Periodic waves in the theory of self-induced transparency

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A general singly periodic solution of the Maxwell–Bloch equations describing self-induced transparency in a medium composed of two-level atoms has been obtained. The solution is represented in effective form and depends on all four parameters of the problem (the Riemann invariants). The Whitham equations are derived, which give the evolution of the Riemann invariants for modulated periodic solutions.

1. INTRODUCTION

Since its theoretical prediction in Ref. 1, a large number of papers [see, e.g., Ref. 2 (review)] have been dedicated to self-induced transparency (SIT), and have investigated the behavior of SIT solitons (also called 2π pulses) in various concrete situations. However, the theory of periodic pulses or self-induced transparency waves has been substantially less well-developed, although experiments have been carried out (see, e.g., Ref. 3) which have studied the evolution of this type of periodic pulse.

Since the SIT equations are exactly integrable,[4] their periodic solutions can, in principle, be found by numerical integration using the finite-zone method,[5,6] which, however, for a number of equations, including the SIT equations, turns out to be ineffective. As a consequence, previous studies have been devoted either to particular periodic solutions obtained by elementary means[7–9] or simplifying assumptions have been introduced into the method of finite-zone integration,[10] as a consequence of which the solutions that have been obtained depend on fewer parameters than is necessary in the case of the general one-phase solution. For this reason it has been impossible to use such solutions to describe the evolution of a sufficiently generally modulated periodic wave. Reference 11 proposes a modification of the finite-zone integration method which allows one to obtain periodic solutions of a broad class of integrable equations in effective form. This approach has been used to describe periodic waves in fiber-optic waveguides[12] and in magnets—both isotropic magnets[13] and in magnets with a single-axis anisotropy.[14] In the present paper we apply the method developed in Ref. 11 to the SIT equations, thanks to which we obtain a general periodic solution of these equations in effective form. In addition, in contrast with Ref. 10, we take account of inhomogeneous broadening, which is very important experimentally. We neglect relaxation processes, so that the field intensity of the wave is assumed to be strong enough that the wave period is much shorter than the relaxation time (see below). Sections 2 and 3 are devoted to periodic solutions of the SIT equations.

In real physical problems, periodic waves are invariably inhomogeneous. If the degree of inhomogeneity is relatively small, the evolution of such a wave is described by Whitham's modulation equations.[15,16] In Section 4 we obtain the Whitham equations for the case of self-induced transparency, which generalize the equations of Ref. 10.

2. PERIODIC SOLUTIONS OF THE SIT EQUATIONS

We take the SIT equations in the Lamb form[17,18]

$$E_\xi=\langle d\rangle,\quad d_\tau+2i\Delta d=En,\quad n_\tau=-\frac{1}{2}(Ed^*+E^*d),\tag{1}$$

where E is the electric field intensity of a wave propagating in a resonant medium of two-level atoms along the ξ axis, Δ is the frequency offset parameter of the atomic transitions from the oscillation frequency of the electromagnetic field of the wave, and d and n are the dipole moment of the transitions (polarization) and the population of the atoms, respectively. They are related by the equation

$$|d|^2+n^2=1,\tag{2}$$

which reflects the conservation of probability: the total probability that an atom can be found in the upper or lower level is equal to unity. In a unit volume the atoms are distributed over the frequency offset parameter Δ according to the distribution function $\tilde{g}(\Delta)$, so that the total polarization of the medium is given by

$$\langle d\rangle=\int d(\Delta)\tilde{g}(\Delta)d\Delta.\tag{3}$$

In what follows, angular brackets always denote this kind of average. System (1) is written in the characteristic variables

$$\xi=x,\quad \tau=t-x,\tag{4}$$

where x and t are the dimensionless spatial coordinate and time.

