Surface-induced nonlocal effects of classical electrodynamics and their application

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The complex mass shift Δm of a classical charge moving in external fields of a special type near conducting boundaries has been studied. The general formulas for the real and imaginary parts of the shift have been obtained for the simplest case of plane motion parallel to a "mirror." The geometric (causal) properties of the world lines lead to dispersion relations that define the relationship between the real and imaginary parts of Δm. The general formulas have been illustrated analytically and numerically in specific examples. The method has been applied to the problem of the shift of the cyclotron frequency of an electron confined in a magnetic field within a flat cavity. The expressions found for the real ($\delta\omega_c'$) and imaginary ($\delta\omega_c''$) parts of the cyclotron frequency shift become equivalent to the known expressions in special cases of the position of the orbit and the orientation of the magnetic field. The resonant (with respect to the magnetic field) behavior of $\delta\omega_c'$ and $\delta\omega_c''$ has been studied. It has been shown that when the resonance condition $\omega_c = N\pi/l$ (l is the distance between the plates, $N=1,2,...$) is satisfied, no logarithmic divergences appear in $\delta\omega_c'$ and no finite discontinuous jumps appear in $\delta\omega_c''$ at the $(N-1)$ points dividing the interval l into N equal parts and that the threshold behavior of $\delta\omega_c''$ is maintained, regardless of the position of the cyclotron orbit.

1. INTRODUCTION

The influence of boundaries on one-particle states in quantum field theory is manifested in two ways, viz., directly through the boundary conditions (independently of the field interaction constants, see Ref. 1) and through boundary-modified radiation effects. The latter include the self-energy effects of quantum electrodynamics, which were first examined in Ref. 2, where the contribution of conducting "walls" to the mass and magnetic moment of an electron was studied. This subject has been widely discussed in recent years[3–8] in connection with probable applications to high-precision measurements of $g-2$ of an electron confined in a cavity resonator.[9,10] The small dimensions of the confinement region compared with the dimensions of the cavity permit neglect of the influence of the boundaries on the field modes of the (massive) electronic field. At the same time, its electromagnetic self-field is modified by the boundaries, altering the character of the self-action of the particle. When certain conditions are satisfied [see Eq. (36) below], the self-action is amenable to a classical treatment. In fact, it was found that the "instrumental" additions to the mass, cyclotron frequency, and magnetic moment calculated in the context of quantum electrodynamics are not dependent on $\hbar$.[4,5,8,11,12] Thus, nonlocal effects of quantum electrodynamics are also exhibited in the limit $\hbar \to 0$ owing to the "global" properties of the self-field of a classical charge. We note, in passing, that the contributions of the boundaries to Δm and Δg for nonlocalized electronic states located between two "mirrors" were considered in Refs. 7, 13, and 14[1)] and were found to have a quantum character (the variation of the modes of the electron-positron field was likewise not taken into account here).

In the present paper we obtain general formulas for the mass shift of a classical charge induced by a plane, ideally conducting boundary and external fields, as well as for the shift of the cyclotron frequency of an electron confined by a uniform magnetic field in a flat cavity. The concept of a mass shift forms the foundation of the approach used,[15,16] and the modified version associated with consideration of the influence of the conducting surfaces[17,18] has several advantages over the other classical methods.[4,6,8] In particular, owing to the original gauge invariance of the method, the problem of selecting a gauge for a "photon" Green's function that depends on the boundary conditions[4,7,13,19,20] can be circumvented, and relatively compact formulas suitable for calculating the nonrelativistic, ultrarelativistic, and "retarded" asymptotes of the mass shift and the cyclotron frequency shift (in the case of motion in a magnetic field) can be obtained.

Self-action effects have been traditionally treated with the Abragam–Lorentz–Dirac equation, in which the influence of boundaries can be taken into account[4,6,8] owing to the nonlocal behavior appearing in the following limiting cases:[2)]

$$V \ll 1, \tag{1a}$$

$$R \gg T, \tag{1b}$$

$$R \ll T. \tag{1c}$$

Here V is the velocity of the charge; R is the (doubled) distance to the boundary; T is the characteristic time of the motion. In addition, the perturbation introduced by the boundary and the reactive force are considered to be small. At the same time, the mass-shift method, which is also essentially a perturbation-theory method, makes it possible to dispense with the restrictions (1). This is achieved at the expense of a detailed picture of the motion based on the

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solution of the Abragam–Lorentz–Dirac equation and a transition to a description which utilizes averaged corrections to the parameters of the unperturbed motion.

In Sec. 2.1 we study the mass shift of electric and scalar charges undergoing planar motion near a mirror for a specific class of external fields. The real part of the shift is expressed in terms of the geometric characteristics of the motion and is related to its imaginary part by means of dispersion relations, which follow from the causality of the world lines. These properties of the mass shift have their own analogies in the case of charges (electric and scalar charges) that are accelerated in the absence of boundaries, but have a massive self-field.[16,21,22] Section 2.2 is devoted to the analytical results, and Sec. 2.3 is devoted to the numerical results for three possible cases of the plane motion of a charge near a mirror. Section 3.1 describes the effective-Hamiltonian method[18] as applied to the (complex) cyclotron frequency shift in a flat cavity and the derivation of fundamental equations (58) and (63), which generalize the previously published results. In Sec. 3.2 we analyze the known "resonances"[6,7,9] and reproduce the proof of the existence of "antiresonance" points[23] between the plates, at which infinities do not appear in the real part of the cyclotron frequency shift and there are no jumps in the imaginary part. Because a wave cannot propagate with frequency smaller than the critical frequency of a waveguide ($\omega_{cr}=\pi/l$), the imaginary part of the cyclotron frequency shift exhibits a threshold,[6] which, as is shown below, persist for an arbitrary position of the orbit in the cavity.

