritic growth

E. Brener

*Institute of Solid State Physics, Russian Academy of Sciences, 142432, Chernogolovka, Moscow region, Russia*

V. I. Melnikov

*Institute for Theoretical Physics, Russian Academy of Sciences, 142432 Chernogolovka, Moscow Region, Russia*

(Submitted 6 October 1994)
Zh. Eksp. Teor. Fiz. **107**, 628–636 (February 1995)

The problem of needle crystal and its stability in dendritic growth is considered. The analytical theory for steady-state growth[4] is extended to the case of a nonstationary perturbation. It is shown that, as in the two-dimensional (2*D*) case, in the discrete spectrum of steady-state solutions only the unique solution, which corresponds to the highest velocity, is stable against the tip splitting perturbations. The instability spectrum for the solutions with lower velocities is enriched compared to 2*D* due to angular modes. The most unstable modes correspond to the eigensolutions which are localized near the extremum of the anisotropy of surface energy in the azimuthal direction. 

## 1. INTRODUCTION

The pattern selection in dendritic crystal growth was considered first by Ivantsov.[1] The interest in this problem was revived later with the important discovery that the unique solution from the continuous family of solutions found by Ivantsov is physically selected by effects of anisotropic surface tension.[2] Mathematically, the selection mechanism is provided by a singular perturbation of the initial integral equation by differential terms that come from the curvature of the liquid-crystal interface. These results were obtained primarily for 2*D* dendrite. The solution of the 3*D* problem turned out to be more intricate since, for a realistic crystal anisotropy, the shape of a dendrite deviates from a paraboloidal shape and the mathematical structure of the theory becomes more involved. A numerical approach to the nonaxisymmetric problem was presented by Kessler and Levine,[3] who pointed out the following aspect of the problem. In the 2*D* case or in the axisymmetric case the selection of the growth velocity follows from the solvability condition of smoothness of the dendrite tip. In the 3*D* nonaxisymmetric case a solvability condition must be satisfied for each of the azimuthal harmonics. Kessler and Levine made several approximations and performed only a two-mode calculation, but the crucial point of their analysis is that they found enough degrees of freedom to satisfy all solvability conditions.

Recently, an analytic theory of three-dimensional dendritic growth has been developed by Ben Amar and Brener.[4] In the framework of asymptotics beyond all orders, they derived the inner equation in the complex plane for the nonaxisymmetric shape correction to the Ivantsov paraboloid. The solvability condition for this equation provides selection of both the velocity of the dendrite and the interface shape. Below, we reexamine this approach in order to elucidate some points of the derivation and extend it to include a nonstationary perturbation.

## 2. VELOCITY SELECTION

The Stephan problem for dendritic growth consists in solving the stationary heat-diffusion equation

$$D\nabla^2 T + v\,\partial T/\partial z = 0, \tag{1}$$

where $D$ is the diffusivity coefficient, $v$ is the stationary growth velocity in the $z$ direction, with the boundary condition

$$cD\mathbf{n}(\nabla T_{\text{liq}} - \nabla T_{\text{cryst}}) = -L\mathbf{v}_{\text{n}}, \tag{2}$$

where $T_{\text{liq}}$ and $T_{\text{cryst}}$ are the temperature profiles of the liquid and crystal, respectively, $c$ is the specific heat, $L$ is the latent heat, $\mathbf{n}$ is a unit vector normal to the interface, and $\mathbf{v}_{\text{n}}$ is the normal velocity. The temperature at the interface should be equal to the melting point. It is convenient to introduce the dimensionless field,

$$\Delta(\mathbf{r},\mathbf{z}) \equiv c[T(\mathbf{r},z) - T_\infty]/L, \tag{3}$$

where $T_\infty$ is the temperature far ahead of the dendrite. The shape of a dendrite is given by $z=\zeta(\mathbf{r})$, where $\mathbf{r}\equiv(x,y)$. At the tip the dendrite is close to a paraboloid,

$$\zeta \approx -r^2/2\rho. \tag{4}$$

Rescaling all the lengths by $\rho$, we can solve Eq. (1) with the boundary condition (2), obtaining the temperature distribution

$$\Delta(\mathbf{r},z) = \frac{p}{2\pi}\int d^2\mathbf{r}'\,G[\mathbf{r},z;\mathbf{r}',\zeta(\mathbf{r}')], \tag{5}$$