The solution of Eqs. (1) by the inverse scattering transform method is based on their representation as the two consistent linear systems[4,18]

$$\begin{aligned}&\partial\psi_1/\partial\tau=F\psi_1+G\psi_2,\quad \partial\psi_1/\partial\xi=A\psi_1+B\psi_2,\\ &\partial\psi_2/\partial\tau=H\psi_1-F\psi_2,\quad \partial\psi_2/\partial\xi=C\psi_1-A\psi_2,\end{aligned}\tag{5}$$

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where in the given case

$$F=-i\lambda,\quad G=E/2,\quad H=-E^*/2, \tag{6}$$

$$A=\frac{i}{4}\left\langle\frac{n}{\lambda-\Delta}\right\rangle,\quad B=-\frac{i}{4}\left\langle\frac{d}{\lambda-\Delta}\right\rangle,$$

$$C=-\frac{i}{4}\left\langle\frac{d^*}{\lambda-\Delta}\right\rangle, \tag{7}$$

and λ is the spectral parameter. It is convenient to transform to the spherical vector (f,g,h) whose components are built up from the two basis solutions (ψ_1,ψ_2) and (ϕ_1,ϕ_2) of systems (5):

$$f=-\frac{i}{2}(\psi_1\phi_2+\psi_2\phi_1),\quad g=\psi_1\phi_1,\quad h=-\psi_2\phi_2. \tag{8}$$

Their evolution in τ and ξ is described by the following linear systems of equations:

$$\partial f/\partial\tau=-iHg+iGh,\quad \partial f/\partial\xi=-iCg+iBh,$$

$$\partial g/\partial\tau=2iGf+2Fg,\quad \partial g/\partial\xi=2iBf+2Ag,$$

$$\partial h/\partial\tau=-2iHf-2Fh,\quad \partial h/\partial\tau=-2iCf-2Ah. \tag{9}$$

It is easy to convince oneself that the length of the vector (f,g,h)

$$f^2-gh=P(\lambda) \tag{10}$$

is an invariant of this evolution.

Periodic solutions are distinguished by the condition that $P(\lambda)$ be a polynomial in λ. For many physical applications it is enough to know only the one-phase solutions, for which $P(\lambda)$ is (in our case) of the fourth degree:

$$P(\lambda)=\prod_{i=1}^{4}(\lambda-\lambda_i)=\lambda^4-s_1\lambda^3+s_2\lambda^2-s_3\lambda+s_4. \tag{11}$$

It is natural to seek the solution of systems (9) in the form of a polynomial of second degree in λ for f and first degree in λ for g and h. Equations (9) then yield

$$f=\lambda^2-\Delta^2-f_1(\lambda-\Delta)+an,\quad g=\frac{E}{2}(\lambda-\Delta)+iad, \tag{12}$$

$$h=-\frac{E^*}{2}(\lambda-\Delta)+iad^*,$$

here f_1 and a are as yet unspecified constants. Substituting expressions (12) in Eq. (10) and equating the coefficients of the terms in λ^3 and the free terms allows us to express f_1 and a in terms of the zeros λ_i of the polynomial $P(\lambda)$:

$$f_1=\frac{s_1}{2}=\frac{1}{2}\sum_{i=1}^{4}\lambda_i,\quad a^2=\prod_{i=1}^{4}(\Delta-\lambda_i)=P(\Delta). \tag{13}$$

In what follows it will be convenient to introduce the points μ and μ^* of the so-called auxiliary spectrum at which g and h, taken as functions of λ, vanish:

$$g=\frac{E}{2}(\lambda-\mu),\quad h=-\frac{E^*}{2}(\lambda-\mu^*), \tag{14}$$

comparison with Eq. (12) then gives

$$2ad=iE(\mu-\Delta). \tag{15}$$

Substituting relations (14) into Eqs. (9) leads to the evolution equation for μ:

$$\frac{\partial\mu}{\partial\tau}=-2if(\mu)=-2i\sqrt{P(\mu)}, \tag{16}$$

$$\frac{\partial\mu}{\partial\xi}=-2i\left\langle\frac{1}{4a}\right\rangle f(\mu)=\left\langle\frac{1}{4a}\right\rangle\frac{\partial\mu}{\partial\tau}. \tag{17}$$