2. MASS SHIFT OF A CHARGE IN THE CASE OF PLANE MOTION PARALLEL TO A CONDUCTING BOUNDARY

2.1. General equations and dispersion relations

The variation of the self-action of a charge caused by the presence of boundaries and external fields is described by the following correction to the classical action of a particle:

$$\Delta W=\frac{e^2}{2}\int\int \dot{x}_\alpha(\tau)\dot{x}_\beta(\tau')D^{(B)}_{\alpha\beta}(x,x')d\tau d\tau'\Big|_0^{F,B}, \quad (2)$$

where $|_0^{F,B}$ indicates subtraction of the "vacuum" contribution (see Refs. 16 and 17). Here $x=x(\tau)$ is the world line of the particle, τ is its proper time, and $D^{(B)}_{\alpha\beta}$ is the causal Green's function of the "photon," which takes into account the boundary conditions imposed on the self-field of the charge. The causality of the Green's function $D^{(B)}_{\alpha\beta}$ means that $\exp(i\Delta W)$ is the classical limit of the amplitude of the elastic scattering of the electron by the external field, so that $\mathrm{Im}\,\Delta W$ specifies the probability of radiation and must be positive.[16,22] The Green's function $D^{(B)}_{\alpha\beta}$ for a flat cavity of width l can be found by separation of variables (the spatial and Lorentzian variables)[20] in the following manner:

$$D^{(B)}_{\alpha\beta}(x,x')=\delta_{\alpha\beta}D^{(\alpha)}(x,x'), \quad (3)$$

where

$$D^{(0)}(x,x')=\sum_{-\infty}^{\infty}(-1)^N\frac{i/4\pi^2}{(x-x'^{(N)})^2}=D^{(1)}=D^{(2)} \quad (4)$$

satisfies the Dirichlet boundary conditions at the ends of the interval $0\leqslant x_3\leqslant l$, and

$$D^{(3)}(x,x')=\sum_{-\infty}^{\infty}\frac{i/4\pi^2}{(x-x'^{(N)})^2} \quad (5)$$

is a solution of the Neumann boundary-value problem in the same region. Here $x^{(N)}$ denotes the 4-vector $(x_1,x_2,x_3^{(N)},ix_0)$, in which

$$x_3^{(N)}=(-1)^N\left(x_3-\frac{l}{2}\right)+l\left(N+\frac{1}{2}\right), \quad (6)$$

N is an integer, and the origin of coordinates is located on one of the walls. In the limit $l\to\infty$ we obtain the Green's function for the case of one mirror. As is seen from Eqs. (6), only two terms, which correspond to the charge and its image ($N=0,-1$) remain in Eqs. (4) and (5) in this limit.

We consider plane motion in a constant uniform field near (one) mirror. In this case the functions $D^{(\alpha)}$ in (3) ($l=\infty$) are expressed in terms of the intervals

$$(x(\tau)-x(\tau'))^2=f(\tau-\tau'), \quad (7)$$

$$(x(\tau)-\tilde{x}(\tau'))^2=f(\tau-\tau')+R^2, \quad (8)$$

where $\tilde{x}\equiv(x_1,x_2,-x_3,ix_0)$, and $R/2$ is the (constant) distance from the charge to the boundary. The isometric nature of world lines, which is expressed by Eq. (7), permits (2) to be rewritten in the form

$$\Delta W=-\Delta m\tau \quad (9)$$

(with separate translational divergence with respect to τ). For the mass shift Δm we obtain

$$\Delta m=\frac{-i\alpha}{2\pi}\int_0^\infty d\tau\left[\frac{f''(\tau)}{f(\tau)}-\frac{2}{\tau^2}-\frac{f''(\tau)}{f(\tau)+R^2+i0}\right]. \quad (10)$$

The principal properties of integral function $f(\tau)$ follow from Eq. (7):

$$f(\tau)=f(-\tau)<0, \quad \tau\neq 0, \quad (11a)$$

$$f'(\tau-\tau')=2\dot{x}(\tau)(x(\tau)-x(\tau'))\neq 0, \quad \tau\neq\tau', \quad (11b)$$

$$f(\tau)=-\tau^2+O(\tau^4), \quad \text{when } \tau \text{ or } F\to 0. \quad (11c)$$

As a consequence of relations (11a) and (11b), the equation

$$f(\tau)+R^2=0 \quad (12)$$

has only two nonmultiple roots $\tau=\tau_\pm$ and $\tau_+=-\tau_-$; $\tau_+>0$ corresponds to the retarded proper-time interval between the emission of a photon and its absorption after reflection from the mirror.

According to (10), the real part of Δm equals ($\tau-\tau'=\tau_+$)

$$\mathrm{Re}\,\Delta m=\frac{\alpha}{2}\frac{f''(\tau_+)}{f'(\tau_+)}=\frac{\alpha}{2}\frac{\dot{x}\dot{x}'}{|\dot{x}(x-\dot{x}')|}<0 \quad (13)$$

[see (11)]. The nonlocal geometric structure on the left-hand side of the last inequality reduces to the Coulomb multiplier $-\alpha/2R$ only for rectilinear motion between x and x'. This becomes possible either for $\tau_+\to 0$ (i.e., $R\to 0$) or when the first derivatives may be neglected in the Taylor series

$x(\tau')=x(\tau+\tau_+)=x(\tau)+\dot{x}(\tau)\tau_+ + ...$ (i.e., $F\to 0$). We note that a term dependent on R cannot be isolated in Eq. (10) in the general case: in the case of motion in an electric field this term ensures infrared convergence of integral (10). This example was examined in detail in Ref. 17, and its analog with a massive "photon" in the absence of boundaries was treated in Refs. 16 and 22.

