The QED two-loop thermodynamic potential in a constant magnetic field

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(Submitted 29 July 1994)
Zh. Eksp. Teor. Fiz. **107**, 299–306 (February 1995)

The QED thermodynamic potential in a constant magnetic field is calculated in the two-loop approximation. The two-loop contribution to the amplitude of electron-gas magnetization oscillations is shown to be much higher than the monotonic part of the magnetization in the one-loop approximation. The applicability of perturbation theory in the limit of a strong magnetic field and a degenerate electron gas is investigated. © *1995 American Institute of Physics.*

1. INTRODUCTION

Akhiezer and Peletminskiĭ were the first to use the two-loop approximation with relativistic effects to calculate the QED thermodynamic potential without an external field. Other researchers[2,3] have also examined this problem. The QED one-loop thermodynamic potential in a constant magnetic field, i.e., the thermodynamic potential of an ideal gas of electrons, positrons, and photons, has also been studied in detail (see, e.g., Refs. 4 and 5).

The effective Lagrangian of an arbitrary constant field has been calculated in the two-loop approximation by Ritus.[6] The analogous problem in QED at a finite temperature and a nonzero chemical potential in a constant magnetic field is important. For instance, it would be interesting, from the standpoint of specific physical applications, to resolve the question posed by Akhiezer and Peletminskiĭ for a model problem about the way in which the electron–electron interaction influences the Landau magnetization oscillation effect in an electron gas. Leaving the scope of the one-loop approximation makes it possible to solve the problem of the limits of applicability of perturbation theory methods in strong external fields. Also, calculating the two-loop thermodynamic potential in a constant magnetic field will help to establish the ranges of the characteristic parameters outside which the higher order terms in the known asymptotic expansions for the one-loop thermodynamic potential lose all physical meaning.[5]

In this paper we employ the method of temporal Green's functions to calculate the QED thermodynamic potential in a constant magnetic field to within terms on the order of the fine-structure constant (Sec. 2). In Sec. 3 we use the example of a degenerate electron gas to arrive at asymptotic representations of the result in the limit of a strong magnetic field and in the case of a relatively weak magnetic field. We study the oscillating part of the two-loop thermodynamic potential and examine the question about the perturbation-theory expansion parameter in a strong magnetic field. In conclusion we discuss the results.

2. THE TWO-LOOP CONTRIBUTION TO THE THERMODYNAMIC POTENTIAL

The effective Lagrangian of a constant homogeneous field in the two-loop approximation at absolute zero T and at zero chemical potential μ is given by the following formula:[6]

$$\mathscr{L}^{(2)}=\frac{ie^2}{2}\int dxdx'tr[\gamma^{\mu}G_0(x,x')\gamma_{\mu}G_0(x',x)]D_0(x-x'), \tag{1}$$

where $G_0(x,x')$ and $D_0(x-x')$ are the electron and photon Feynman propagators at $T=\mu=0$ (the two-loop thermodynamic potential is the statistical analog of this Lagrangian).

To allow for finite-temperature effects we employ the real-time approximation, in which the Green's function is represented as a sum of two terms, the causality propagator at $T=\mu=0$ and a purely temperature-dependent part:[8–10]

$$\begin{aligned}G(x,x')&=G_0(x,x')+G^{T,\mu}(x,x')\\&=G_0(x,x')+i\sum_{s,\varepsilon=\pm1}\frac{\varepsilon\Psi_s^{(\varepsilon)}(x)\Psi_{s'}^{(\varepsilon)}(x')}{\exp\{\beta(E_n-\varepsilon\mu)\}+1},\end{aligned} \tag{2}$$

$$\begin{aligned}D(x-x')&=D_0(x-x')+D^T(x-x')\\&=\frac{1}{(2\pi)^4}\int d^4k\,\exp\{-ik(x-x')\}\biggl[-\frac{i}{k^2+i0}\\&\quad-2\pi\frac{\delta(k^2)}{\exp\{\beta|k_0|-1\}}\biggr],\end{aligned} \tag{3}$$

where $\beta=T^{-1}$ is the reciprocal temperature, and the summation in (2) is over all the quantum states $s=(n,p_3,p_2,\xi)$ of the electrons ($\varepsilon=+1$) and the positrons ($\varepsilon=-1$).

