Equilibrium and nonequilibrium mean-field dynamics of quantum spin cluster

M. Chertkov

Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

I. Kolokolov

Budker Institute for Nuclear Physics, 630090 Novosibirsk Russia and INFN, Sezione di Milano, 20133 Milano, via Celoria 16, Italy

(Submitted 22 June 1994)
Zh. Eksp. Teor. Fiz. **106**, 1525–1541 (November 1994)

Using the functional integration method we compute exactly the time-dependent correlation functions of local spins in a finite quantum Heisenberg magnet with a position-independent exchange between spins. Explicit expressions for such correlation functions for an arbitrary number (N) of spins in the cluster and for all temperatures are presented. The spin correlation function for a cluster placed in an external time-dependent magnetic field is obtained under the same conditions. At large N the correlation function is found to have a Gaussian bump and a nonvanishing tail at large times. It is in good agreement with the finite-time domain of the experimental curve. A possible application of the exact results, which form a starting dynamical mean-field approximation for long-range magnets and an accompanied physical picture, are discussed. © *1994 American Institute of Physics.*

1. INTRODUCTION

The theory of quantum magnets stretched due to a very long period in the framework and on the basis of the mean-field approach is well known.[1–3] Magnets that allow a consistent mean-field approximation must be characterized by a large number of spins in the exchange sphere. But even in the case of the nearest-neighbor exchange, the inverse number of those "nearest neighbors" turns out to be reasonably small parameters at least in 3D. The main purpose of the zeroth-order approximation in the method of static homogeneous mean field is to replace many-spin exchange of a fixed spin with its surrounding onto an action of an auxiliary homogeneous magnetic field on the spin. This field, in turn, must be expressed, in terms of the average value of the spin with respect to a corresponding density matrix. Formally, it results in the fact that we can replace the Heisenberg Hamiltonian (we restrict the analysis here to the Heisenberg type of exchange only) with a cluster Hamiltonian

$$\hat{H}_0 = -\frac{J}{2N}\,\hat{\sigma}^2, \tag{1}$$

where N is the number of spins in the exchange sphere of an original magnet, $\hat{\sigma}=\Sigma_j^N \hat{\mathbf{s}}_j$ is the total spin operator of the system, and J is the strength of exchange between the spin and its surroundings. The position-independent feature of the exchange between spins in the model renders spatial correlations homogeneous, allowing one to express any spin correlator in terms of the autocorrelation function. The case of an external, spatially homogeneous, time-independent magnetic field is also described by the method, with the addition of $\hat{\Delta}=-h\hat{\sigma}$ to the Hamiltonian (1). The standard approach, which works well at low temperatures, at least for 3D magnets, is to build on the basis of this static mean-field theory a spin-wave approximation.[4] But in the vicinity of the Curie point, in the paramagnetic phase those techniques have failed together with the proposition about the presence of well-defined excitations. In the vicinity of the Curie point from the ordered phase side the situation can be improved by means of the first-order static approximation with respect to $1/N$ (Refs. 3 and 5), but in the paramagnetic phase anything "static" cannot help, because the average spin (or the average spin of the sublattice in the case of an antiferromagnet) is zero. We are therefore dealing with a problem of finding a dynamic mean-field approximation, which would describe the paramagnetic phase in an intermediate interval of times, and which would improve the static mean-field theory in the ordered phase.

In this paper we present for the first time a nonequilibrium dynamical mean-field theory for a long-range quantum magnet in a homogeneous (but arbitrary changing in time) magnetic field. Our results are rigorous in the sense that we solved explicitly the dynamical problem for the cluster of N quantum spins 1/2 whose pair interactions are independent of separations, with the Hamiltonian

$$\hat{H}(t)=\hat{H}_0+\Delta(t),\Delta(t)=-h(t)\hat{\sigma}. \tag{2}$$

A functional integral formalism is used with this cluster model to express the transverse pair autocorrelator in terms of a matrix element for one spin equal to 1/2. This reduction can be performed at an arbitrary moment of time, temperature, number of spins, and arbitrary magnetic field varying in time. In the absence of an external magnetic field, as in the case of some specific feature of its temporal behavior, this matrix element is calculated explicitly. The sketch of the equilibrium variant (no magnetic field) of the present theory was published elsewhere.[6] We show that the cluster approximation, formed by the model (1) for a real magnet, explains some short- and intermediate-time (up to a time of a spin diffusion regime formation) experimental measurements[7] in the paramagnetic phase.

 1063-7761/94/110824-09$10.00

Let us formulate the problem. We consider the transverse pair autocorrelation function, which is defined as follows:

$$\mathcal{K}[\mathbf{h}(t)](T;\beta;N)$$

$$\equiv\frac{1}{2}\frac{\mathrm{Tr}[(\hat{s}^+(T)\hat{s}^-(0)+\hat{s}^-(0)\hat{s}^+(T))e^{-\beta\hat{H}(\mathbf{h}_0)}]}{\mathrm{Tr}[e^{-\beta\hat{H}(\mathbf{h}_0)}]}. \tag{3}$$

Here $\hat{s}^{\pm}(T)$ is the usual notation for up- and down- spin operators in the Heisenberg representation $\hat{s}^{\pm}(T)=\mathcal{T}\exp[-i\int_0^T dt\hat{H}(t)]\hat{s}^{\pm}\tilde{\mathcal{T}}\exp(i\int_0^T dt\hat{H}(t))$, $\hat{s}^{\pm}=\hat{s}^x\pm i\hat{s}^y$; $\mathcal{T}\exp$ and $\tilde{\mathcal{T}}\exp$ are time-chronological and antichronological exponentials, respectively; $\hat{H}(\mathbf{h}_0)\equiv\hat{H}_0-\mathbf{h}\hat{\boldsymbol{\sigma}}$, and $\mathbf{h}_0\equiv\mathbf{h}(t=0)$. The lattice site index of the local spin operators $\hat{s}^{\pm}$ here and below is omitted and the coordinate axes are chosen. Our aim is to rigorously calculate the many-spin trace (3) for an arbitrary function $\mathbf{h}(t)$, $0<t<T$ at arbitrary values of time T, of the inverse temperature β, and of the number of spins N.

The article is structured as follows. In Sec. 2 we develop the functional integral method to study the real time evolution of the quantum spin cluster. Using this method, we obtain explicit expressions for the correlator (3) without an external field. The case of antiferromagnetic exchange is also discussed in Sec. 2. Section 3 is devoted to the spin cluster in an external, uniform magnetic field. We show that the field-dependent part of the evolution operators is factorized. However, the effect of magnetic field in the initial Gibbs distribution turns out to be very nontrivial. We present the expression for (3) in two cases. First, when the initial field $\mathbf{h}$ is constant in direction and, second, for a magnetic field which rotates uniformly along some axes with a constant angular velocity. In the last part of the article we discuss the correspondence between our results and the published experimental data. Appendices A and B contain some technical details.