where $p \equiv v\rho/2D$ is the Peclet number, $G$ is the Green's function,

$$G[\mathbf{r},z;\mathbf{r}',z'] \equiv R^{-1}\exp[p(z'-z-R)], \tag{6}$$

and

$$R^2 \equiv (\mathbf{r}-\mathbf{r}')^2 + (z-z')^2. \tag{7}$$

 1063-7761/95/020341-05$10.00  

Ignoring the surface energy, the temperature of the interface should be constant,

$$\Delta[\mathbf{r},\zeta(\mathbf{r})]=\Delta_0=c(T_m-T_\infty)/L, \tag{8}$$

where $T_m$ is the equilibrium melting point.

Among solutions with circular symmetry, only the paraboloid $\zeta=-r^2/2$ provides a constant $\Delta$ throughout the interface, with a unique relation between the undercooling $\Delta$ and the Peclet number $p$:[1]

$$\Delta_0=p\,\exp(p)E_1(p), \tag{9}$$

where $E_1(p)$ is the exponential integral function.

However, a solution of the form

$$\zeta_0=-r^2/2+\sum_m A_m r^m \cos m\varphi, \tag{10}$$

where $\varphi$ is the azimuthal angle in the $x,y$ plane, retains a constant $\Delta$ throughout the interface, when the Green's function in (5) is expanded to linear terms in $A_m$'s. We believe that Eq. (10) provides a satisfactory solution of the Stephan problem with a vanishing surface energy even for finite $A_m$'s, which should be found from a solvability solution with allowance for the surface energy.[3,4]

For a small effect of the surface energy, the interface shape is only weakly perturbed,

$$\zeta(\mathbf{r})=\zeta_0(\mathbf{r})+\zeta_1(\mathbf{r}), \tag{11}$$

where $\zeta_1\ll\zeta_0$. The left-hand side of Eq. (5) should be corrected by the Gibbs–Thomson shift of the melting point due to the finite curvature of the interface,

$$\Delta=\Delta_0-(d_0/\rho)\Delta_\mu(\zeta), \tag{12}$$

where $d_0$ is a microscopic length which specifies the scale of the surface energy. The expression for $\Delta_\mu(\zeta)$ will be discussed below. Equation (5), taken at $z=\zeta(\mathbf{r})$, after linearization of its right-hand side, is written as

$$-\sigma\Delta_\mu(\zeta)=\frac{1}{2\pi}\int[\zeta_1(\mathbf{r})-\zeta_1(\mathbf{r}')]\frac{\partial G}{\partial z}\,d^2\mathbf{r}', \tag{13}$$

where $\sigma\equiv d_0/p\rho$ is a small parameter of our problem.

We consider separately the local term, which is proportional to $\zeta(\mathbf{r})$, and the integral term. The coefficient of $\zeta_1(\mathbf{r})$ can be calculated, in general,[2] from the following considerations. From Eq. (5) it follows that

$$\frac{1}{2\pi}\int\frac{\partial G}{\partial z}\,d^2\mathbf{r}'=\frac{1}{p}\frac{\partial\Delta}{\partial z},\quad z>\zeta_0(\mathbf{r}), \tag{14}$$

$$\frac{1}{2\pi}\int\frac{\partial G}{\partial z}\,d^2\mathbf{r}'=0,\quad z<\zeta_0(\mathbf{r}). \tag{15}$$

The derivative $\partial\Delta/\partial z$ at the interface can be calculated from the boundary condition (2),

$$\frac{1}{p}\frac{\partial\Delta}{\partial z}=-2\cos^2\theta=-\frac{2}{1+(\nabla\zeta_0)^2}, \tag{16}$$

where $\theta$ is the angle between the normal and the $z$ axis. Equation (15) reflects the fact that, for a vanishing surface energy, the temperature inside the crystal is constant. The coefficient of $\zeta_1(\mathbf{r})$ in Eq. (13) is given by the principal value of the corresponding integral, which in turn, should be calculated as half of the sum of Eqs. (14) and (15). We thus obtain

$$\frac{\zeta_1}{2\pi}\,\mathrm{V.P.}\int\frac{\partial G}{\partial z}=-\frac{\zeta_1}{1+(\nabla\zeta_0)^2}. \tag{17}$$