The equations like (10) and (13) for a scalar charge e follow from expression (2), in which the replacement $\dot{x}_\alpha \dot{x}'_\beta \to 1$ must be made[16,21,24] and one of the D functions (4) must be taken as the Green's function (if the Dirichlet boundary conditions are adopted for the self-field of the charge and $l=\infty$). Thus, we obtain

$$\Delta m_{\rm sc}=\frac{-i\alpha}{\pi}\int_0^\infty d\tau\left[\frac{1}{f(\tau)}+\frac{1}{\tau^2}-\frac{1}{f(\tau)+R^2+i0}\right], \qquad (10')$$

$$\text{Re } \Delta m_{\rm sc}=\frac{\alpha}{|f'(\tau_+)|}>0. \qquad (13')$$

A comparison of Eqs. (13) and (13′) reveals that $|\text{Re}\Delta m|\geqslant \text{Re}\Delta m_{\rm sc}$, since $\dot{x}\dot{x}'\leqslant -1$. The direction of the inequality in (13′) depends on the type of boundary condition. In particular, a Neumann boundary condition imposed on the self-field of the particle would change the sign in front of $\text{Re}\Delta m_{\rm sc}$.

We note that an example of an external scalar field $\varphi^{\rm ex}(x)$ in which the motion of the charge would have the property (7) can be devised. {The action $-\int[m+e\varphi^{\rm ex}(x)]d\tau$ gives the equation of motion

$$\ddot{x}_\alpha(\tau)=-(\delta_{\alpha\beta}+\dot{x}_\alpha\dot{x}_\beta)\partial_\beta\Psi, \quad \Psi=\ln[m+e\varphi^{\rm ex}(x)],$$

whose solutions include hyperbolic world lines satisfying condition (7) when $\Psi(x)=\Phi(x_\alpha^2)$ ($\Phi'<0$)}.

The relationship between the radiation process and the reactive change in the energy of the self-field of an accelerated charge is expressed by the dispersion relations. Let us examine the analytical properties of $\Delta m(R^2)$ specified by expressions (10) and (10′): a) Δm is an analytic function in a complex R^2 plane with a cut along the positive real semiaxis; b) $\Delta m(z)^*=-\Delta m(z^*)$ [Riemann–Schwarz symmetry, which follows from the imaginary and analytic nature of $\Delta m(R^2)$ on the negative real semiaxis]. These properties, supplemented by the asymptotic restrictions

$$\Delta m=\frac{\alpha}{2\sqrt{z}}(-1)^s+O(\sqrt{z}) \quad (s=0,1) \qquad (14)$$

as $z\to 0$ and

$$|\Delta m|\leqslant \text{const}\cdot\ln|z| \qquad (15)$$

as $|z|\to\infty$, give the dispersion relations

$$\text{Re } \Delta m_a(R^2)=\frac{2R}{\pi}⨍_0^\infty \frac{\text{Im } \Delta m_a(u^2)du}{(u^2-R^2)}, \qquad (16)$$

$$\text{Im } \Delta m_a(R^2)=-\frac{2R^2}{\pi}⨍_0^\infty \frac{\text{Re } \Delta m_a(u^2)du}{u(u^2-R^2)}, \qquad (17)$$

in which $\Delta m_a=\Delta m\pm\alpha/2R$ [as in (14), the signs correspond to the spin of the self-field $s=1$ and 0]. The proof of the dispersion relations (16) and (17) is standard (see, for example, Ref. 25) and uses the Cauchy theorem for the contour depicted in Fig. 1. Conditions (14) and (15) [the former is the Coulomb asymptote for $z=(R^2+iO)\to 0$, and the latter is a restriction on the infrared behavior[16,17]] make it possible to discard the integrals over the small and large circles in Fig. 1. We stress that all three properties (11) are essential for obtaining the dispersion relations.

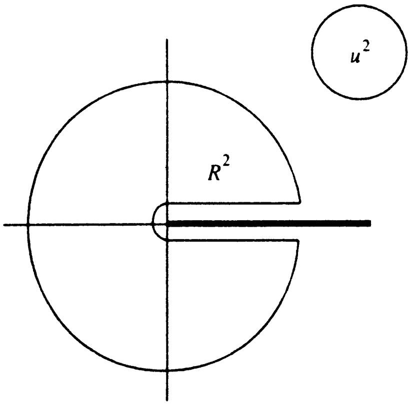

FIG. 1.

It would be useful to compare Eqs. (16) and (17) with the dispersion relations for the square of the "photon mass" (μ) obtained in so general a case for the mass shift of electric and scalar charges.[22] The qualitative correspondence $\mu\sim 1/R$, which was verified numerically for uniformly accelerated motion in Ref. 17, was found to hold with a high accuracy over a fairly broad region, including values far from $\mu=0$.

2.2. Special cases of plane motion near a mirror

There are only three configurations of a constant uniform electromagnetic field that cannot be brought into coincidence with one another and allow two-dimensional 3-trajectories. These configurations are defined by the values of the field invariants: $G\equiv(1/4)\tilde{F}_{\mu\nu}F_{\mu\nu}=0$, $2F\equiv(1/2)F_{\mu\nu}F_{\mu\nu}=0$ (a crossed field); $G=0$, $2F=-\varepsilon^2<0$ (an electric field ε); $G=0$, $2F=\eta^2>0$ (a magnetic field η). The remaining configurations can be obtained by applying Lorentz transformations (i.e., with the aid of boosts parallel to the boundary; the invariance of the boundary condition $n_\mu\tilde{F}_{\mu\nu}|_S=0$ for the self-field of the charge, where n_μ is the 4-vector of a normal to the surface S, is significant here[3)]).