2. EQUILIBRIUM CLUSTER DYNAMICS WITHOUT AN EXTERNAL FIELD

Let us first describe why even without a magnetic field it is difficult to calculate the autocorrelator (3). The spectrum of many-spin Hamiltonian (1) commutates with the total spin of the system $\hat{\sigma}$. This means that we can classify the eigenstates of the Hamiltonian by the value of the spin angular momentum L, which can take the positive value $L=N/2$, $N/2-1,...$, and its projection onto the z axis M, which can take the values $M=-L,-L+1,...,L$. The energy level $E_L=-JL(L+1)/2N$, which corresponds to the value of the total spin L, thus has $(2L+1)$-fold degeneracy. This $|L,M\rangle$ representation of the eigenstates would be very convenient for a calculation of a diagonal matrix element (or a trace) which is defined in terms of the total spin operators only. However, for the calculation of a diagonal matrix element of a one-spin operator, which does not commute with the Hamiltonian, as in (3), this representation is not suitable. In addition, this eigenstate expansion would represent the result as a sum of exponentials oscillating in time, which is inappropriate for an analysis of long-term behavior.

To avoid the effect of mixing of the $|L,M\rangle$ eigenstates by the one-spin operators $\hat{s}^{\pm}$, we will formulate below a technique that maps the initial $\hat{\sigma}$ phase space into an unbounded functional phasespace. The resulting one-particle quantum mechanics will be shown to be exactly solvable. We note that the method which we use has nothing to do with the Bethe-ansatz or inverse-scattering approaches.

Infinite Temperature

As a first equilibrium step we consider the transverse pair autocorrelation function (3) at $\mathbf{h}=0$ and infinite temperature $\beta=0$, when it takes the form

$$\mathcal{K}^0(T;N)=\mathrm{Tr}[e^{-iT\hat{H}_0}\hat{s}^+e^{iT\hat{H}_0}\hat{s}^-]. \tag{4}$$

Hubbard–Stratonovich transformations of the evolution operators from the definition (4)

$$e^{-it\hat{H}_0}\propto\int\mathcal{D}\varphi_1(t)\exp\left(-\frac{iN}{2}\int_0^T dt\varphi_1^2\right)\prod_j\mathcal{A}_j^{(1)}(T), \tag{5}$$

$$e^{it\hat{H}_0}\propto\int D\varphi_2(t)\exp\left(\frac{iN}{2}\int_0^T dt\varphi_2^2\right)\prod_j\mathcal{A}_j^{(2)}(T), \tag{6}$$

$$\mathcal{A}_j^{(1,2)}(T)=\mathcal{T}\exp\left[i\int_0^T dt\varphi_{1,2}(t)\hat{\mathbf{s}}_j\right] \tag{7}$$

factorize the initial trace to a product of local traces (here and below we measure t in units of J). These equalities are products of the operator-valued Gaussian integrals. When the time discretization interval goes to zero, the noncommutativity of the spin operators in each "infinitesimal" factor can be ignored. However, the properties of the total product are determined by this noncommutativity. Because of the absence of a time dependence in the Hamiltonian (1), we had no time-ordered exponentials in (4) but only the usual kind. After the Hubbard-Stratonovich transformation, however, the $\mathcal{T}$ exponents appeared again [in $\mathcal{A}_j^{(1,2)}(T)$]. This stems from the noncommutativity of the total spin of the system $\hat{\sigma}$ with an arbitrary one-site operator $\hat{s}_i$ and it reflects the multiplicity (in discrete time) of the functional measures in (5) and (6). At first glance, these time-ordered exponentials create some difficulties. Indeed, we cannot calculate them explicitly as functionals of $\varphi_{1,2}$. However, there is a substitution which recasts the time-ordered exponents (7) into the products of the usual exponents (see Refs. 8–10 and Appendix A) by means of the following change of variables $\varphi\to(\rho,\psi^{\pm})$ in the functional integrals (5) and (6):

$$\varphi_{1,2}^z=\rho_{1,2}-2\psi_{1,2}^+\psi_{1,2}^-,$$

$$\varphi_1^-=\psi_1^-,\quad \varphi_1^+=-i\dot{\psi}_1^++\rho_1\psi_1^+-(\psi_1^+)^2\psi_1^-,$$

$$\varphi_2^+=\psi_2^+,\quad \varphi_2^-=i\dot{\psi}_2^-+\rho_2\psi_2^--(\psi_2^-)^2\psi_2^+, \tag{8}$$

where $\varphi^{\pm}=(\varphi^x\pm i\varphi^y)/2$. The map (8) faces the above-mentioned difficulty that it is impossible to express T exponentials in terms of the usual function of initial variables φ. In general, ρ and $\psi^{\pm}$ cannot be expressed solely in terms of φ from (8). But to perform changes of variables $\varphi\to(\rho,\psi^{\pm})$ in the functional integrals (5) and (6), it is not necessary to

invert (8). The map (8) contains time derivatives of the fields ψ_1^+ and ψ_2^-. It should therefore be supplemented with the initial or boundary conditions. It was shown[9] that only the initial conditions provide a one-to-one correspondence between these two sets of fields. Moreover, initial conditions are natural for the real-time evolution problem. In this paper we impose the constraints

$$\psi_1^+(0)=0, \quad \psi_2^-(0)=0. \tag{9}$$

Explicit expressions for $\mathscr{A}^{(1,2)}(T)$ and the Jacobian of the map (8) in terms of the new variables ρ and $\psi^{\pm}$ are given in Appendix A [see Eqs. (A3), (A4), and (A7), respectively]. After a calculation of the N local traces, they yield for (4) the following functional representation for $\mathscr{K}^0(T;N)$:

$$\mathscr{K}^0=\text{const}\int \mathscr{D}\rho_{1,2}\mathscr{D}\psi_{1,2}^{\pm}e^{\mathscr{S}_0}\mathscr{B}^{N-1}\mathscr{C}, \tag{10}$$

$$\mathscr{S}_0=-\frac{iN}{2}\int_0^T[\rho_1^2-\rho_2^2-4i\dot{\psi}_1^+\psi_1^- -4i\psi_2^+\dot{\psi}_2^-]dt + \frac{i}{2}\int_0^T(\rho_1-\rho_2)dt, \tag{11}$$

$$\mathscr{B}=\mathrm{Tr}[\mathscr{A}^{(1)}(T)\mathscr{A}^{(2)}(T)] =2\cos\left[\frac{i}{2}\int_0^T(\rho_1+\rho_2)dt'\right]+e^{i/2\int_0^T(\rho_1-\rho_2)dt'} \times\left(\psi_1^+ +i\int_0^T\psi_2^+e^{i\int_0^t\rho_2dt'}dt\right) \times\left(-\psi_2^- +i\int_0^T\psi_1^-e^{-i\int_0^t\rho_1dt'}dt\right),$$

$$\mathscr{C}=\mathrm{Tr}[\mathscr{A}^{(1)}(T)\hat{s}^+\mathscr{A}^{(2)}(T)\hat{s}^-]=e^{i/2\int_0^T(\rho_1-\rho_2)dt'}. \tag{12}$$

Normalization of the functional integral (11) depends on N only and it can be fixed by the condition $\mathscr{K}^0(T=0)=1$.