This expression, which is considered as a function of the complex variable, is singular when

$$(\nabla\zeta_0)^2=-1, \tag{18}$$

but this singularity is canceled by a contribution from the integral term. The latter cannot be calculated in general. However, it can be calculated near the singular point (or a line of singular points), defined by Eq. (18). To further simplify our task, we consider the limit of a small Peclet number, $p\to0$. In this regime we obtain

$$-\frac{1}{2\pi}\,\mathrm{V.P.}\int\zeta_1(\mathbf{r}')\frac{\partial G}{\partial z}\,d^2\mathbf{r}'\approx\frac{1}{2\pi}\,\mathrm{V.P.}\int\frac{\zeta_1(\mathbf{r}')[\zeta_0(\mathbf{r})-\zeta_0(\mathbf{r}')]}{R^3}\,d^2\mathbf{r}'dz'. \tag{19}$$

It is well known that, for the point $\mathbf{r}$ which is displaced to the neighborhood vicinity of the singular point (18), the leading terms coming from this integral have a local form,[5] i.e., they are proportional to $\zeta_1(\mathbf{r})$ and its derivatives. To calculate Eq. (19) in the singular region, we assume that there is a large difference of scales in the coordinate dependences of $\zeta_1(\mathbf{r})$ in the lateral and transverse directions to the singular line. Introducing a fast variable $u$ and a smooth variable $v$, we first ignore the dependence of $\zeta_1$ on $v$, $\zeta_1=\zeta_1(u)$. Substituting into Eq. (19) the expansions

$$(\mathbf{r}-\mathbf{r}')^2=(u-u')^2+(v-v')^2 \tag{20}$$

and

$$\zeta_0(\mathbf{r})-\zeta_0(\mathbf{r}')\approx\frac{\partial\zeta_0}{\partial u}(u-u')+\frac{\partial\zeta_0}{\partial v}(v-v'), \tag{21}$$

after integration over $v$ we obtain

$$\frac{1}{2\pi}\,\mathrm{V.P.}\int\frac{\zeta_1(\mathbf{r}')[\zeta_0(\mathbf{r})-\zeta_0(\mathbf{r}')]}{R^3}\,d^2\mathbf{r}'\approx\frac{1}{2\pi}\oint\frac{\zeta_1(u')du'}{u-u'}\frac{\partial\zeta_0/\partial u}{[1+(\nabla\zeta_0)^2][1+(\partial\zeta_0/\partial v)^2]^{1/2}}, \tag{22}$$

where the contour integral around the point $u'=u$ has its origin in the contour representation of the principal value integral. We have retained the local contribution and dropped out the integral term which turns out to be small for small $\sigma$. Finally, we obtain

$$\frac{1}{2\pi}\,\mathrm{V.P.}\int\frac{\zeta_1(\mathbf{r}')[\zeta_0(\mathbf{r})-\zeta_0(\mathbf{r}')]}{R^3}\,d^2\mathbf{r}'\approx-\frac{i\zeta_1(u)\partial\zeta_0/\partial u}{[1+(\nabla\zeta_0)^2][1+(\partial\zeta_0/\partial v)^2]^{1/2}}. \tag{23}$$

This expression cancels the pole in Eq. (17) since in the singular point we have

$$\frac{\partial \zeta_0}{\partial u}=i\left[1+\left(\frac{\partial \zeta_0}{\partial v}\right)^2\right]^{1/2}. \tag{24}$$

The resulting equation is

$$\sigma\Delta_\mu(\zeta)=\frac{\zeta_1(u)}{2[1+(\partial\zeta_0/\partial v)^2]}. \tag{25}$$

A natural fast variable in the singular region is

$$t\equiv 1+i|\nabla\zeta_0|, \tag{26}$$

where

$$|\nabla\zeta_0|\equiv\left[\left(\frac{\partial \zeta_0}{\partial u}\right)^2+\left(\frac{\partial \zeta_0}{\partial v}\right)^2\right]^{1/2}. \tag{27}$$

With allowance for the identity

$$dt=i|\nabla|\nabla\zeta_0||du, \tag{28}$$

we obtain

$$1+(\partial\zeta_0/\partial v)^2=\cos^2\beta, \tag{29}$$

where $\beta$ is the angle between the vectors $\nabla\zeta_0$ and $\nabla|\nabla\zeta_0|$.