Note that in our case of a constant homogeneous magnetic field $\mathbf{H}\|0Z$, specified by the potential

$$A^{\mu}(x_1)=(0,0,x_1H,0), \tag{4}$$

the electron wave function defined by the four quantum numbers n, p_3, p_2, and ξ has the form[11]

$$\Psi_s^{(\varepsilon)}(x)=\frac{(eH)^{1/4}}{L}\exp\{-i\varepsilon E_n+ip_2x_2+ip_3x_3\}\Phi_s, \tag{5}$$

where

$$\Phi_s=\begin{pmatrix}C_1 & U_{n-1}(\eta)\\ iC_2 & U_n(\eta)\\ C_3 & U_{n-1}(\eta)\\ iC_4 & U_n(\eta)\end{pmatrix}, \tag{6}$$

L is the normalization length along the x_2 and x_3 axes, and $U_n(\eta)$ is the Hermite function with argument

$$\eta=\sqrt{eH}\left(x_1+\frac{p_2}{eH}\right), \tag{7}$$

with the explicit form of the coefficients C_i $(i=1,2,3,4)$ not given here (see Ref. 12).

The quantum numbers specifying the electron state have the following physical meaning: $n=0,1,2,...$ is the principal quantum number determining the magnitude of the transverse momentum $p_\perp$ and hence the radius R of the quasiclassical orbit (for $n\gg 1$),

$$p_\perp=\sqrt{2eHn},\quad R=\frac{p_\perp}{eH}=\left(\frac{2n}{eH}\right)^{1/2}; \tag{8}$$

p_3 is the momentum projection in the direction of the magnetic field $\mathbf{H}$; p_2 fixes the position of the orbit center,

$$\langle x_1\rangle=-\frac{p_2}{3H}, \tag{9}$$

and the spin quantum number ξ determines a state with the spin oriented along the field $(\xi=+1)$ or opposite the field $(\xi=-1)$.

The energy levels of an electron in a constant homogeneous magnetic field are given by the formula[11]

$$E(n,p_3)=\sqrt{m^2+2eHn+p_3^2} \tag{10}$$

and are independent of p_2 and ξ, i.e., the levels are infinitely degenerate.

The two-loop contribution to the effective Lagrangian at a finite temperature is obtained by replacing the Green's functions $G_0(x,x')$ and $D_0(x-x')$ in Eq. (1) with their expressions (2) and (3) for $T\neq 0$ and $\mu\neq 0$ (Ref. 8). After this replacement we are interested only in the temperature-dependent part of (1). As noted in the Introduction, the temperature-independent part has been studied in detail by Ritus.[6]

If we use the temporal Green's functions approach in the proper-time representation obtained by Gavrilov *et al.*,[10] the physical meaning of the quantity of interest to us is not as clear as it is in the approach employed below, where the quantum numbers determining the energy spectrum of a particle are explicitly incorporated into the general expression for the two-loop thermodynamic potential. This requires calculating the quantity

$$F_{s,s'}=\int dx_1dx_1'\mathrm{tr}[\gamma^\mu K_s(x_1,x_1')\gamma_\mu K_{s'}(x_1',x_1)]\times\exp^{ik_1(x_1-x_1')}, \tag{11}$$

where the matrix $K(x_1,x_1')$ has the form

$$K^s_{\alpha\beta}(x_1,x_1')=[\Phi_s^{(\varepsilon)}(x_1)]_\alpha[\bar\Phi_s^{(\varepsilon)}(x_1')]_\beta. \tag{12}$$

To evaluate (11) we expand K in the complete set of matrices $\{\gamma^A\}=\{1,\gamma^5,\gamma^\mu,i\gamma^\mu\gamma^5,i\sigma^{\mu\nu}\}$:

$$K=\sum_{A=1}^{16}C_A\gamma^A,C_A=\tfrac{1}{4}\mathrm{Tr}\gamma_AK. \tag{13}$$

We then have

$$\mathrm{tr}[\gamma^\mu K(x_1,x_1')\gamma_\mu K(x_1',x_1)]=SS'-\tfrac{1}{2}(VV')-\tfrac{1}{2}(AA'), \tag{14}$$

where

$$S=\mathrm{tr}\,K(x_1,x_1'),V^\mu=\mathrm{tr}\gamma^\mu K(x_1,x_1'),$$
$$A^\mu=\mathrm{tr}\gamma^\mu\gamma^5K(x_1,x_1'). \tag{15}$$