The fields ψ_1^+ and ψ_2^- enter the expression for $\mathscr{B}$ with the final time argument T only. On the other hand, the bare propagators $\langle\psi_1^+(T)\psi_1^-(t)\rangle\propto\langle\psi_2^-(T)\psi_2^+(t)\rangle\propto\theta(T-t)=1$, given by the action $\mathscr{S}_0$, do not depend on $t<T$. This means that the functional integral in (11) remains the same if the fields $\psi_2^+(t'),\psi_1^-(t')$ are replaced by $\psi_2^+(T)$ and $\psi_1^-(T)$, respectively. Thus the functional integral over the fields $\psi_1^{\pm}$ and $\psi_2^{\pm}$ can be replaced by the usual integral by simply omitting the time arguments and by replacing the measure:

$$\mathscr{D}\psi_{1,2}^{\pm}e^{\mathscr{S}_0}\to d\psi_1^{\pm}d\psi_2^{\pm}\ \exp(-2N\psi_1^+\psi_1^- -2N\psi_2^+\psi_2^-). \tag{13}$$

The dynamics of the ρ fields remains a very nontrivial one. It turns out, however, that we can calculate explicitly the resulting functional integral. We show in Appendix B that the problem transforms to a calculation of a matrix element of the accompanied one-dimensional quantum mechanics with the Hamiltonian

$$\hat{\mathscr{H}}_{\mathrm{ac}}=-\frac{1}{2N}\partial_\xi^2+\frac{N}{2}e^{-\xi}. \tag{14}$$

We note that a similar matrix element with respect to exactly the same quantum mechanics appears in the calculation of the multipoint density correlator in the $1D$ localization.[12] The wave function (and all the matrix elements) is calculated exactly. In total we therefore obtain the following answer for the equilibrium transverse pair correlator at infinite temperature

$$\mathscr{K}^0(T;N)=\frac{2}{3}\left\{\frac{1}{2}+\left(\cos\frac{T}{2N}\right)^N-\frac{1}{N}\left[\left(\cos\frac{T}{2N}\right)^N-1\right] -(N-1)\sin^2\frac{T}{2N}\left(\cos\frac{T}{2N}\right)^{N-2}\right\}. \tag{15}$$

$\mathscr{K}^0(T;N)$ is periodic in time; starting with unity at zero time it relaxes to a minimum, then recovers up to 1/3 and becomes unity again at $T_{\mathrm{per}}=4\pi N$.

In the limit of large N, when T_{per} is not reached, an intermediate asymptotic behavior takes place. Thus, at $T=\tau\sqrt{N}$, $\tau\sim1$, $N\gg1$ we have a smooth relaxation, which depends on N in terms of τ only:

$$\mathscr{K}^0(T)\approx\frac{1}{3}\left\{1+2e^{-\tau^2/8}-\frac{\tau^2}{2}e^{-\tau^2/8}\right\}. \tag{16}$$

The result (16) is characterized by a Gaussian bump at small times, a minimum at intermediate times, and a nonvanishing tail in the limit $\tau\to\infty$. We note that the Gaussian bump, which takes place, as we will see below, at lower temperatures, has also been obtained by Belinicher and L'vov for the Green's function of dispersion-free magnons in a long-range quantum magnets.[3] The asymptotic value of $K^0(\tau)$ at $\sqrt{N}\ll\tau\ll N$ (1/3 at $N\to\infty$) stems from the eigenstates of the initial quantum mechanics (1) with a zero total spin $\sigma=0$. The nonvanishing tail is just an artifact of the nonergodicity of our model. Thus, in the case of a more realistic long-range model, the 1/3-plateau can be realized in the thermodynamic limit only as an intermediate-time asymptotic relation.

The present result gives an approximation for real long-range quantum magnets. It is zeroth order with respect to the small parameter $1/N$ cluster approximation (the number of spins in the exchange sphere plays the role of N). Experimental observations[7] supports this statement. The experimental curve for the transverse autocorrelation function in $Rb_2CuBr_4\times H_2O$ (which is three-dimensional BCC, $s=1/2$) repeats dependence (16) with a very good agreement in the interval of times from zero up to the instant of time $\mathscr{K}^0(T)$ from (16) reaches its minimal value. Later in time the dependence (16) deviates from the experimental curve, which restores the value approximately at 0.05 units to cross over further into a spin-diffusion tail at the largest times.

The behavior (16) looks similar to the result of calculations[13] for spin in a classical random field. In our case, however, there was no external randomness at all. The constant of Gaussian relaxation, which was external in Ref. 13, is defined in our analysis by means of the dynamics itself.

In a recent paper,[14] we investigated the long-term dynamics of an arbitrary-exchange quantum Heisenberg model at the infinite temperature. It was shown that the quantum

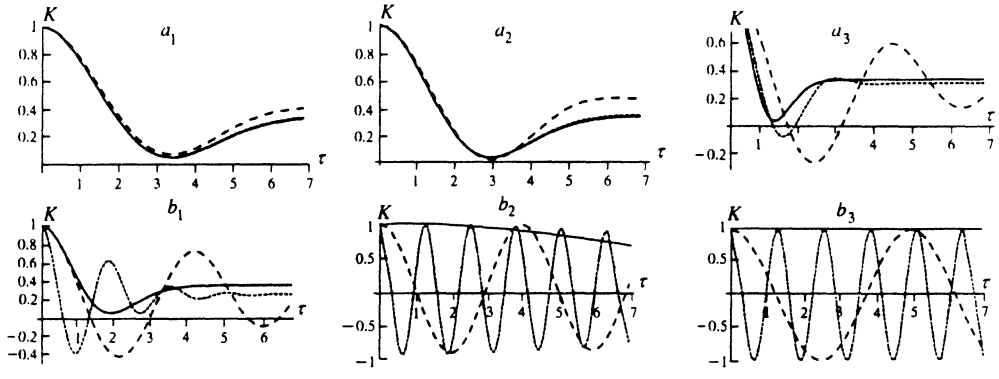

FIG. 1. Transverse pair autocorrelation function versus τ ($\tau=t/\sqrt{N}$, t is measured in units of J) at various temperatures: in the paramagnetic phase (a_1) $\beta=0$, (a_2) $\beta=1$, (a_3) $\beta=3.5$; in the ferromagnetic phase (b_1) $\beta=4.5$, (b_2) $\beta=10$, (b_3) $\beta=\infty$. The three plots correspond to different number of spins in the cluster: the straight line-asymptotic in the limit $N\to\infty$; the long-dashed line $-N=10$; the dashed line $N=100$.

spin pair correlation function (in the paramagnetic phase due to unbroken symmetry of the Hamiltonian it is a transverse correlation function multiplied by 3/4) is equal to the correlation function of a classically evaluated vector field $\phi_k(t)$

$$\dot{\phi}_k=\sum_j J_{kj}[\phi_k\times\phi_j], \tag{17}$$

averaged over the initial conditions $\phi_k(0)=p_k$ with respect to the Gaussian measure

$$\prod_k d\mathbf{p}_k \exp\left\{-\frac{1}{2s(s+1)}\sum_i \mathbf{p}_i^2\right\}. \tag{18}$$

This "classical" problem, which is strongly nonlinear at an arbitrary exchange J_{ij}, becomes linear and exactly solvable for the position-independent exchange, $J_{ij}=J/N$. In this case the right-hand side of (17) is $J[\phi_k\times\mathbf{P}]/N$, where $\mathbf{P}=\Sigma_k\mathbf{p}=\Sigma_k\phi$ is the integral of motion. Classical motion of a spin turns out to be just the uniform precession around the total spin of the system. It yields the same answer (16) for spin $s=1/2$ at $N\gg1$, $T=\tau\sqrt{N}$ for the transverse autocorrelation function. We therefore see that (first) a good physical picture results in (16) and (second) the transition to the classical model is valid not only at a large enough time,[14] but also for a long enough exchange rate.