The constant $1/a$ turns out to be related (after averaging over Δ) to the phase velocity of the wave. In order that this velocity (in the variables x and t) not exceed the light velocity, it is necessary to take the minus sign when extracting the root in Eq. (13):

$$a=-\sqrt{P(\Delta)}. \tag{18}$$

The phase velocity in the variables ξ and τ is equal to V, where

$$\frac{1}{V}=\left\langle\frac{1}{4a}\right\rangle=-\int\frac{\tilde{g}(\Delta)d\Delta}{4\sqrt{\prod_{i=1}^{4}(\lambda_i-\Delta)}}. \tag{19}$$

The variable μ depends only on the phase

$$W=\tau+\xi/V \tag{20}$$

and is a solution of the equation

$$\frac{d\mu}{dW}=-2i\sqrt{P(\mu)}. \tag{21}$$

If μ is known, then the field E can be found with the help of equations which follow from Eqs. (9) and (15):

$$\frac{\partial E}{\partial\tau}=-2if_1E+2i\mu E, \tag{22}$$

$$\frac{\partial E}{\partial\xi}=\langle d\rangle=\frac{i}{2}\left\langle\frac{\mu-\Delta}{a}\right\rangle E=-\frac{i}{2}\left\langle\frac{\Delta}{a}\right\rangle E+\left\langle\frac{1}{4a}\right\rangle 2i\mu E.$$

Hence we see that

$$E=\exp\left(-is_1\tau-\frac{i}{2}\left\langle\frac{\Delta}{a}\right\rangle\xi\right)\tilde{E}, \tag{23}$$

where $\tilde{E}$ depends only on the phase W and satisfies the equation

$$\frac{d\tilde{E}}{dW}=2i\mu\tilde{E}. \tag{24}$$

We will carry out the actual integration of the equations by the method of Ref. 11. The initial condition for the SIT master equations should be matched with the initial condition for Eq. (21) in such a way that identity (10) is satisfied, where f, g, and h are defined in agreement with Eqs. (12) and (14). Therefore it is convenient to parametrize $\mu(\nu)$ in such a way that the indicated equality is satisfied automatically. We take as the parameter ν the quantity

$$\nu=|E^2|/4 \qquad (25)$$

and rewrite the identity (10) in the form

$$[(\lambda-\Delta)^2-(f_1-2\Delta)(\lambda_1-\Delta)+an]^2+\nu[(\lambda-\Delta)^2-(\mu+\mu^*-2\Delta)(\lambda-\Delta)+(\mu-\Delta)(\mu^*-\Delta)]=\prod_{i=1}^{4}[\lambda-\Delta-(\lambda_i-\Delta)]=(\lambda-\Delta)^4-\tilde{s}_1(\lambda-\Delta)^3+\tilde{s}_2(\lambda-\Delta)^2-\tilde{s}_3(\lambda-\Delta)+\tilde{s}_4,$$

where the $\tilde{s}_i$ can be expressed according to the formulas of Viète in terms of $\lambda_i-\Delta$ in the same way as s_i are expressed in terms of λ_i, in particular, $\tilde{s}_4=\Pi(\lambda_i-\Delta)=a^2$. Equating the coefficients of the powers of $\lambda-\Delta$, we obtain the algebraic system

$$\begin{aligned}
&2(f_1-2\Delta)=\tilde{s}_1=s_1-4\Delta,\\
&(f_1-2\Delta)^2+2an+\nu=\tilde{s}_2,\\
&2(f_1-2\Delta)an+\nu(\mu+\mu^*-2\Delta)=\tilde{s}_3,\\
&(an)^2+\nu(\mu-\Delta)(\mu^*-\Delta)=\tilde{s}_4,
\end{aligned} \qquad (26)$$

which coincides, except for the notation, with system (2.6) in Ref. 11. We will implement the solution found there:

$$f_1=\frac{s}{2},\quad an=\frac{s^2}{16}+\frac{p-\nu}{2}, \qquad (27)$$

$$\mu=\frac{s}{2}-\frac{q+i\sqrt{-R(\nu)}}{2\nu}. \qquad (28)$$

Here $R(\nu)$ is the cubic resolvent of $P(\lambda)$:

$$R(\nu)=\nu^3-2p\nu^2+(p^2-4r)\nu+q^2, \qquad (29)$$

whose zeros are expressed in terms of the zeros λ_i of $P(\lambda)$ by the simple symmetric formulas

$$\begin{aligned}
\nu_1&=-\frac{1}{4}(\lambda_1-\lambda_2+\lambda_3-\lambda_4)^2,\\
\nu_2&=-\frac{1}{4}(\lambda_1-\lambda_2-\lambda_3+\lambda_4)^2,\\
\nu_3&=-\frac{1}{4}(\lambda_1+\lambda_2-\lambda_3-\lambda_4)^2,
\end{aligned} \qquad (30)$$

The quantities s, p, q, and r are expressed in terms of the coefficients of $P(\lambda)$ as follows:

$$s=s_1,\quad p=s_2-\tfrac{3}{8}s_1^2,\quad q=\tfrac{1}{2}s_1(s_2-\tfrac{1}{4}s_1^2)-s_3,$$
$$r=s_4+\tfrac{1}{16}s_1^2(s_2-\tfrac{3}{16}s_1^2)-\tfrac{1}{4}s_1s_3. \qquad (31)$$

In writing Eqs. (27) and (28) we have made use of the circumstance, obvious from Eqs. (29) and (30), that p, q, and r do not vary when the zeros λ_i are shifted by the constant term Δ. This also ensures that μ and $\nu=|E|^2/4$ are independent of the frequency offset Δ.

The zeros λ_i comprise, as in Ref. 11, two complex-conjugate pairs

$$\lambda_1=\alpha+i\gamma,\quad \lambda_2=\beta+i\delta,\quad \lambda_3=\alpha-i\gamma,\quad \lambda_4=\beta-i\delta, \qquad (32)$$

whence formulas (30) become

$$\nu_1=-(\alpha-\beta)^2,\quad \nu_2=(\gamma-\delta)^2,\quad \nu_3=(\gamma+\delta)^2. \qquad (33)$$

The quantity ν, positive by definition, varies within the interval $\nu_2\leq\nu\leq\nu_3$, so that μ, according to Eq. (28), describes an oval in the complex plane enclosing the zeros λ_1 and $\lambda_3=\lambda_1^*$ (see Ref. 11). The dependence of ν on the phase W is determined by[11]

$$\frac{d\nu}{d(2W)}=\sqrt{-R(\nu)}. \qquad (34)$$

Integrating this equation, we obtain a simple expression for the dependence of the intensity $|E|^2$ on the phase W:

$$|E|^2=4\nu=4[\nu_3+(\nu_2-\nu_3)\mathrm{sn}^2(\sqrt{\nu_3-\nu_1}W,k)]=4(\gamma+\delta)^2-16\gamma\delta\,\mathrm{sn}^2\{[\sqrt{(\gamma+\delta)^2+(\alpha-\beta)^2}]W,k\}, \qquad (35)$$

where the parameter of the elliptic function k is given by

$$k^2=\frac{\nu_3-\nu_2}{\nu_3-\nu_1}=\frac{4\gamma\delta}{(\gamma+\delta)^2+(\alpha-\beta)^2} \qquad (36)$$

and the initial condition is chosen such that the intensity takes its maximum $4(\gamma+\delta)^2$ at $W=0$. Knowing ν, it is not hard to find n with the help of the second of relations (27).

It is convenient to express the field E in terms of the Weierstrass elliptic functions in analogy with the solution of the nonlinear Schrödinger equation studied in Ref. 11. Formula (35) is then found to correspond to the expression

$$\nu=\tfrac{2}{3}p-4\wp(2W+\omega'), \qquad (37)$$

where ω' is one of the half-periods of the Weierstrass function $\wp$ (we follow the notation and definitions of Ref. 19). Substituting Eqs. (34) and (37) in Eq. (28) gives the following expression for μ:

$$\mu=\frac{s}{4}+\frac{q}{8}\frac{1}{\wp(2W+\omega')-p/6}-\frac{i}{2}\frac{1}{\nu}\frac{d\nu}{d(2W)}. \qquad (38)$$