To write an explicit expression for $\Delta_\mu(\zeta)$, we need to specify the expression for the surface energy. For a cubic crystal the simplest anisotropy of the surface energy is given by the expression

$$\begin{aligned}\gamma(\theta,\tilde{\phi})&=1+4\varepsilon[\cos^4\theta+\sin^4\theta(\cos^4\tilde{\phi}+\sin^4\tilde{\phi})]\\&=1+4\varepsilon(\cos^4\theta+(3/4)\sin^4\theta\\&\quad+(1/4)\sin^4\theta\cos 4\tilde{\phi}),\end{aligned} \tag{30}$$

where $\theta$ and $\tilde{\varphi}$ are the spherical angles made by the normal to the interface. The general expression for $\Delta_\mu(\zeta)$ is rather complicated.[3] In the singular region, where $\tan\theta\approx|\nabla\zeta_0|\approx i$, it simplifies to[4]

$$\Delta_\mu\approx-\left(1-\frac{\alpha(7+\cos 4\tilde{\phi})}{[1+(\nabla\zeta)^2]^2}\right)\frac{\nabla\zeta\cdot\nabla(|\nabla\zeta|)}{[1+(\nabla\zeta)^2]^{3/2}|\nabla\zeta|}, \tag{31}$$

where $\alpha=15\varepsilon$ and $\varepsilon$ is the small anisotropy parameter in Eq. (30). From this expression it follows that typically $t=1+i|\nabla\zeta_0|\propto\alpha^{1/2}$. We assume and verify it later that $\zeta_1\propto\alpha$, $\partial\zeta_1/\partial t\propto\alpha^{1/2}$, and $\partial^2\zeta_1/\partial t^2\propto 1$. Near the point where $(\nabla\zeta_0)^2=-1$, we obtain

$$1+(\nabla\zeta)^2\approx 2t+2\nabla\zeta_0\cdot\nabla\zeta_1=2(t-qd\zeta_1/dt), \tag{32}$$

with

$$q=\nabla\zeta_0\cdot\nabla(|\nabla\zeta_0|)|\nabla\zeta_0|^{-1}\equiv|\nabla(|\nabla\zeta_0|)|\cos\beta. \tag{33}$$

The parameter $q$ accounts both for the length scale on the curved interface $z=\zeta_0(\mathbf{r})$ and for a finite angle between the lines $\zeta_0(\mathbf{r})=\text{const}$ and $|\nabla\zeta_0(\mathbf{r})|=\text{const}$. In the same way as Eq. (32), we can rewrite Eq. (31) in terms of the variable $t$, and after substituting it into Eq. (25) we obtain

$$\begin{aligned}&\sigma\frac{q(1-qd^2\zeta_1/dt^2)}{[2(t-qd\zeta_1/dt)]^{3/2}}\left[1-\frac{\alpha(7+\cos 4\tilde{\varphi})}{4(t-qd\zeta_1/dt)^2}\right]\\&\quad=-\frac{\zeta_1}{2\cos^2\beta}.\end{aligned} \tag{34}$$

After substituting $t=\tilde{\alpha}^{1/2}\tau$ and $\zeta_1=-\tilde{\alpha}f/q$, where $\tilde{\alpha}\equiv(\alpha/4)(7+\cos 4\tilde{\varphi})$, we obtain the final equation

$$\left(1-\frac{1}{(\tau+f_\tau)^2}\right)\frac{1+f_{\tau\tau}}{(\tau+f_\tau)^{3/2}}-f\lambda=0, \tag{35}$$

where

$$\lambda=\frac{2^{1/2}\tilde{\alpha}^{7/4}}{\sigma q^2\cos^2\beta}. \tag{36}$$

This result differs from that in Ref. 4 by $\cos^2\beta$ in the denominator, which gives a small correction for small $A_m$'s. The difference comes from the fact that the integral term has been linearized in Ref. 4 around the Ivantsov paraboloid, while here we have linearized it around the nonaxisymmetric solution (10).

Equation (35) is identical to the solvability equation in the $2D$ case[5] and generates a discrete spectrum of $\lambda$, if one requires that its solution be finite on the rays $\arg\tau=0$, $\pm 4\pi/7$.