Crossed field. The function $f(\tau)$ has the form[16]

$$f(\tau)=-\tau^2-a^2\tau^4/12, \quad a^2=(eF_{\mu\nu}\dot{x}_\nu/m)^2, \qquad (18)$$

so that in accordance with (10), (12), and (13) we obtain

$$\tau_+=a^{-1}[6(\sqrt{1+R^2a^2/3}-1)]^{1/2}, \qquad (19)$$

$$\text{Re } \Delta m^{\rm cr}=-\frac{\alpha}{2\tau_+}\frac{1+a^2\tau_+^2/2}{1+a^2\tau_+^2/6}, \qquad (20)$$

$$\text{Im } \Delta m^{\rm cr}=-\frac{5\alpha a}{4\sqrt{3}}+\frac{\alpha a}{4\sqrt{3}}\frac{5+a^2\tau_+^2/2}{(1+a^2\tau_+^2/6)\sqrt{1+a^2\tau_+^2/12}}. \qquad (21)$$

The first term in (21) corresponds to an infinite distance from the boundary.[16] Expanding the numerator and denominator in Eq. (13) in powers of τ_+ and neglecting terms $\sim\tau_+^3$ in comparison to unity, we obtain Eq. (20), in which $a^2=[\ddot{x}(\tau)]^2=-(1/2)f^{\mathrm{IV}}(0)$ [see (7)]. Thus, if motion occurs sufficiently close to the mirror, the crossed-field conditions are satisfied for the real part of the mass shift (13). This property is not realized for the imaginary part of the mass shift in the general case: the integral like (10) representing it is determined by values of τ from an infinite range.

Electric field ε ($\nu\equiv e\varepsilon$). The real parts of the boundary-induced mass shifts in electric and magnetic fields were investigated in Ref. 18. To complete the picture, we present the principal equations from Refs. 18 and 23 here. Using $u_\perp$ to denote the persistent 4-velocity of an electron in the direction perpendicular to the direction of the electric field, we have

$$f(\tau)=u_\perp^2\tau^2-4(m\gamma_0/\nu)^2\,\mathrm{sh}^2(\nu\tau/2m), \tag{22}$$

$$\mathrm{Re}\;\Delta m^{\mathrm{el}}=-\frac{\alpha\nu}{2m}\,\frac{\mathrm{ch}\;\vartheta-V_{0\perp}^2}{\mathrm{sh}\;\vartheta-V_{0\perp}^2\vartheta}, \tag{23}$$

$$\mathrm{Im}\;\Delta m^{\mathrm{el}}=-\frac{\alpha\nu}{2m\pi}\,⨍_0^{\infty}\left[\frac{V_{0\perp}^2-\mathrm{ch}\;2w}{V_{0\perp}^2w^2-\mathrm{sh}^2\;w}-\frac{1}{w^2}-\frac{V_{0\perp}^2-\mathrm{ch}\;2w}{V_{0\perp}^2w-\mathrm{sh}^2\;w+(R\nu/2\gamma_0m)^2}\right]dw. \tag{24}$$

In Eq. (23) ϑ is determined from the equation

$$4\;\mathrm{sh}^2\,\frac{\vartheta}{2}=V_{0\perp}^2\vartheta^2+\left(\frac{\nu R}{m\gamma_0}\right)^2,\qquad \vartheta=\frac{\nu\tau_+}{m}, \tag{25}$$

and the Lorentz factor γ_0 and $V_{0\perp}^2$ equal, respectively,

$$\gamma_0=(1-V_{0\perp}^2)^{-1/2},\quad V_{0\perp}^2=2(\tilde{F}_{\mu\nu}\dot{x}_\nu)^2[2(\tilde{F}_{\mu\nu}\dot{x}_\nu)^2+\tilde{F}_{\mu\nu}^2]^{-1}. \tag{26}$$

Magnetic field η ($\kappa\equiv e\eta$). The equations corresponding to (22)–(26) for the case of a magnetic field have the form

$$f(\tau)=-\gamma_\perp^2\tau^2+(2mu_\perp/\kappa)^2\,\sin^2(\kappa\tau/2m),$$
$$u_\perp=V_\perp\gamma_\perp, \tag{27}$$

$$\mathrm{Re}\;\Delta m^{\mathrm{mag}}=-\frac{\alpha\kappa}{2m}\,\frac{1-V_\perp^2\cos\vartheta}{\vartheta-V_\perp^2\sin\vartheta},\qquad \vartheta=\frac{\kappa\tau_+}{m}, \tag{28}$$

$$\mathrm{Im}\;\Delta m^{\mathrm{mag}}=\mathrm{Im}\;\Delta m_b^{\mathrm{mag}}(V_\perp^2,\,\kappa R/m)+\mathrm{Im}\;\Delta m_\infty^{\mathrm{mag}}(V_\perp^2), \tag{29}$$

where

$$\mathrm{Im}\;\Delta m_b^{\mathrm{mag}}=\frac{\alpha}{2R}\,\frac{\kappa R}{\pi m}\,⨍_0^{\infty}\frac{1-V_\perp^2\cos 2w}{w^2-V_\perp^2\sin^2 w-(\omega_cR/2)^2}\,dw, \tag{30}$$

$$\mathrm{Im}\;\Delta m_\infty^{\mathrm{mag}}=-\frac{\alpha}{2R}\,\frac{\kappa R}{\pi m}\int_0^{\infty}\left[\frac{1-V_\perp^2\cos 2w}{w^2-V_\perp^2\sin^2 w}-\frac{1}{w^2}\right]dw \tag{31}$$

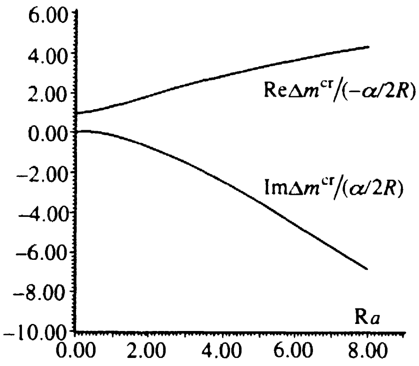

FIG. 2.