Integrating Eq. (24), we obtain with the help of Eq. (38)

$$\tilde{E}=2\sqrt{\nu}\exp\left[\frac{is}{2}W+\frac{iq}{8}\int_0^W\frac{d(2W)}{\wp(2W+\omega')-p/6}\right], \qquad (39)$$

where the integral can be calculated and transformed in exactly the same way as was done in Ref. 11. The final expression for the electric field of the wave has the form

$$E(\tau,\xi)=2(\gamma+\delta)\exp\left[-2i(\alpha+\beta)\tau+\frac{i}{2}\left\langle\frac{\Delta}{\sqrt{P(\Delta)}}\right\rangle\xi+i(\alpha+\beta)W-2W\left(\zeta(\chi)-\frac{\eta\chi}{\omega}+\frac{i(\gamma^2-\delta^2)}{2|\alpha-\beta|}\right)\right]\frac{\theta_4(0)\theta_3[(2W+\chi)/2\omega]}{\theta_4(W/\omega)\theta_3(\chi/2\omega)}, \qquad (40)$$

where θ_k is the Jacobi theta function, $\zeta(\chi)$ is the Weierstrass ζ function, and the parameter χ is defined by the relation

$$\sqrt{(\gamma+\delta)^2+(\alpha-\beta)^2}\chi=2i\int_0^{\sin\varphi}\frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}}$$
$$=2iF(\varphi,k'), \tag{41}$$

$k'^2=1-k^2$, and the angle φ has the simple geometric meaning

$$\sin\varphi=\frac{|\alpha-\beta|}{\sqrt{(\alpha-\beta)^2+(\gamma-\delta)^2}}. \tag{42}$$

The polarization d can be found with the help of (15). The expressions obtained above give the general periodic solution of the SIT equations, which depends on the four parameters λ_i, $i=1,2,3,4$.

3. SPECIAL CASES

Let us consider some special cases in which the general solution simplifies substantially.

Let $\beta=\alpha$, i.e., all the λ_i lie on one vertical line. Then, by analogy with the nonlinear Schrödinger equation,[11] we find

$$E(\xi,\tau)=2(\gamma+\delta)\exp\left[-2i\alpha\tau-\frac{i\xi}{2}\left\langle\frac{\alpha-\Delta}{\sqrt{((\alpha-\Delta)^2+\gamma^2)((\alpha-\Delta)^2+\delta^2)}}\right\rangle\right]\times\mathrm{dn}\left((\gamma+\delta)W,\frac{2\sqrt{\gamma\delta}}{\gamma+\delta}\right), \tag{43}$$

where $W=\tau+\xi/V$ and

$$\frac{1}{V}=-\frac{1}{4}\left\langle\frac{1}{\sqrt{[(\alpha-\Delta)^2+\gamma^2][(\alpha-\Delta)^2+\delta^2]}}\right\rangle. \tag{44}$$

If we set $\delta=\gamma$ in these expressions, so that the two pairs of zeros λ_i merge into one pair, then we obtain the well-known one-soliton solution[18]

$$E(\tau,\xi)=4\gamma\exp\left(-2i\alpha\tau-\frac{i\xi}{2}\left\langle\frac{\alpha-\Delta}{(\alpha-\Delta)^2+\gamma^2}\right\rangle\right)\mathrm{sech}[2\gamma(\tau+\xi/V)], \tag{45}$$

where

$$\frac{1}{V}=-\frac{1}{4}\left\langle\frac{1}{(\alpha-\Delta)^2+\gamma^2}\right\rangle=-\frac{1}{4}\int\frac{\tilde{g}(\Delta)d\Delta}{(\alpha-\Delta)^2+\gamma^2}. \tag{46}$$