In contrast to the $2D$ case, the parameter $\lambda$ is not a combination of constants, but should be calculated as a function of the smooth variable on the singular line (18). On the other hand, it should be equal to the eigenvalue in Eq. (35). This condition represents the $3D$ self-consistency equation for the determination of $\sigma$ and the shape of the dendrite by choosing appropriate values of the coefficients[4] $A_m$. This problem of the shape selection apparently contains no small parameter. However, from the relation $\tilde{\alpha}=(\alpha/4)(7+\cos 4\tilde{\varphi})$ we realize that the azimuthal anisotropy of $\tilde{\alpha}$ is numerically small, so that 1/7 can be effectively exploited as a small parameter. In the linear approximation with respect to this small parameter, we can satisfy the solvability condition by the correction $A_4r^4\cos(4\varphi)$ with $A_4=1/96$. In this approximation no correction to the growth velocity compared to the axisymmetric approach is obtained:

$$\begin{aligned}v&=2Dp^2(\Delta_0)\sigma(\alpha)/d_0=2^{3/2}Dp^2(\Delta_0)\\&\quad\times(7\alpha/4)^{7/4}/(\lambda_j d_0),\end{aligned} \tag{37}$$

where $p(\Delta_0)$ is given by Eq. (9).

An important aspect of Eq. (10) is that the shift vector $r^m\cos m\phi$ grows at a faster rate than the underlying Ivantsov solution. This means that only the tip region, where the anisotropy correction is still small, can be described by the approximation used by us. This is the crucial difference between the $3D$ nonaxisymmetric case and the $2D$ case. In the latter case the small anisotropy implies that the shape of the selected needle crystal is close to the Ivantsov parabola everywhere; in the former case the strong deviations from the Ivantsov paraboloid appear for any anisotropy. The description of the dendritic tail and the matching of the nonaxisymmetric shape (10) in the tip region to the asymptotic shape in the tail region is given by Brener.[6]

## 3. INSTABILITY SPECTRUM

In the $2D$ dendritic growth the solution with the smallest value of $\lambda$, $\lambda=\lambda_0$ (with the highest velocity), is stable against small perturbations. For larger $\lambda_j$ the dendritic growth becomes unstable, where the exact number of unstable modes is equal to $j$.[7–9] In the $3D$ growth the situation is more complicated because of the larger number of degrees of freedom. To demonstrate this point we derive an equation for the time-dependent perturbations with higher accuracy than above.

We start with the general time-dependent equation,

$$\frac{\Delta_0}{p}-\sigma\Delta_\mu=\int_0^\infty \frac{p^{1/2}dt'}{(2\pi t')^{3/2}}\int d^2\mathbf{r}'\dot{\zeta}(\mathbf{r}',t-t')\times\exp\left\{-\frac{p}{2t'}[(\mathbf{r}-\mathbf{r}')^2+[\zeta(\mathbf{r},t)-\zeta(\mathbf{r}',t-t')]^2]\right\}, \tag{38}$$

where $\dot{\zeta}\equiv d\zeta/dt$. Substituting into this equation

$$\zeta(\mathbf{r},t)=t+\zeta(\mathbf{r})+\zeta_\Omega(\mathbf{r})\exp(\Omega t), \tag{39}$$

and expanding it linearly in $\zeta_\Omega$ in the regime of a small Peclet number, we obtain

$$-\sigma\frac{\delta\hat{\Delta}_\mu(\zeta)}{\delta\zeta}\zeta_\Omega=-\frac{1}{2\pi}\int\frac{[\zeta_\Omega(\mathbf{r})-\zeta_\Omega(\mathbf{r}')][\zeta_0(\mathbf{r})-\zeta_0(\mathbf{r}')]}{R^3}d^2\mathbf{r}'+\frac{\Omega}{2\pi}\int\frac{\zeta_\Omega(\mathbf{r}')}{R}d^2\mathbf{r}', \tag{40}$$

where $\hat{\Delta}_\mu(\zeta)$ is considered as a nonlinear differential operator.