[expressions (22), (27), and (31) were given in Ref. 16 with somewhat altered notation]. The Lorentz factor is $\gamma_\perp=(1-V_\perp^2)^{-1/2}$, $\omega_c=\kappa/m\gamma_\perp$, and the invariant is

$$V_\perp^2=2(F_{\mu\nu}\dot{x}_\nu)^2[2(F_{\mu\nu}\dot{x}_\nu)^2+F_{\mu\nu}^2]^{-1}. \tag{32}$$

The parameter ϑ in (28) is given by an equation similar to (25):

$$\vartheta^2=4V_\perp^2\,\sin^2(\vartheta/2)+(\kappa R/m\gamma_\perp)^2. \tag{33}$$

The ultrarelativistic asymptotes of the real parts of mass shifts (20), (23), and (28) coincide:

$$\mathrm{Re}\;\Delta m^{\mathrm{cr}}\simeq-\frac{3\alpha}{2R}\,(Ra)^{1/2}6^{-1/4},\quad Ra\gg1, \tag{34a}$$

$$\mathrm{Re}\;\Delta m^{\mathrm{el}}\simeq-\frac{3\alpha}{2R}\left(\frac{\nu R}{m}\,\gamma_0\right)^{1/2}6^{-1/4},\quad \gamma_0\gg\nu R/m, \tag{34b}$$

$$\mathrm{Re}\;\Delta m^{\mathrm{mag}}\simeq-\frac{3\alpha}{2R}\left(\frac{\kappa R}{m}\,\gamma_\perp\right)^{1/2}6^{-1/4},\quad \gamma_\perp\gg\kappa R/m. \tag{34c}$$

The correspondence

$$a\sim\frac{\nu}{m}\,\gamma_0\sim\frac{\kappa}{m}\,\gamma_\perp \tag{35}$$

is not surprising, since a relativistic particle perceives any field as a crossed field in its own reference system[4)] (Ref. 26).

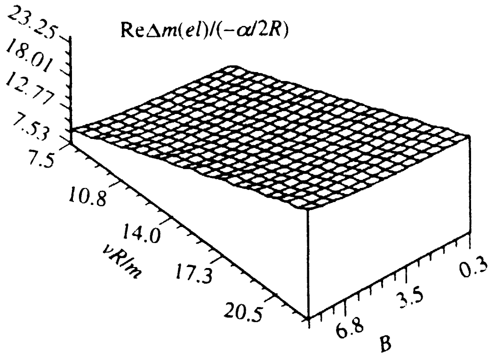

FIG. 3.

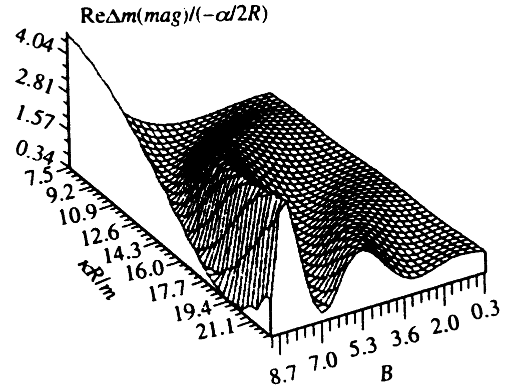

FIG. 4.

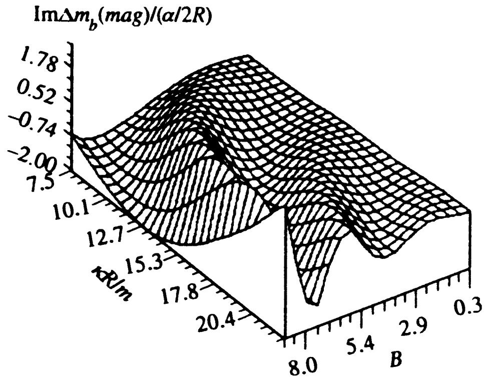

FIG. 6.

The boundary-induced mass shift (23) has a remarkable property: it does not tend to zero with increasing distance to the boundary. However, its nonlocal nature, which is obvious in (13), then becomes hidden (the limit $R \to \infty$ gives $\mathrm{Re}\,\Delta m_\infty^{\mathrm{el}} = -\alpha\nu/2m$, which is consistent with the mass shift found in Ref. 15). In our approximation with respect to the radiation field, $\mathrm{Re}\,\Delta m_\infty^{\mathrm{el}}$ specifies a correction to the rate of pair formation by an electric field.[27]

2.3. Numerical results

The real parts of electron mass shifts (20), (23), and (28) are presented in Figs. 2, 3, and 4, respectively, in units of $(-\alpha/2R)$. The parameter B equals $10V_{0\perp}^2$ (Fig. 3) or $10V_\perp^2$ (Fig. 4). Figures 2, 5, 6, and 7 show the imaginary parts $\mathrm{Im}\,\Delta m^{\mathrm{cr}}$ (21), $\mathrm{Im}\,\Delta m^{\mathrm{el}}/4$ (24), $\mathrm{Im}\,\Delta m_b^{\mathrm{mag}}$ (30), and $\mathrm{Im}\,\Delta m_\infty^{\mathrm{mag}}$ (31). The absolute value of the Coulomb shift $\alpha/2R$ was taken here as the unit of measure, although $\mathrm{Im}\,\Delta m_\infty^{\mathrm{mag}}$ is actually not dependent on R. A comparison of the plots of (30) and (31) reveals that the presence of a conducting surface can result in weak inhibition or stimulation of radiation, since $\mathrm{Im}\,\Delta m_b^{\mathrm{mag}}$ is not sign-invariant [in contrast, of course, to the complete sum in (29)]. The behavior of the real part $\mathrm{Re}\,\Delta m^{\mathrm{mag}}$ (Fig. 4) is also of interest: the oscillations exhibit unequal accumulation of reactive energy by the "elementary contour" for different values of η, $V_\perp$, and R. The range of values of $\kappa R/m$ corresponds qualitatively to the experimental conditions in Penning traps (see below), and the values of $\nu R/m$ were set arbitrarily. As we have already mentioned, there is no division like (29) for $\mathrm{Im}\,\Delta m^{\mathrm{el}}$.