When we let $\delta\to\gamma$ in Eq. (40), we obtain

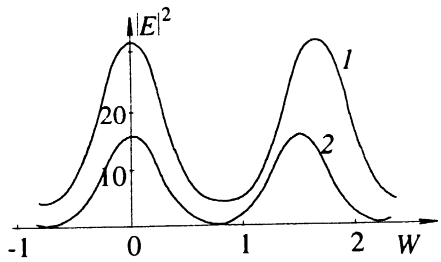

FIG. 1. Distribution of the electric field intensity of a wave for $\lambda_1=1+i$, $\lambda_2=1+2i$ (curve *1*) and $\lambda_1=1+i$, $\lambda_2=2+i$ (curve *2*). The slow evolution of the parameters λ_i (in comparison with the oscillations of the wave field) is described by the Whitham equations.

$$E(\tau,\xi)=4\gamma\exp\left[-i(\alpha+\beta)\tau-\frac{i}{4}\left\langle\frac{\alpha+\beta-2\Delta}{\sqrt{[(\alpha-\Delta)^2+\gamma^2][(\beta-\Delta)^2+\gamma^2]}}\right\rangle\xi\right]\times\mathrm{cn}\left(\sqrt{4\gamma^2+(\alpha-\beta)^2}W,\frac{2\gamma}{\sqrt{4\gamma^2+(\alpha-\beta)^2}}\right), \tag{47}$$

where $W=\tau+\xi/V$ and

$$\frac{1}{V}=-\frac{1}{4}\left\langle\frac{1}{\sqrt{[(\alpha-\Delta)^2+\gamma^2][(\beta-\Delta)^2+\gamma^2]}}\right\rangle. \tag{48}$$

If now we set $\alpha=\beta$, we regain the soliton solution (45). It is clear from the formulas that the period of the wave decreases with increasing field intensity. Hence it follows that the field intensity must be large enough that it will be possible to ignore relaxation processes (see the Introduction).

By way of illustration, Fig. 1 shows the intensity distribution $|E|^2$ in the wave for two sets of values of the parameters: $\lambda_1=1+i$, $\lambda_2=1+2i$ (curve *1*) and $\lambda_1=1+i$, $\lambda_2=2+i$ (curve *2*).

4. MODULATED WHITHAM EQUATIONS

In weakly inhomogeneous periodic solutions, the parameters λ_i become slowly varying functions of the spatial coordinate and time. The smallness of the variation of the λ_i over one wavelength and during one period of the wave allows us to average the equations of motion for λ_i over the rapid oscillations and obtain what we may call the modulated Whitham equations[15,16] for the case of a periodic solution of the SIT equations.

To derive the Whitham equations, we will make use of the method developed in Refs. 10, 12, 13, and 20 and based on writing the generating function of the conservation laws of the equations, given as a condition of consistency of linear systems (5) and (9), in the form

$$\frac{\partial}{\partial\xi}\left(\frac{G}{g}\right)-\frac{\partial}{\partial\tau}\left(\frac{B}{g}\right)=0. \tag{49}$$

Substituting Eqs. (6), (7), and (14) in Eq. (49) and making use of relation (16), we rewrite Eq. (49) in the form

$$\frac{\partial}{\partial \xi}\left(\frac{\sqrt{P(\lambda)}}{\lambda-\mu}\right)-\frac{\partial}{\partial \tau}\left(\sqrt{P(\lambda)}\left\langle\frac{\mu-\Delta}{4a(\lambda-\mu)(\lambda-\Delta)}\right\rangle\right)=0, \quad (50)$$

where the functions f, g, and h are normalized by the condition $f^2-gh=1$. We average (50) over one period

$$\frac{2\pi}{\Omega}=\frac{1}{2}\int\frac{d\mu}{\sqrt{-P(\mu)}}. \quad (51)$$

and, to distinguish this average from the average over the inhomogeneous broadening, we will denote this average by a horizontal bar over the quantity to be averaged. The condition of vanishing of the singular terms which arise as a result of differentiating $\sqrt{P(\lambda)}$ with respect to the "slow" variables T and X gives the equations

$$\overline{\left(\frac{1}{\lambda_i-\mu}\right)}\frac{\partial\lambda_i}{\partial X} -\left[\left\langle\frac{1}{4a}\right\rangle\overline{\left(\frac{1}{\lambda_i-\mu}\right)}-\left\langle\frac{1}{4a(\lambda_i-\Delta)}\right\rangle\right]\frac{\partial\lambda_i}{\partial T}=0. \quad (52)$$