Paramagnetic Phase Dynamics at Finite Temperatures

It is possible to show that the approach taken in (15) is generalized to finite temperatures. Substitution of $T_1=T-i\beta$ instead of T in the first exponential on the right-hand side of (4) produces $\mathcal{K}^+(T;\beta;N)$, whose real part gives $\mathcal{K}(T;\beta;N)$ (3). Omitting the details of the calculations in Appendix B (they are performed exactly in the manner discussed above for the infinite temperature), we write the result

$$\mathcal{K}^+(T;\beta;N)=\frac{e^{\beta/8N}}{\sqrt{\beta}Z(\beta)}\int_{-\infty}^{+\infty}dx(\cosh x)^N e^{-2x^2N/\beta}$$
$$\times\left[\frac{2Ne^{-x-\beta/8N}}{\beta}\left(\frac{4x^2N}{\beta}-x-1\right)\right.$$
$$-4\left(\frac{1}{4}+\frac{N}{\beta}-\frac{4x^2}{\beta^2}\right.$$
$$\left.+\frac{(4ixN+T-i\beta)(T-i\beta)}{\beta^2}\right)$$
$$\left.\times\exp\left(\frac{(T-i\beta+2ixN)(T-i\beta)}{2N\beta}\right)\right], \tag{19}$$

where $Z(\beta)$ is the partition function defined, for example, from this expression under the condition Re $[\mathcal{K}^+(0;\beta;N)]=1$ [note that expression (19) is valid in the entire temperature interval, not just in the paramagnetic phase).

The saddle-point equation

$$\tanh t^*=4t^*/\beta \tag{20}$$

for $\beta>4$ has nonzero solutions corresponding to the ferromagnetic phase. At the end of Sec. 3, we consider this low-temperature region in the context of a more-general case of a finite external field. We now set $\beta<4$, $(4-\beta)N\gg1$ and $t^*=0$. The saddle-point calculation of the integral (19) gives us the following expression for the correlation function in the paramagnetic phase:

$$\mathcal{K}(T;\beta)\approx\frac{1}{3}\left\{1+2\left(1-\frac{\tau^2}{4-\beta}\right)e^{-\tau^2/2(4-\beta)}\right\}. \tag{21}$$

In the case $\beta<4$ we have $Z(\beta)\propto(4-\beta)^{-3/2}$. We thus conclude that at $\beta=4$ there is a peculiarity of the usual phase-transition type (of course, there is a phase transition only in the limit of a large number of spins, $N\gg1$). It is clear that the static critical exponent 3/2 is the mean-field (by construction) exponent, but its evaluation is useful for a control of the complicated dynamical calculations. We see that the squared inverse time of the Gaussian relaxation (21) goes linearly to infinity as $4-\beta\to0$.

The results for $\mathcal{K}(T;\beta;N)$ [defined in (19) in the entire temperature interval] are illustrated graphically in Fig. 1.

The finite range of the exchange leads to the existence of propagating excitations. In the ferromagnetic phase their contribution to thermodynamic quantities can be found by perturbation theory methods.[2,5,15] However, one cannot maintain this perturbation approach (at least directly) in studying the temporal correlations, because of the strong influence of the dynamic mean-field background.[9,16–18] The exact treatment of the latter would allow us to reach a new understanding of the problem.

Antiferromagnet Cluster

The position-independent antiferromagnetic exchange is described by the Hamiltonian (1), where J has a negative value. In the expression (19) β and T are measured in units of J. Thus, performing an analytical continuation $\beta\to-\beta$ and $T\to-T$ in (19), we obtain an analogous representation for a transverse correlation function in an antiferromagnet cluster, where β and T are measured in units of $|J|$. In moving from the positive semiaxis of β to the negative one, we must rotate the line of the dx-integration toward the imaginary axis. In the thermodynamic limit $N/\beta\gg1$, $T=\tau\sqrt{N}$ we obtain

$$\mathscr{K}(T;\beta)\approx\frac{1}{3}\left\{1+2\left(1-\frac{\tau^2}{4+\beta}\right)e^{-\tau^2/2(4+\beta)}\right\}. \tag{22}$$

There is no phase transition in this case. At the strictly zero temperature limit $\beta=\infty$, only states with a zero total spin survive. Each such state $|0\rangle$ satisfies the equality

$$\hat{H}_0\hat{s}_j^-|0\rangle=-\frac{J}{N}\hat{s}_j^-|0\rangle, \tag{23}$$

where j is an arbitrary lattice site number. This immediately gives

$$\mathscr{K}^+(T;\beta=\infty,N)=\exp\left(i\frac{T}{N}\right). \tag{24}$$

3. CLUSTER IN A MAGNETIC FIELD

Let us first consider the value

$$\mathscr{K}^+[\mathbf{h}(t)]=\frac{\mathrm{Tr}[\hat{s}^+(T)\hat{s}^-e^{-\beta\hat{H}(\mathbf{h}_0)}]}{\mathrm{Tr}[e^{-\beta\hat{H}(\mathbf{h}_0)}]}. \tag{25}$$

We recall that the time dependence of the operator $\hat{s}^+(T)$ is governed by the Hamiltonian $\hat{H}(t)$, $0<t<T$ [see Eq. (2)], the same as the initial value h_0 of the magnetic field determines the initial Gibbs density matrix. The correlation function (25) can be considered as a retarded Green's function for spin operators. In general, the Hamiltonian $\hat{H}(t)$ at some instant of time does not commute with itself. But the perturbation $\Delta(t)$ and the Heisenberg Hamiltonian (1), which does not depend on time, commute with each other. This means that we can rewrite (25) in the following way:

$$\mathscr{K}^+[\mathbf{h}(t)]=\frac{\mathrm{Tr}[e^{-iT\hat{H}_0}\hat{\tilde{s}}^+e^{iT\hat{H}_0}\hat{s}^-e^{-\beta\hat{H}_0}e^{-\beta\mathbf{h}_0\hat{\varphi}}]}{Z(\mathbf{h}_0)},$$

$$\hat{\tilde{s}}^+(T)=\mathscr{T}e^{i\int_0^T\mathbf{h}(t)\hat{s}dt}\hat{s}^+\tilde{\mathscr{T}}e^{-i\int_0^T\mathbf{h}(t)\hat{s}dt},$$

$$Z(\mathbf{h}_0)=\mathrm{Tr}[e^{-\beta\hat{H}}e^{-\beta\mathbf{h}_0\hat{\sigma}}]\ . \tag{26}$$