The first term in the right-hand side is calculated in the manner described above. The term proportional to $\zeta_\Omega$ can be restored by comparison with Eq. (25). To take into account the dependence of $\zeta_\Omega$ on the smooth variable we can, in first approximation, identify the latter with the angle $\varphi$. A simple estimate shows that the derivatives $\partial^2\zeta_\Omega/\partial\varphi^2$ come in both from the left-hand side and from the first term on the right-hand side of this equation, but the latter is greater by the factor $\alpha^{-1/2}$. To calculate this contribution, we need to expand $\zeta_\Omega$ quadratically in $\varphi'-\varphi$, obtaining an integral that diverges logarithmically for $|r-r'|\ll r$,

$$\frac{1}{2\pi}\text{V.P.}\int\frac{[\zeta_\Omega(\mathbf{r})-\zeta_\Omega(\mathbf{r}')][\zeta_0(\mathbf{r})-\zeta_0(\mathbf{r}')]}{R^3}d^2\mathbf{r}'\approx-\frac{\zeta_\Omega(r,\varphi)}{2}+\frac{\partial\zeta_0}{\partial r}\frac{1}{2\pi}\text{V.P.}\int\frac{\partial^2\zeta_\Omega(r',\varphi)}{\partial\varphi^2}\times(r-r')\ln|r-r'|dr'. \tag{41}$$

Similarly to Eq. (22), this integral, which is continued to the singular region, contains contributions of a different nature. One of them comes from integration along the real axis of $r'$ and can be disregarded. The second one comes from integration in the singular region and can be singled out with use of the relation

$$\text{V.P.}\int(x-x')\ln|x-x'|\psi(x')dx'\to\pi i\int_x^\infty dy'\int_{y'}^\infty\psi(-iy'')dy''. \tag{42}$$

The second term also contains a logarithmic divergence and can be calculated in close analogy to the $2D$ case.[9] Bringing together all these results, we obtain the equation

$$\sigma\frac{\delta\hat{\Delta}_\mu(\zeta)}{\delta\zeta}\zeta_\Omega=-\frac{\zeta_\Omega}{2}-\frac{\tilde{\alpha}(\varphi)}{2}\int_\tau^\infty d\tau'\int_{\tau'}^\infty\frac{\partial^2\zeta_\Omega(\tau'')}{\partial\varphi^2}d\tau''+\omega g(\varphi)\int_\tau^\infty\zeta_\Omega(\tau')d\tau', \tag{43}$$

where the left-hand side is still written in a symbolic form, while on the right-hand side the fast variable is introduced, $r=i(1-\tilde{\alpha}^{1/2}\tau)$. Here we introduce the notation $\omega\equiv\Omega\alpha^{1/2}$, $g(\varphi)\equiv(7+\cos 4\varphi)^{1/2}/2$. The left-hand side of Eq. (43) can easily be obtained by linearization of Eq. (35). The final differential equation is rather simple,

$$\frac{\partial^2}{\partial\tau^2}\left[B(\tau)\frac{\partial^2\zeta_\Omega}{\partial\tau^2}-\zeta_\Omega\right]-2\omega g(\varphi)\frac{\partial\zeta_\Omega}{\partial\tau}=\tilde{\alpha}(\varphi)\frac{\partial^2\zeta_\Omega}{\partial\varphi^2}, \tag{44}$$

where

$$B(\tau)=\left(1-\frac{1}{(\tau+f_\tau)^2}\right)\frac{1}{\lambda(\tau+f_\tau)^{3/2}}. \tag{45}$$

This equation can be treated in the spirit of the adiabatic perturbation theory, since its right-hand side is small, while the left-hand side contains no derivatives in $\varphi$. The difference from the familiar approach to the solution of the Schrödinger equation is that the operator on the left-hand side is not a self-adjoint operator. From a formal point of view, we need to find the spectrum and the eigenfunctions of the equation

$$\frac{\partial}{\partial\tau}\left[B(\tau)\frac{\partial^2\psi_n}{\partial\tau^2}-\psi_n\right]-2\omega_n\psi_n=0, \tag{46}$$

and the solutions $\tilde{\psi}_n$ of the adjoint equation, which have no rising exponents along the rays $\arg(\tau)=\pm4\pi/7$. This equation is exactly the same as in the $2D$ case and it has been considered in Ref. 9. The solution with the smallest value of $\lambda$, $\lambda=\lambda_0$ (with the highest velocity) is stable and has no unstable modes. For larger $\lambda_j$, growth is unstable. Here the exact number of unstable modes is equal to $j$. Substituting $\zeta_\Omega(\tau,\varphi)=\psi_n(\tau)A(\varphi)$ into Eq. (44), multiplying the resulting equation by $\tilde{\psi}_n(\tau)$ from the left, and integrating it in $\tau$, we obtain the equation for $A$,

$$\tilde{\alpha}(\varphi)\frac{d^2A}{d\varphi^2}=-2k_n[\omega g(\varphi)-\omega_n]A(\varphi), \tag{47}$$

where the numerical factor $k_n$ is given by the ratio

$$k_n=\frac{\int_C\tilde{\psi}_n \frac{d\psi}{d\tau}d\tau}{\int_C\tilde{\psi}_n\psi_n d\tau}. \quad (48)$$