Calculating the quantities specified in Eqs. (15), combining (11) and (14), and integrating with respect to x_1 and x_1', we get

$$\begin{aligned}F=&(C_1^2-C_3^2)(C_1'^2-C_3'^2)I^2_{n-1,n'-1}+(C_2^2-C_4^2)(C_2'^2-C_4'^2)I^2_{n,n'}\\&-2I^2_{n-1,n'-1}(C_1^2C_4'^2+C_3^2C_2'^2)-2(C_2^2C_3'^2+C_4^2C_1'^2)I^2_{n,n'-1}\\&+(C_1C_4+C_2C_3)(C_1'C_4'+C_2'C_3')I_{n,n'}I_{n-1,n'-1},\end{aligned} \tag{16}$$

where the argument of the Laguerre functions $I_{n,n'}$ is

$$z=\frac{k_1^2+k_2^2}{2eH}. \tag{17}$$

As a result the two-loop contribution to the QED thermodynamic potential in a constant magnetic field can be written as follows:

$$\Omega^{(2)}=\Omega^{(F)}+\Omega^{(\gamma)}. \tag{18}$$

The electron–positron contribution in (18) is

$$\Omega^{(F)}=\sum_{s,\varepsilon=\pm1}(\exp[\beta(E_n-\varepsilon\mu)]+1)^{-1}[\delta E_n^{(F-F)}+\delta E_n^{(F-\gamma)}+\delta E_n(T=\mu=0). \tag{19}$$

The last term in the numerator is the radiative shift of the energy of an electron $(\varepsilon=+1)$ or a positron $(\varepsilon=-1)$ in a constant magnetic field and has been studied in detail (see, e.g., Ref. 11). For the other two terms we have

$$\delta E_n^{(F-F)}=-\frac{\alpha}{4\pi^2}\sum_{s',\varepsilon'=\pm1}\times\int\frac{d^4k}{k^2}\frac{\varepsilon F\,\delta(\varepsilon E_n-k_0-\varepsilon' E_{n'})}{\exp[\beta(E_{n'}-\varepsilon'\mu)]+1}, \tag{20}$$

$$\delta E_n^{(F-\gamma)}=-\frac{\alpha}{4\pi^2}\sum_{s',\varepsilon'=\pm1}\times\int d^4k\frac{F\,\delta(k^2)}{(k_0-\varepsilon E_n+\varepsilon' E_{n'})[\exp(\beta|k_0|)]}. \tag{21}$$

Equation (20) describes the temperature shift in the energy of an electron $(\varepsilon=+1)$ or a positron $(\varepsilon=-1)$ caused by the interaction with the plasma electrons $(\varepsilon=+1)$ and the plasma positrons $(\varepsilon=-1)$. Equation (21) describes the shift in the energy of an electron or a positron caused by the interaction with equilibrium radiation. The two contributions to the shift in the energy of an electron in a constant magnetic field were studied in Refs. 9 and 13. As for the quantity $\Omega^{(\gamma)}$ in (18), which is the two-loop contribution to the thermodynamic potential of the photon gas, it has already been discussed in the literature (see, e.g., Ref. 14), so we do not consider it here.

3. LIMITING CASES

We start with the two-loop contribution to the thermodynamic potential of a nonrelativistic degenerate electron gas in which the following conditions are maintained:

$$\frac{T}{m} \lesssim \frac{H}{H_0} \ll \frac{\mu - m}{m} \ll 1, \tag{22}$$

where $H_0 = m^2/e = 4.41 \times 10^{13}$G is the Schwinger magnetic field. Note that the effect of magnetization oscillations in an ideal electron gas is discussed in classic treatments (see, e.g., Ref. 15) precisely using the example of a nonrelativistic electron gas in conditions specified by (22). Then the main contribution to (19) is provided by the exchange part of the shift in the electron energy, and in the first approximation we can ignore $\delta E_n^{(F-\gamma)}$ and δE_n $(T=\mu=0)$ in comparison to $\delta E_n^{(F-F)}$. Using the Poisson summation formula,[15] we can write (19) in the form