We direct the initial field $\mathbf{h}_0$ along the z axis: $\mathbf{h}_0\hat{\sigma}=h_0\hat{\sigma}_z\equiv h_0\Sigma_j\hat{s}_j^z$. There is a unitary operator that transforms s^- into s^+:

$$\hat{P}=2^N\prod_j\hat{s}_j^x,\quad \hat{P}\hat{s}^-\hat{P}=\hat{s}^+. \tag{27}$$

It allows us to establish a relationship between the correlation function (25) and

$$\mathscr{K}^-[\mathbf{h}(t)]=\frac{\mathrm{Tr}[\hat{s}^-\hat{s}^+(T)e^{-\beta\hat{H}(\mathbf{h}_0)}]}{\mathrm{Tr}[e^{-\beta\hat{H}(\mathbf{h}_0)}]}. \tag{28}$$

Substituting the identity $\hat{P}^2=1$ under the sign of the trace in (25) and performing the Hermitian conjugation, we obtain

$$\mathscr{K}^-[\mathbf{h}(t)]=(\mathscr{K}^+[\tilde{\mathbf{h}}(t)])^*, \tag{29}$$

where $\tilde{\mathbf{h}}=(h_x,-h_y,-h_z)$. Like in the equilibrium case, we discuss first the case of the infinite temperature (specifically, the unit density matrix). Making the Hubbard-Stratonovich transformations (5)–(7) and performing the change of variables (8), we obtain the expression which differs from Eq. (11) only in the factor $\mathscr{C}$. Instead of $\mathscr{C}$ in (11), we have $\mathscr{C}[\mathbf{h}(t)]$ for $\mathscr{K}[\mathbf{h}(t)](T;\beta=0;N)$, where

$$\mathscr{C}[\mathbf{h}(t)]=\mathrm{Tr}[\mathscr{A}^{(1)}\hat{\tilde{s}}^+\mathscr{A}^{(2)}\hat{s}^-]$$
$$=\mathscr{C}\langle\uparrow|\exp\left(\hat{s}^+i\int_0^T dt\psi_1^-(t)e^{-i\int_0^t\rho_1(t')dt'}\right)\hat{\tilde{s}}^+(T)$$
$$\times\exp(\hat{s}^+\psi_2^-)|\downarrow\rangle. \tag{30}$$

Terms having nonzero "cumulative charge" with respect to the phase transformation $\psi^\pm\to e^{\pm i\alpha}\psi^\pm$ give a zero contribution to the result. Thus, $\mathscr{C}[\mathbf{h}(t)]$ can be replaced by

$$\mathscr{C}[\mathbf{h}(t)]\to\mathscr{C}\langle\uparrow|\hat{\tilde{s}}(T)|\downarrow\rangle=\mathscr{C}\langle\downarrow|\mathscr{T}e^{i\int_0^T\mathbf{h}(t)\hat{s}dt}|\downarrow\rangle$$
$$\times\langle\uparrow|\tilde{\mathscr{T}}e^{-i\int_0^T\mathbf{h}(t)\hat{s}dt}|\uparrow\rangle \tag{31}$$

and

$$\mathscr{K}^+[\mathbf{h}(t)](T;\beta=0;N)=\mathscr{K}^+(T;\beta=0;N)$$
$$\times\langle\downarrow|\mathscr{T}e^{i\int_0^T\mathbf{h}(t)\hat{s}dt}|\downarrow\rangle$$
$$\times\langle\uparrow|\tilde{\mathscr{T}}e^{-i\int_0^T\mathbf{h}(t)\hat{s}dt}|\uparrow\rangle. \tag{32}$$

Returning to the case of finite temperatures, we note that the initial field $\mathbf{h}_0$ directed along the z axis does not destroy the phase invariance of the averaging measure. Thus the field-dependent part of the evolution operator can be factorized in this case:

$$\mathscr{K}^+[\mathbf{h}(t),h_0](T;\beta;N)=\mathscr{K}^+_{h_0}(T;\beta;N)\alpha[\mathbf{h}(t)], \tag{33}$$

where

$$\alpha[\mathbf{h}(t)]=\langle\downarrow|\mathscr{T}e^{i\int_0^T\mathbf{h}(t)\hat{s}dt}|\downarrow\rangle\langle\uparrow|\tilde{T}e^{-i\int_0^t\mathbf{h}(t)\hat{s}dt}|\uparrow\rangle \tag{34}$$

and

$$\mathscr{K}^+_{h_0}(T;\beta;N)=\frac{\mathrm{Tr}[e^{-iT\hat{H}_0}\hat{s}^+e^{iT\hat{H}_0}\hat{s}^-e^{-\beta\hat{H}_0}e^{-\beta h_0\hat{\sigma}_z}]}{Z(\mathbf{h}_0)}. \tag{35}$$

The Hubbard–Stratonovich transformation (5)–(7) for $\exp(-iT_1\hat{H}_0)$, $T_1=T-i\beta$, and $\exp(iT\hat{H}_0)$, with the change of variables (8), gives us a representation for $\mathscr{K}^+_{h_0}(T;\beta;N)$ in the form of the right-hand side of (10), where the functionals $\mathscr{B}$ and $\mathscr{C}$ are replaced by

$$\mathscr{B}_{h_0}=\mathrm{Tr}[\exp(\beta h_0\hat{s}_z)\mathscr{A}^{(1)}(T_1)\mathscr{A}^{(2)}(T)],$$

$$\mathscr{C}=\frac{1}{Z(h_0)}\,\mathrm{Tr}[\exp(\beta h_0\hat{s}_z)\mathscr{A}^{(1)}(T_1)\hat{s}^+\mathscr{A}^{(2)}(T)\hat{s}^-]. \tag{36}$$

Substituting for $\mathscr{A}^{(1)}(T_1)$ and $\mathscr{A}^{(2)}(T)$ the explicit expressions (A3) and (A4) and retracing the procedure of Appendix B, we obtain the following quantum-mechanical representation:

$$\begin{aligned}\mathscr{K}^+_{h_0}(T;\beta;N)=\frac{e^{\beta h_0/2}}{Z(h_0)}\int_\Gamma\frac{dz}{z^N}\int_0^\infty dy e^{-4N^2y}\cdot\\ \times\int d\xi_1 d\xi_2\delta(\xi_1+\xi_2)\langle\xi_1|\\ \times\exp\left[-2z\cosh\left(\xi+\frac{\beta h_0}{2}\right)\right]\\ \times e^{-i\hat{H}_1T_1}e^{-2\xi}e^{i\hat{H}_2T}|\xi_2\rangle.\end{aligned} \tag{37}$$

This representation differs from representation (B4) in the operators $\hat{\mathscr{H}}_{1,2}$ only:

$$\hat{H}_{1,2}=-\frac{1}{2N}\partial_\xi^2+2Nzye^{\mp\beta h_0/2}e^{-\xi}. \tag{38}$$