3. MASS AND CYCLOTRON FREQUENCY SHIFTS FOR AN ARBITRARY POSITION OF THE ORBIT IN A FLAT CAVITY

The problem of finding the "instrumental" contributions to the cyclotron frequency ω_c of an electron confined in a "trap" became interesting after record precision was achieved in the determination of its anomalous magnetic moment. Detailed descriptions of the University of Washington experiments were given in Refs. 9 and 10. The quantity $(\omega_s - \omega_c)/\omega_c = (g-2)/2$, where ω_s is the electron spin precession frequency, was measured in these experiments. As the calculations of various investigators[4,6,8] showed, when the size of the cavity is $l \sim 0.1$ cm and the magnetic field is $\eta \sim 50$ kG, the boundary contribution to ω_s may be completely neglected. The value of the cyclotron frequency shift ($\delta\omega_c/\omega_c \sim 10^{-12}$) approximates the experimental error and should be taken into account only when the latter is reduced. The resonant (with respect to the magnetic field) behavior of the cyclotron frequency shift and the boundary addition to

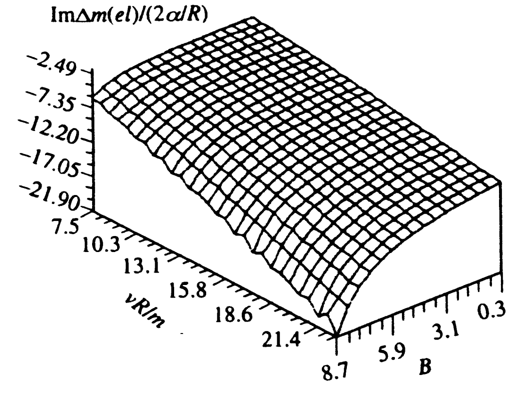

FIG. 5.

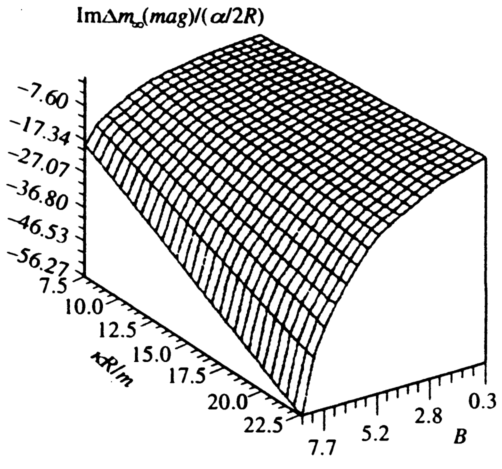

FIG. 7.

$g-2$ discovered for cavities with simple geometry in Refs. 5 and 6 has aroused special interest.[5)] Although under ideal boundary conditions the cyclotron frequency shift logarithmically diverges at a resonance, the detailed analysis of the problem performed for the case of two "mirrors" in Ref. 8 shows that consideration of the finite conductivity of the plates eliminates the infinity when $|\ln(0)|$ is replaced by a factor of order $|\ln \omega_c\delta|\sim 10$ (δ is the depth of the skin layer).

The instrumental mass and cyclotron frequency shifts have been calculated for a simplified cavity model using quantum electrodynamics under the restrictions[4,5,8,11,12]

$$\frac{1}{ml} \ll \frac{R_c^q}{l} \ll 1, \tag{36}$$

which follow from the conditions of experiments with an isolated electron. Here m^{-1}, R_c^q, and l are, respectively, the Compton wavelength of the electron, the quantum cyclotron radius $(e\eta)^{-1/2}$, and the distance between the conducting walls of the flat cavity. The first of inequalities (36) signifies nonrelativistic conditions, and the second inequality simulates the condition of confinement of the electron in a trap mentioned in the introduction. The validity of the classical treatment of the problem follows from the fact that the quantum conditions (36) leave a degree of freedom for the classical parameter $\omega_c l$: along with (36) the two possibilities

$$\frac{1}{ml} \ll \left(\frac{R_c^q}{l}\right)^2 \quad (\omega_c l \ll 1), \tag{37a}$$

$$\frac{1}{ml} \gg \left(\frac{R_c^q}{l}\right)^2 \quad (\omega_c l \gg 1) \tag{37b}$$

are equally permissible. "Retarded" conditions (37b) are realized in $g-2$ experiments [$R_c^q/l \sim 10^{-6}$, $1/ml \sim 10^{-11}$, $\omega_c l \sim 25$ (Refs. 6 and 8)].

3.1. Effective Hamiltonian and the cyclotron frequency shift

Like $V_\perp^2$, the mass shift alters the "dispersion law" $E(V_\perp^2)$, where E is the energy of the electron in the rest system of the cyclotron orbit. As clearly follows from the mathematical operations presented below [see (56) and (60)], the addition $\Delta L = -\Delta m^{\mathrm{mag}}/\gamma_\perp$ to the Lagrangian of a particle in the nonrelativistic limit has the form

$$\Delta L = -\delta m_C + \frac{\delta m}{2} V_\perp^2. \tag{38}$$

The real constant δm_C coincides with the Coulomb addition to the rest energy (it can be calculated by simple application of the image method), and the total energy of the charge is

$$E(V_\perp^2) = m + \delta m_C + \frac{m+\delta m}{2} V_\perp^2. \tag{39}$$

The value of δm specifies the cyclotron frequency shift (in the nonrelativistic approximation $\omega_c = \kappa/m$)

$$\delta\omega_c = -\omega_c \delta m/m \tag{40}$$

and, in the general case, is not equal to δm_C.[6)] Unlike δm_C, δm depends not only on the position of the orbit relative to the boundaries, but also on the magnitude of the magnetic field, as well as on the orientation of the latter relative to the cavity plates. It is important that the sign of the imaginary part of $\delta\omega_c$, which is determined by the causality of the Green's function $D_{\alpha\beta}^{(B)}$, correctly describes the decay of the cyclotron motion of the electron.