Thus we obtain the Whitham equations in diagonal Riemannian form

$$\frac{\partial\lambda_i}{\partial X}+\frac{1}{v_i}\frac{\partial\lambda_i}{\partial T}=0, \quad i=1,2,3,4, \quad (53)$$

where the characteristic velocities are given by

$$\frac{1}{v_i}=-\left\{\left\langle\frac{1}{4a}\right\rangle-\left\langle\frac{1}{4a(\lambda_i-\Delta)}\right\rangle\left[\overline{\left(\frac{1}{\lambda_i-\mu}\right)}\right]^{-1}\right\}. \quad (54)$$

Invoking the obvious relations [see Eqs. (51) and (19)]

$$\overline{\left(\frac{1}{\lambda_i-\mu}\right)}=\frac{2}{\Omega}\frac{\partial\Omega}{\partial\lambda_i}, \quad \left\langle\frac{1}{4a(\lambda_i-\Delta)}\right\rangle=-2\frac{\partial}{\partial\lambda_i}\frac{1}{V},$$

we rewrite Eq. (54) in the simple form

$$\frac{1}{v_i}=-\left(1+\frac{\Omega}{\partial_i\Omega}\partial_i\right)\frac{1}{V}, \quad i=1,2,3,4, \quad \partial_i\equiv\partial/\partial\lambda_i. \quad (55)$$

Introducing the wave vector K of the nonlinear wave

$$K=\frac{\Omega}{V}, \quad (56)$$

we rewrite Eq. (55) in the universal form

$$v_i=\left(1+\frac{K}{\partial_i K}\partial_i\right)V, \quad i=1,2,3,4, \quad (57)$$

analogous to that derived earlier for other equations.[12,13,20–23] In all these cases, the same differential operator appears in the expressions for the Whitham velocities, acting on the phase velocity of the nonlinear periodic wave. In essence, formula (57) signifies conservation of "the number of waves"

$$\frac{\partial\Omega}{\partial X}=\frac{\partial K}{\partial T} \quad (58)$$

when the wave number K and the phase velocity $V=\Omega/K$ are expressed in terms of the Riemann invariants λ_i (see Refs. 22 and 23).

Note that the Whitham equations for self-induced transparency we have derived contain as special cases many previously considered systems. If inhomogeneous broadening is absent, so that the distribution function $\tilde{g}(\Delta)=\delta(\Delta-\Delta_0)$ and the average over Δ vanishes, then as $\Delta_0\to 0$ we obtain from Eqs. (57) the Whitham equations for a so-called *AB* system,[21] and as $\Delta_0\to\infty$, Eq. (57) goes over to the Whitham equation for the nonlinear Schrödinger equation.[24] It can be shown that this generality is due to the fact that the indicated integrable equations correspond in the inverse scattering transform method to the same Zakharov–Shabat spectral problem.

5. DISCUSSION

We have obtained a periodic solution of the SIT equations which depends on all four parameters of the problem, λ_i, $i=1-4$. In the weakly inhomogeneous case, the parameters λ_i play the role of the Riemann invariants whose evolution is described by the Whitham equations. The fact that the parameters λ_i are complex points to the instability of the periodic solutions, analogous to the instability of the periodic waves described by the nonlinear Schrödinger equation. However, as the modulus of the elliptic functions approaches unity, the instability growth rate approaches zero, and the solitons which arise in the one-dimensional case are stable. Hence, a long light pulse is unstable against decay into solitons in a resonant medium. The qualitative picture of the evolution of the leading edge of the pulse is analogous to that considered in Ref. 10, and the evolution of the perturbation far from the leading edge is analogous to the case of the nonlinear Schrödinger equation studied in Ref. 25. Note that the indicated modulation pulse instability can serve as a means of experimental verification of the theory. Another qualitative confirmation is the slow increase with time of the wavelength in a periodic pulse sequence detected in Ref. 3.

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