The identities (B7)–(B9) allow us to resolve the quantum-mechanical problem. After some computations we obtain

$$\begin{aligned}\mathscr{K}^+_{h_0}(T;\beta;N)=\frac{e^{\beta h_0/2}}{Z(h_0)}\int_{-\infty}^\infty dx(\cosh x)^N\int_{-\infty}^\infty d\nu\\ \times\exp\left[-\frac{\beta}{2N}\left(\nu-\frac{i}{2}\right)^2+2i\nu x\right]\\ \times\left\{\frac{\sin\nu\beta h_0}{\nu\sinh^3\frac{\beta h_0}{2}}\right.\\ \times(e^{-\nu T/N}-1)-2e^{-\nu T/N+i\nu\beta h_0}\\ \times\left(\frac{\cosh\frac{\beta h_0}{2}}{\sinh^2\frac{\beta h_0}{2}}-\frac{4i\nu}{\nu\sinh\frac{\beta h_0}{2}}\right)\\ \left.+\frac{2\cosh(\beta h_0/2+i\nu\beta h_0)}{\sinh^2\frac{\beta h_0}{2}}\right\}.\end{aligned} \tag{39}$$

In the limit $N\gg1$, $\beta h_0\gg1$ this integral is given by a saddle-point contribution;[1)]

$$\begin{aligned}\mathscr{K}^+_{h_0}(T;\beta)=\frac{e^{\beta h_0/2+\beta\tilde{\phi}}}{\cosh[\beta(h_0/2+\tilde{\phi})]}\exp(-\tau^2D_{\beta,h_0}\\ +i\tau N^{1/2}\tilde{\phi}).\end{aligned} \tag{40}$$

Here $\tilde{\phi}$ is the solution of mean-field equation:

$$2\tilde{\phi}=\tanh(\beta(\tilde{\phi}+h_0)/2), \tag{41}$$

$T=\tau N^{1/2}$, and the relaxation constant D_{β,h_0} is

$$D_{\beta,h_0}=\frac{1-4\tilde{\phi}^2}{2(4-\beta+4\beta\tilde{\phi}^2)}. \tag{42}$$

At small temperatures and in high fields h_0 the constant D_{β,h_0} decays exponentially. At the phase-transition point $\beta=4$ for small h_0 we have $D_{\beta,h_0}\sim h_0^{-2/3}$. In the absence of a "dynamical" field $\mathbf{h}(t)$, the complete correlation function has the form

$$\mathscr{K}_{h_0}(T;\beta)=\exp(-\tau^2D_{\beta,h_0}+i\tau N^{1/2}\tilde{\phi}). \tag{43}$$

In general, we can use the relation (28). The expressions for an antiferromagnetic cluster can be obtained from (40) and (42) by a formal substitution $\beta\to-\beta$. Note that in the ferromagnetic phase in the limit $1/N\ll h_0\ll1$ the spin correlation functions do not depend on h_0; the corresponding answer for the transverse spin correlation function is obtained after setting the limit $h_0\to0$ in Eqs. (40)–(43) and choosing one of the solutions of Eq. (20) [which is the zero field limit of Eq. (41)]. This asymptotic relation breaks down at strictly zero temperature. One of the saddle point solutions of (20) is $t^*=\beta/4=\infty$, and (19) gives us

$$\mathscr{K}(T;\beta=\infty;N\gg1)=\cos\frac{(N-2)t}{2N}. \tag{44}$$

Let us now turn to the calculation of the one-spin matrix element $\alpha[\mathbf{h}(t)]$. In general, the problem of one-spin quantum dynamics in an arbitrary, spatially homogeneous magnetic field cannot be solved exactly. There are, however, at least two examples[19] which can be analyzed explicitly. First, in the case of a spatially homogeneous magnetic field, which is constant in direction but varies in absolute magnitude, according to an arbitrary law $h=h(t)$, we obtain

$$\alpha[\mathbf{h}(t)]=\exp\left[-i\int_0^T h(t)\right]. \tag{45}$$

Second, in the case of a uniform magnetic field which is constant in absolute value, whose direction rotates uniformly with an angular velocity ω around the z axis and at an angle θ to it, we obtain

$$\begin{aligned}\alpha[\mathbf{h}(t)]=e^{i\omega T}\left[\cos\frac{\Omega T}{2}-i\frac{\omega+h\cos\theta}{\Omega}\sin\frac{\Omega T}{2}\right]^2,\\ \Omega=\sqrt{(\omega+h\cos\theta)^2+h^2\sin^2\theta}.\end{aligned} \tag{46}$$

CONCLUSION

In summary, we have introduced an N-spin quantum model with a position-independent exchange, for which we have rigorously calculated the temporal dependence of the transverse pair autocorrelation function with an arbitrary number of spins and temperature for a spatially homogeneous magnetic field applied to the system.

We have shown that the dependence of the correlation function $\mathscr{K}^+[h](t;\beta;N)$ [whose real part gives the sought-for transverse autocorrelation function $\mathscr{K}[h](t;\beta;N)$] on a spatially homogeneous magnetic field is "factorized" com-

pletely in the one multiplier $\alpha(\mathbf{h};\beta)$. $\alpha(\mathbf{h};\beta)$ is expressed in terms of well-defined, one-spin matrix element which is possible to calculate at least for two nontrivial time dependences of the magnetic field: a) the case of a magnetic field which is constant in direction and b) the case of a magnetic field constant in absolute value, whose direction rotates uniformly.

The results are obtained by a method which is a dynamic mean-field method and which by construction turns out to be exact. As seems to us that this is the first example of this kind.

From the point of view of possible applications to long-range quantum magnets, these exact results are unique as a basis for generating a starting dynamical approximation. The possibility of having an exact result for an arbitrary N is very important. Indeed, result (19) gives an approximation to the problem with a large but finite radius of exchange in an infinite magnet, when N plays the role of a number of spins in the exchange sphere. The physical picture drawn by this approximation is clear from the classical model:[17,18] dynamics of a spin is defined by its uniform precession around the total spin of the system. A comparison of the approximation, which gives for the equilibrium autocorrelation function in the paramagnetic phase a Gaussian relaxation from 1 at $t=0$ through a minimum at $t=t_m$ (for $\beta=0$, ≈ 0.04 at $t_m \approx 3.5\sqrt{NJ}$) to the universal plateau 1/3 at t, $\sqrt{N} \ll t \ll N$, with the corresponding experimental curve,[7] gives a remarkable coincidence at short and intermediate (up to t_m) times; the experimental results show that the cluster approximation breaks down immediate after t_m, when a spin diffusion regime begins. Precise analytical calculations describing a crossover between the cluster and the spin-diffusion regimes must be carried out.

We wish to thank V. Cherepanov, A. Larkin, V. Lebedev, and V. L'vov for useful discussions.