$$\Omega^{(F)} = \frac{Vm}{2\pi^2} \left\{ \int_{-\infty}^{\infty} dp_3 \int_0^{\infty} dy \left[\frac{\delta E(y,p_3;H)}{\exp[\beta(p_3^2/2m + y - \mu_0)] + 1} + 2\mathrm{Re} \sum_{k=1}^{\infty} \frac{\delta E(y,p_3;H) \exp\{i2\pi k m y/eH\}}{\exp[\beta(p_3^2/2m + y - \mu_0)] + 1} \right] \right\}, \tag{23}$$

$$\mu_0 = \mu - m.$$

To obtain the main contribution to the monotonic part of (19) it is enough in the first term in (23) to use the well-known expression for the shift in the energy of a nonrelativistic electron at $T=0$ and $H=0$ (Refs. 13 and 16):

$$\delta E(p) = -\frac{\alpha p_F}{2\pi} \left[2 + \frac{p_F^2 - p^2}{p_F p} \ln \left| \frac{p_F + p}{p_F - p} \right| \right], \tag{24}$$

where p_F is the Fermi momentum.

Allowing for (24), we see that Eq. (23) yields

$$\Omega^{(F)}(H=0) = -V\alpha \frac{p_F^4}{4\pi^3}. \tag{25}$$

This result (25) coincides with the exchange correction to the thermodynamic potential of a nonrelativistic degenerate gas in the free case with $H=0$ (Refs. 1 and 15). The main contribution to the oscillating part of (23) is provided by electrons whose energies are in the vicinity of $\mu_0 = \mu - m$ with a spread of order T. Also, the smearing of the discrete energy levels of an electron in a constant magnetic field caused by the Coulomb electron–electron interaction can smooth out the oscillations. Hence we assume, in addition to the conditions specified in (22), that the energy of this interaction e^2/a, where a is a quantity on the order of the mean electron separation, is small compared to the distance between energy levels of an electron in a constant magnetic field:

$$\frac{e^2}{a} \ll \frac{eH}{m}. \tag{26}$$

Then, using the dispersion law near the Fermi surface (see Eq. (24)), from (23) we find the following representation for the oscillating part of the two-loop thermodynamic potential:

$$\Omega_{\mathrm{osc}}^{(F)} = V\alpha m p_F T \frac{\sqrt{2m\mu_B H}}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin(\pi k \mu_0/\mu_B H - \pi/4)}{\sqrt{k}\,\sinh(\pi^2 kT/\mu_B H)}, \tag{27}$$

where $\mu_B = e/2m$ is the Bohr magneton.

We now turn to the case of a completely degenerate electron gas in a relatively strong magnetic field,

$$2eH > \mu^2 - m^2, \tag{28}$$

i.e., a situation in which the electrons fill only the ground level with the principal quantum number $n=0$ and their spin points in the direction opposite the magnetic field. Here the electron number density is related to the Fermi energy by

$$n_e = \frac{eH}{2\pi^2} \sqrt{\mu^2 - m^2} \tag{29}$$

and (28) is equivalent to the condition

$$n_e < \frac{1}{\sqrt{2}\pi^2} \left(\frac{H}{H_0} \right)^3, \tag{30}$$

which is realized, for instance, in the magnetosphere of pulsars.

In this case it is possible to express the contribution of the exchange interaction to the thermodynamic potential of the electron gas as a one-dimensional integral:

$$\Omega^{(F)} = V \frac{\alpha m}{32\pi^3} \frac{H}{H_0} \int_0^{2a} d\tau (2a - \tau) \exp\left\{ \frac{1}{2} \frac{H_0}{H} (\cosh \tau - 1) \right\} \mathrm{Ei}\left(-\frac{1}{2} \frac{H_0}{H} (\cosh \tau - 1) \right), \tag{31}$$

where $\mathrm{Ei}(-x)$ is the exponential integral, and the parameter a is related to the Fermi energy by

$$\sinh a = \sqrt{(\mu/m)^2 - 1}. \tag{32}$$

In the limit of an ultrahigh magnetic field the condition (30) yields

$$\Omega^{(F)} = -V \frac{\alpha e H}{8\pi^3} (\mu^2 - m^2) \ln \frac{\mu^2 - m^2}{eH}, \qquad 2eH \gg \mu^2 - m^2. \tag{33}$$

In addition to the exchange correction (30), an essential contribution to the two-loop thermodynamic potential of a degenerate electron gas is provided by the correction caused by the radiative shift of the electron energy at $T=\mu=0$, i.e., the last term in Eq. (19).