Let us proceed to the calculation of the correction δm. Let φ be the angle between the plane of the electron orbit and the planes of the boundaries $x_3=0$ and $x_3=l$, and let $R/2$ be the distance from the center of the orbit to the $x_3=0$ plane. Then the interval $(x-x'^{(N)})^2$ equals

$$(x-x'^{(2k)})^2 \equiv g(w,u,Z_k) = \frac{4}{\omega_c^2}[-w^2+V_\perp^2 \sin^2 w - 2Z_k V_\perp \sin\varphi \sin u \sin w + Z_k^2] \tag{41}$$

for even $N=2k=0,2,4,\ldots$, or

$$(x-x'^{(2k+1)})^2 \equiv \tilde{f}(w,u,X_k^\pm) = 4\omega_c^{-2}[-w^2+V_\perp^2 \sin^2 w + \tfrac{1}{2}V_\perp^2 \sin^2\varphi(\cos 2w + \cos 2u) \mp 2X_k^\pm V_\perp \sin\varphi \cos u \cos w + (X_k^\pm)^2] \tag{42}$$

for odd $N=2k+1=1,3,\ldots$. The value $k=0$ corresponds to the contribution of one boundary ($x_3=0$). The following notation was introduced in Eqs. (41) and (42):

$$Z_k = kl\omega_c \equiv 2\Lambda k, \quad X_k^\pm = 2\Lambda(k\pm t), \quad t=R/2l, \tag{43}$$

$$w=\kappa(\tau-\tau')/2m, \quad u=\kappa(\tau+\tau')/2m. \tag{44}$$

Thus, using (2)–(6), we arrive at the expression

$$\Delta W = \left(\frac{\kappa}{2m}\right)^2 \int_{-\tau_0}^{\tau_0} d\tau \int_{-\tau_0}^{\tau_0} d\tau' F_1(w,u), \tag{45}$$

in which

$$\tau_0 = (2\pi m/\kappa)M, \quad M(\text{integer}) \to \infty, \tag{46}$$

$$F_1(w,u) = F(w,u) + \frac{i\alpha}{4\pi}\sum_{k=1}^{\infty}\left[2\,\frac{g''_{ww}(w,u,Z_k)-g''_{uu}(w,u,Z_k)}{g(w,u,Z_k)+i0} - \frac{\tilde{f}''_{ww}(w,u,X_k^+)-\tilde{f}''_{uu}(w,u,X_k^+)}{\tilde{f}(w,u,X_k^+)+i0} - \frac{\tilde{f}''_{ww}(w,u,X_k^-)-\tilde{f}''_{uu}(w,u,X_k^-)}{\tilde{f}(w,u,X_k^-)+i0}\right]. \tag{47}$$

The function which does not depend on l

$$F(w,u) = \frac{i\alpha}{4\pi}\left[\frac{f''(w)}{f(w)} - \frac{2}{w^2} - \frac{\tilde{f}''_{ww}(w,u)-\tilde{f}''_{uu}(w,u)}{\tilde{f}(w,u)+i0}\right], \tag{48}$$

and the functions $f(w)$ and $\tilde{f}(w,u)$ are equal, respectively, to $g(w,u,Z_0)$ and $\tilde{f}(w,u,X_0^\mp)$ [see (41)–(43)]. Utilizing the periodicity of $F_1(w,u)$ with respect to u, we obtain the mass shift averaged with respect to the proper time period $2\pi m/\kappa$:

$$\Delta m^{\mathrm{mag}} = -\frac{1}{2\tau_0}\,\Delta W = -\frac{\kappa}{2\pi m}\int_0^{2\pi} du \int_0^{\infty} dw F_1(w,u). \tag{49}$$

It is convenient to perform the further calculations for the real and imaginary parts of Δm^{mag} separately. Contributions to the real part are made by the (nonmultiple) poles of the expressions in the square brackets on the right-hand sides of Eqs. (47) and (48). The half-residues in these poles are determined by the roots of transcendental equations like (12). To obtain the nonrelativistic expansion (38), these roots must be found with an accuracy of $V_\perp^2$. We demonstrate the method for the case of a single mirror ($l=\infty$), in which the function F from (48) appears instead of F_1 under the integral sign in (49). We have

$$\mathrm{Re}\,\Delta m^{\mathrm{mag}}(l=\infty) = \frac{\alpha\kappa}{8\pi m}\int_0^{2\pi} du\, \frac{\tilde{f}''_{ww}(w_+,u) - \tilde{f}''_{uu}(w_+,u)}{|\tilde{f}'_w(w_+,u)|}. \tag{50}$$

The positive root w_+ of the equation for $\tilde{f}(w,u)=0$ allows the nonrelativistic representation

$$\begin{aligned} w_+(u) = X_0^+\{1 &- (V_\perp/X_0^+)\sin\varphi \cos u \cos X_0^+ \\ &+ (V_\perp/2X_0^+)^2[2\sin^2 X_0^+ + \sin^2\varphi(\cos 2u \\ &+ \cos 2X_0^+ - 2\cos^2 u \cos^2 X_0^+ \\ &- 2X_0^+ \sin 2X_0^+ \cos^2 u)]\}, \end{aligned} \tag{51}$$

so that, expanding the integrand in (50) in powers of $V_\perp$ and integrating by parts, we find the cyclotron frequency shift (Ref. 23) in accordance with (38) and (40) ($\delta m = \delta m' + i\,\delta m''$, $\delta\omega_c = \delta\omega_c' + i\,\delta\omega_c''$):

$$\frac{\delta\omega_c'}{\omega_c} = -\frac{\alpha}{2Rm}\,(\Delta_1(X) + \sin^2\varphi\,\Delta_3(X)), \tag{52}$$

where $X = X_0^+ = R\omega_c/2$, and

$$\Delta_1(X) = 2\left(\frac{\sin 2X}{2X} - \cos 2X - \frac{1-\cos 2X}{4X^2}\right), \tag{53}$$

$$\Delta_3(X) = \cos 2X + \frac{\sin 2X}{2X} + \frac{3+\cos 2X}{4X^2}. \tag{54}$$