APPENDIX A: AVERAGING OF TIME-ORDERED EXPONENTIALS

Time ordered exponentials

$$\mathscr{A}^{(1,2)}(t)=T\exp\left(i\int_0^t dt'\,\varphi_{1,2}(t'),\hat{s}\right) \tag{A1}$$

are defined by the equations

$$\dot{\mathscr{A}}^{(1,2)}(t)=i\varphi(t)\hat{s}\mathscr{A}^{(1,2)}(t) \tag{A2}$$

and by the initial conditions $\hat{A}^{(1,2)}(0)=\hat{1}$. Let us consider another operators $\hat{B}^{(1,2)}(t)$ given in the explicit form

$$\hat{B}^{(1)}(t)=\exp[\hat{s}^-\psi_1^+(t)]\exp\left(\hat{s}^z i\int_0^t \rho_1(t')dt'\right)$$
$$\times\exp\left(\hat{s}^+ i\int_0^t dt'\,\psi_1^-(t')e^{-i\int_0^{t'}\rho_1(t'')dt''}\right)$$
$$\times\exp[-\hat{s}^-\psi_1^+(0)], \tag{A3}$$

$$\hat{B}^{(2)}(t)=\exp[-\hat{s}^+\psi_2^-(t)]\exp\left(\hat{s}^z i\int_0^t \rho_2(t')dt'\right)$$
$$\times\exp\left(\hat{s}^- i\int_0^t dt'\,\psi_2^+(t')e^{i\int_0^{t'}\rho_2(t'')dt''}\right)$$
$$\times\exp[\hat{s}^+\psi_2^-(0)], \tag{A4}$$

where $\psi_{1,2}^{\pm}(t),\rho_{1,2}(t)$ are certain new functions of time. Using the commutation relations for spin operators, we can see that the operators $\hat{B}^{(1,2)}(t)$ satisfy the conditions

$$\hat{B}^{(1)}(t)=\{\hat{s}^-[\dot{\psi}_1^+ + i\rho_1\psi^- - i\psi_1^+(\psi_1^-)^2]+i\hat{s}^-\psi_1^+$$
$$+i\hat{s}^z(\rho_1-2\psi_1^+\psi_1^-)\}\hat{B}^{(1)}(t), \tag{A5}$$

$$\hat{B}^{(2)}(t)=\{\hat{s}^+[-\dot{\psi}_{2-}+i\rho_2\psi^- - i\psi_2^+(\psi_2^-)^2]+i\hat{s}^-\psi_2^+$$
$$+i\hat{s}^z(\rho_2-2\psi_2^+\psi_2^-)\}\hat{B}^{(2)}(t). \tag{A6}$$

The last factors in (A3) and (A4) give the equalities $\hat{B}^{(1,2)}(0)=\hat{1}$. Thus, after comparing (A2), (A5), and (A6) we see that the substitution of (8) transforms $\mathscr{A}^{(1,2)}(t)$ into (A3) and (A4); hence, $\mathscr{A}^{(1,2)}(t)=\hat{B}^{(1,2)}(t)$.

The Jacobian $\mathscr{J}[\rho_{1,2},\varphi_{1,2}^{\pm}]$: $\mathscr{D}\varphi_{1,2}=\mathscr{J}[\rho_{1,2},\varphi_{1,2}^{\pm}]\mathscr{D}\rho_{1,2}\mathscr{D}\psi_{1,2}^{\pm}$ depends on regularization of the map (8). We make use of the regularization from Refs. 9 and 10. It gives for the Jacobian

$$\mathscr{J}=\text{const}\exp\left(\frac{i}{2}\int_0^t(\rho_1-\rho_2)dt\right). \tag{A7}$$

Finally, the constraints $\varphi_{1,2}^+=(\varphi_{1,2}^-)^*$ (*≡complex conjugation) can be replaced by the standard constraint $\psi_{1,2}^+=(\psi_{1,2}^-)^*$ which deforms the integration surface. The corresponding definition of the path integral and the explicit form of this homotopy are presented in Refs. 8 and 11.

APPENDIX B: DERIVATION AND CALCULATIONS IN ACCOMPANIED QUANTUM MECHANICS

1. To describe the case of the infinite temperature, we begin with expression (11) for $\mathscr{K}^0(t;N)$, where the replacement (13) was already done. Using the identity

$$\int_\Gamma \frac{dz}{z^N}e^{-z\mathscr{B}}=\frac{\Gamma(k+1/2)}{2\pi i}\mathscr{B}^{N-1},$$

(here Γ is an arbitrary contour closed near $z=0$ in the z-complex plane), we recast the $d\psi_{1,2}^{\pm}$-integration in a Gaussian form. It can be performed exactly and we lift the result in the exponential by means of the formula

$$\frac{1}{Y}=\int_0^\infty dy e^{-yY}.$$

We thus obtain the expression

$$\mathscr{K}^0(T;N)=\text{const}\int_\Gamma\frac{dz}{z^N}\int_0^\infty dy e^{-4N^2y}\int\mathscr{D}\rho_{1,2}(t)$$
$$\times\exp\left\{-i\frac{N}{2}\int_0^T dt(\rho_1^2-\rho_2^2)\right. \tag{B1}$$
$$\left.+2izN\exp\left(\frac{i}{2}\int_0^T(\rho_1-\rho_2)dt\right)\right.$$

$$\times\left[\int_0^T \exp\left(-i\int_0^t dt'\rho_1\right)dt-\int_0^T \exp\left(i\int_0^t dt'\rho_2\right)dt\right]$$

$$+\int_0^T dt(\rho_1-\rho_2)-2z\cos\left(\frac{1}{2}\int_0^T dt(\rho_1+\rho_2)\right)\Big\}. \qquad \text{(B2)}$$

If we perform a linear change of variables $\rho_{1,2}(t)\to\xi_{1,2}(t)$, where

$$\xi_{1,2}(t)=\pm i\int_0^t \rho_{1,2}dt'-\frac{i}{2}\int_0^T(\rho_1-\rho_2)dt' \qquad \text{(B3)}$$

(note that the upper limit in the last integral is the "final" time argument T), we obtain a path integral over $\mathscr{D}\xi_{1,2}$, which has a Feynman–Kac form. The boundary conditions following from (B3),

$$\xi_1(T)=-\xi_2(T), \quad \xi_1(0)=-\xi_2(0),$$

allow us to rewrite (B2) as the quantum-mechanical matrix element:

$$\mathscr{K}^0(T;N)=\text{const}\int_\Gamma \frac{dz}{z^N}\int_0^\infty dy e^{-4N^2y}\int d\xi_1 d\xi_2\delta(\xi_1+\xi_2)$$

$$\times\langle\xi_1|e^{-2z\cosh\xi}e^{-i\hat{\mathscr{H}}T}e^{-2\xi}e^{i\hat{\mathscr{H}}T}|\xi_2\rangle, \qquad \text{(B4)}$$

where the Hamiltonian of this one-dimensional quantum mechanics is

$$\hat{\mathscr{H}}=-\frac{1}{2N}\partial_\xi^2+2Nzye^{-\xi}. \qquad \text{(B5)}$$