When the conditions

$$2eH \gg \mu^2 - m^2, \quad H \gg H_0 \tag{34}$$

are satisfied simultaneously, the radiative shift in the electron energy in the principal logarithmic approximation is given by[11]

$$\delta E=\frac{m\,\delta m}{E}=\alpha\frac{m^2}{4\pi\sqrt{m^2+p_3^2}}\ln^2\left(\frac{2H}{H_0}\right). \tag{35}$$

Combining this with (19), we arrive at the following expression for the correction:

$$\Omega_{\rm rad}^{(F)}=V\alpha m^4\frac{\ln^2(2H/H_0)}{8\pi^3}\frac{H}{H_0}\ln\left[\frac{\mu}{m}+\sinh a\right]. \tag{36}$$

4. DISCUSSION

Akhiezer and Peletminskiĭ showed that the correlation correction to the one-loop thermodynamic potential is small compared to the exchange correction at fairly low temperatures,

$$T\ll\mu_0=\mu-m. \tag{37}$$

Hence there is every reason to believe that our results in the limit (37) give the main contribution to the QED one-loop thermodynamic potential in a constant magnetic field.

We now compare (27) and (33) with the similar results of the one-loop approximation. The oscillating part of the one-loop thermodynamic potential is given by[14]

$$\Omega_{\rm osc}^{(1)}=V\frac{\sqrt{2}(m\mu_{\rm B}H)^{3/2}T}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(\pi\mu_0k/\mu_{\rm B}H-\pi/4)}{k^{3/2}\sinh(\pi^2kT/\mu_{\rm B}H)}. \tag{38}$$

For $\mu_{\rm B}H\sim T$ the ratio of the amplitudes of the oscillating parts (27) and (28) is

$$r\sim\alpha\frac{p_{\rm F}}{\mu_{\rm B}H}. \tag{39}$$

Allowing for the fact that $p_{\rm F}\propto n^{1/3}$ (here n is the electron number density) and the average distance between the electrons $a\propto n^{-1/3}$, we conclude that the condition $r\ll 1$ coincides with the condition (26) that the electron–electron Coulomb interaction is low compared to the distance between the energy levels of an electron in a magnetic field.

At the same time the ratio of the amplitude of the two-loop contribution to the oscillating part of the magnetic moment, to the monotonic part of the one-loop magnetic moment (see Ref. 15) is of order $\alpha mp_{\rm F}^2(eH)^{-3/2}$. In other words, the two-loop contribution to the amplitude of magnetic-moment oscillations may be considerably higher than the monotonic part of the electron-gas magnetization, as it is in the one-loop approximation.

In the limit of a strong magnetic field, where the condition (28) is met, the one-loop thermodynamic potential of a degenerate electron gas is

$$\Omega^{(1)}=-\frac{VeH}{8\pi^2}\left[\mu\sqrt{\mu^2-m^2}-m^2\ln\left(\frac{\mu+\sqrt{\mu^2-m^2}}{m}\right)\right]. \tag{40}$$

If we assume that $\mu^2-m^2\lesssim O(m^2)$, which agrees, for instance, with the estimates in the literature[17] of the density of a degenerate electron gas in astrophysical conditions, we find

$$\frac{\Omega^{(2)}\,(33)}{\Omega^{(1)}\,(40)}\sim\alpha\ln\frac{eH}{\mu^2-m^2}. \tag{41}$$

Thus, $z=\alpha\ln[eH/(\mu^2-m^2)]$ is the perturbation-expansion parameter, and the result (33) is valid for $z\ll 1$.

We would like to express our gratitude to A. V. Borisov and A. S. Vshivtsev for discussing these results. The work was supported by the Competition Center for the Fundamental Natural Sciences of the Russian State Committee of Institutions of Higher Learning.

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Translated by Eugene Yankovsky