Here the contour $C$ goes from the ray $\arg(\tau)=-4\pi/7$ to the ray $\arg(\tau)=4\pi/7$, circumnavigating the point $\tau=0$ from the right. It is easy to verify that the number of the integration constant is sufficient to provide convergence of the integrals in Eq. (48). Solutions of Eq. (47) have to be periodic in $\varphi$, $A(\varphi+2\pi)=A(\varphi)$. The structure of the instability spectrum is quite simple: for $k_n<0$, the function $A$ is concentrated near the minimum of $g(\varphi)$, where $\varphi\approx\pi/4$ and

$$\omega_{nm}=(2/3)^{1/2}\omega_n-\alpha^{1/2}(\omega_n/3|k_n|)^{1/2}(2m+1); \quad (49)$$

for $k_n>0$, the function $A$ is concentrated near the maximum of $g(\varphi)$, where $\varphi\approx 0$ and

$$\omega_{nm}=\omega_n/2^{1/2}+\alpha^{1/2}(\omega_n/4k_n)^{1/2}(2m+1). \quad (50)$$

As was mentioned above, $\omega_n$ and $k_n$ should be found from the solution of Eq. (46), which is exactly the same as in the $2D$ case. For $\omega\gg 1$ this equation was solved analytically.[9] It was found that

$$\omega_n(j)=(\lambda_j^{1/2}/6)(7/3)^{7/8}-(3/8)^{1/2}(j-n+1/2) \quad (51)$$

and

$$k_n=-3\omega_n. \quad (52)$$

Because $k_n$ is negative, the spectrum $\omega_{nm}$ is given by Eqs. (49), (51), and (52).

In conclusion, we have demonstrated that the stability problem of the needle crystal in the $3D$ dendritic growth is very similar to that for $2D$ dendrites:

1) the only stable solution corresponds to the highest velocity of growth;

2) the $j$th solution, in order of decrease of the velocity, has exactly $j$ unstable modes (for a fixed angular quantum number $m$);

3) with allowance for the angular degrees of freedom, each of these modes splits into a dense spectrum of strongly anisotropic unstable modes.

This work was supported in part by the Russian Basic Research Fund grant No. (93-02-2113).

[1] G. P. Ivantsov, Dokl. Akad. Nauk SSSR **58**, 567 (1947).
[2] For a review see J. S. Langer, in *Chance and Matter*, Les Houches Lectures XLVI, Elsevier, N. Y. (1987); D. Kessler, J. Koplik, and H. Levine, Adv. Phys. **37**, 255 (1988); E. A. Brener and V. I. Melnikov, Adv. Phys. **40**, 53 (1991); Y. Pomeau and M. Ben Amar, in *Solids Far From Equilibrium*, ed. by C. Godréche, Cambridge University Press, Cambridge, (1992).
[3] D. Kessler and H. Levine. Acta metall. **36**, 2693 (1988).
[4] M. Ben Amar and E. Brener, Phys. Rev. Lett. **71**, 589 (1993).
[5] M. Ben Amar and Y. Pomeau, Europhys. Lett. **2**, 307 (1986).
[6] E. Brener, Phys. Rev. Lett. **71**, 3653 (1993).
[7] D. Kessler and H. Levine, Phys. Rev. Lett. **57**, 3069 (1987).
[8] D. Bensimon, P. Pelce, and B. Shraiman, J. Phys. **48**, 2081 (1987).
[9] E. A. Brener, S. V. Iordanskii, and V. I. Mel'nikov, Zh. Eksp. Teor. Fiz. **94**, 320 (1988) [Sov. Phys. JETP **68**, 2574 (1988)].

This article was published in English in the original Russian journal with stylistic changes by the Translation Editor, Dr. S. J. Amoretty.