The delta-functions, the coupled bra- and ket- states in (B4), are determined by the boundary condition at the time T and the initial conditions $\xi_1(0)=\xi_2(0)$ join together the direct and inverse evolution operators in the matrix elements through their averaging weight $e^{-2\xi_1(0)}$. Now we can exclude integration over y in the expression (B4). It can be done in two steps. First, through the shift $\xi\to\xi+\ln(4zy)$ and correspondingly $\hat{\mathscr{H}}\to\hat{\mathscr{H}}_{ac}$ [see (14)]. Second, integration with respect to y. Equation (B4) will then hold if the integration with respect to y and $\exp(-4N^2y)$ with the delta-function of $\xi_1+\xi_2$ are replaced by

$$\frac{F(\eta_1,\eta_2)}{\eta_1\eta_2}\equiv\frac{1}{\eta_1\eta_2}\exp\left[-\frac{\eta_1\eta_2}{4z}-\frac{z(\eta_1^2+\eta_2^2)}{\eta_1\eta_2}\right], \qquad \text{(B6)}$$

where $\eta_{1,2}=2N\exp(-\xi_{1,2}/2)$. The integral representation for $F(\eta_1,\eta_2)$

$$F(\eta_1,\eta_2)$$

$$=\frac{16}{\eta^2}\int_0^\infty d\nu\nu\sinh 2\pi\nu K_{2i\nu}(2z)K_{2i\nu}(\eta_1)K_{2i\nu}(\eta_2), \qquad \text{(B7)}$$

with the identity

$$\eta^{-1}K_{2i\nu}(\eta)=(K_{2i\nu+1}(\eta)-K_{2i\nu-1}(\eta))/4i\nu \qquad \text{(B8)}$$

[$K_\mu(x)$ it is the standard notation for the modified Bessel function of the second kind], makes further progress in the evaluation of $\mathscr{K}^0$. Indeed, $K_{i\mu}(\eta)$ is an eigenfunction of $\hat{\mathscr{H}}_{ac}$

$$\hat{\mathscr{H}}_{ac}K_{i\mu}(\eta)=\frac{\mu^2}{8N}K_{i\mu}(\eta), \qquad \text{(B9)}$$

which immediately gives the expression for $\mathscr{K}^0(t;N)$

$$\mathscr{K}^0(t;N)=\text{const}\int_\Gamma\frac{dz}{z^{N+1}}\int_0^\infty d\nu K_{2i\nu}(2z)$$

$$\times\left\{-1+2\nu^2+(1+4\nu^2)\cosh\frac{\nu t}{N}\right\}. \qquad \text{(B10)}$$

Finally, performing integration with respect to ν and z, we obtain (15).

2. In the case of a finite temperature and zero magnetic field Eq. (B4) for $\mathscr{K}^0(t;N)$ transforms into a corresponding equation for $\mathscr{K}^+(t;\beta;N)$ if $\exp(-i\mathscr{H}T)$ in the matrix element is replaced by $\exp(-i\mathscr{H}T_1)$ [$T_1=T-i\beta$; the exponential $\exp(i\mathscr{H}T)$ remains unchanged]. Using equalities (B6)–(B9), we obtain

$$\mathscr{K}^+(T,\beta,N)=\frac{e^{\beta/8N}}{Z(\beta)}\int_\Gamma\frac{dz}{z^{N+1}}\int_0^\infty d\nu K_{2i\nu}(2z)e^{-\beta\nu^2/2N}$$

$$\times\left\{(1-2\nu^2+3i\nu)\exp\left(\frac{i\beta\nu}{2N}\right)-(1+4\nu^2)\right.$$

$$\left.\times\exp\left(\frac{(2T-i\beta)\nu}{2N}\right)\right\}, \qquad \text{(B11)}$$

which gives rise to (19). (We used the integral representation for the modified Bessel function of the second kind.)

[1)]The expression for the partition function $Z(h_0)$ can be found from (39) with use of the equality: $\mathscr{K}^+_{h_0}(T=0;\beta)+\mathscr{K}^+_{-h_0}(T=0;\beta)=2$.

[1] R. White, *Quantum theory of magnetism*. (McGr.-Hill B.C., New-York, 1970).
[2] V. Vaks, A. Larkin, and S. Pikin, Sov. Phys. JETP **26**, 188 (1968).
[3] V. Belinicher and V. L'vov, Sov. Phys. JETP **59**, 564 (1984).
[4] V. L'vov, *Wave Turbulence under Parametric Excitation. Applications to Magnets*. (Springer-Verlag, Berlin, 1994).
[5] E. Podivilov, Technical Report No. 409, Institute of Automation and Electrometry, Novosibirsk (1990).
[6] M. Chertkov and I. Kolokolov, Technical Report No. 467/FT, IFUM (1994).
[7] J. Labrujere, T. Klaassen, and N. Poulis, J. Phys. C **15**, 999 (1982).
[8] I. Kolokolov, Phys. Lett. A **114**, 99 (1986).
[9] I. Kolokolov and E. Podivilov, Sov. Phys. JETP **68**, 119 (1989).
[10] I. Kolokolov, Ann. of Phys. **202**, 165 (1990).
[11] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Statistics of a passive scalar advected by a large-scale 2*D* velocity field: analytic solution. Preprint WIS-94/7/Jan-PH, (1994).
[12] I. Kolokolov, Ann. of Phys. **231**, 234 (1994).
[13] R. Kubo and T. Toyabe, *Magnetic Resonance and Relaxation*,. ed. (North-Holland, 1967).
[14] M. Chertkov and I. Kolokolov, Phys. Rev. B **49**, 3592 (1994).
[15] I. Kolokolov, V. L'vov, E. Podivilov, V. Cherepanov, A. Shafarenko, Technical Report No. 253, Institute of Automation and Electrometry, Novosibirsk (1984) [in Russian].
[16] R. Fishman and G. Vignale, Phys. Rev. B **44**, 658 (1991).
[17] R. Fishman and S. Liu, Phys. Rev. B **45**, 5414 (1992).
[18] J. Fernandez-Baca and H. Mook, J. Appl. Phys. **73**, 7023 (1993).
[19] L. Landau and E. Lifshitz, *Quantum Mechanics* (Non-relativistic Theory) (Pergamon Press, Oxford, 1977).
[20] R. P. Feynmann and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

Note added in proof (8 September 1994). After this work had been completed and presented for publication we have known about series of papers published by R. Dekeyser and M. H. Lee [Phys. Rev. B **19**, 265 (1979); **43**, 8123 (1991); **43**, 8131 (1991)]. They investigated the more general anisotropic variant of the model discussed in the present paper (so-called van der Waals model). They use the following strategy: first, solve the evolution equation for a single spin (Phys. Rev. B **43**, 8123 (1991)); and then average the solution with respect to the equilibrium density matrix [Phys. Rev. B **43**, 8131 (1991)]. The authors have shown that the dynamics of the single spin is coupled linearly with the dynamics of the rest $N-1$ spins in the system at $N\to\infty$. Then they find the autocorrelation function in this limit analytically. The results obtained are compatible with our Eqs. (16), (21). However, the method used in those papers does not allow to receive our Eqs. (15), (19) which are valid at finite N, as well as our Eqs. (33), (39) for the case of external field and vice versa, the method used in the present publication allows one to generalize all the finite N and external field results to the anisotropic case.

This article was published in English in the original Russian journal with stylistic changes by the Translation Editor, Dr. S. J. Amoretty.