Equation (50) is a generalization of Eq. (28) to the case of arbitrary magnetic field orientation. Just how strongly the mass shift depends on this orientation can be seen from a comparison of Figs. 4 and 8. The latter figure shows $\mathrm{Re}\,\Delta m_\parallel^{\mathrm{mag}}$, i.e., $\mathrm{Re}\,\Delta m^{\mathrm{mag}}$ (50) when $\varphi=\pi/2$ (when the magnetic field is parallel to the boundary). Unlike $\mathrm{Re}\,\Delta m_\perp^{\mathrm{mag}}$ (28), $\mathrm{Re}\,\Delta m_\parallel^{\mathrm{mag}}$ tends to zero as $V_\perp$ increases (in the relativistic limit the self-field of the charge is "attracted" to the force lines of the magnetic field, becoming "insensitive" to the presence of a boundary). It should be noted that due to the restriction $R_c^{\mathrm{cl}} \equiv mV_\perp\gamma_\perp/\varkappa < R/2$, which is natural in the case of $\varphi=\pi/2$, we have the following region in which $\mathrm{Re}\,\Delta m_\parallel^{\mathrm{mag}}(V_\perp, \kappa R/2m)$ is defined:

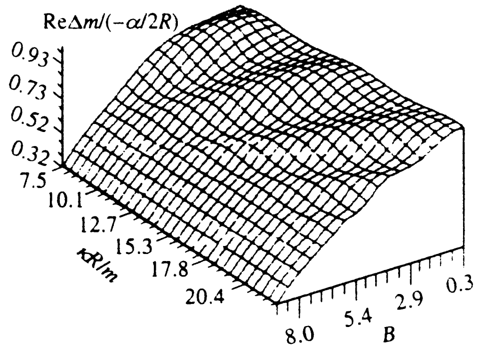

FIG. 8.

$$0 \leqslant V_\perp < (\kappa R/2m)(1+(\kappa R/2m)^2)^{-1/2}, \quad \kappa R/m > 0. \tag{55}$$

The upper limit of $V_\perp$ for the values of $\kappa R/m$ ($\geqslant 7$) chosen from Fig. 8 is close to the absolute maximum $V_\perp = 1$, so that this restriction is not manifested in the figure.

Returning to Eq. (49), we can use the scheme just described to find the nonrelativistic expansion of $\mathrm{Re}\,\Delta m^{\mathrm{mag}}$ in the case of a cavity:

$$\mathrm{Re}\,\Delta m^{\mathrm{mag}} = \gamma_\perp\left(\delta m_C - \frac{V_\perp^2}{2}\,\delta m'\right), \tag{56}$$

where

$$\delta m_C = \frac{\alpha}{4l}\,[\psi(t) + \psi(1-t) + 2\mathscr{C}] \tag{57}$$

is the well known Coulomb shift of the rest energy.[8,11,12] Here $\mathscr{C}=0.577...$ is Euler's constant, and $\psi(t)=\psi(R/2l)$ is the logarithmic derivative of the Γ function. According to (40), the addition $\delta m'$ specifies the real part of the cyclotron frequency shift:[23]

$$\begin{aligned} \frac{\delta\omega_c'}{\omega_c} = \frac{\alpha}{4ml}\sum_{k=1}^{\infty}\Bigg[&(\Delta_1(Z_k) + \sin^2\varphi\,\Delta_2(Z_k))\,\frac{2}{k} \\ &- (\Delta_1(X_k^-) + \sin^2\varphi\,\Delta_3(X_k^-))\,\frac{1}{k-t} - (\Delta_1(X_{k-1}^+) \\ &+ \sin^2\varphi\,\Delta_3(X_{k-1}^+))\,\frac{1}{k-1+t}\Bigg] \end{aligned} \tag{58}$$

[see the notation in (43), (44), (53), and (54)]. Here

$$\Delta_2(X) = \cos 2X - \frac{3\sin 2X}{2X} + \frac{3\sin^2 X}{2X^2}. \tag{59}$$

The calculation of the imaginary part of the cyclotron frequency shift ($\delta\omega_c''$) is more involved; therefore, here we present only the result of the calculations for an arbitrary position of the orbit in the cavity ($0<t<1$), but with $\varphi=0$ (when the magnetic field is orthogonal to the plates). In addition, in (49) the integration with respect to u reduces to the multiplier 2π, and in the nonrelativistic limit the integral with respect to w, which may be understood in the sense of a principal value, is proportional to $V_\perp^2$ [in accordance with the expansion (38)]:

$$\text{Im }\Delta m^{\text{mag}}=\alpha\omega_c\frac{V_\perp^2}{2}\left\{-\frac{2}{3}+\Delta_4(X_0^+)+\sum_{k=1}^{\infty}\left[\Delta_4(X_k^+)+\Delta_4(X_k^-)-2\Delta_4(Z_k)\right]\right\}, \tag{60}$$

$$\Delta_4(X)=(2X)^{-1}\left(\sin 2X+\frac{\cos 2X}{2X}-\frac{\sin 2X}{4X^2}\right). \tag{61}$$

We note several characteristic features of expression (60). The first term in curly brackets, which does not depend on R and l, describes the (nonrelativistic) mass shift in the absence of boundaries and was obtained in Ref. 16; the second term takes into account the influence of one plate, and the sum specifies the contribution of the cavity (which vanishes for $l=\infty$). It is interesting that this contribution becomes dominant in the limit

$$R\ll l, \tag{62a}$$

$$\omega_c R\ll 1, \tag{62b}$$

since the first two terms cancel one another [$\Delta_4(0)=2/3$]. If the second plate is moved away and (62b) is maintained, the radiation vanishes (Im $\Delta m=0$). It is important to stress that the absence of a constant term independent of $V_\perp$ in Eq. (60) for Im Δm is closely related to the nonmultiplicity of the poles of F_1 in (49), i.e., to the property like (11b) in intervals (41) and (42), or, in the final analysis, to the causality of the world line of the charge.

The decay rate (or the imaginary part of the cyclotron frequency shift) obtained using (60) and (40) has the form

$$\delta\omega_c''=-\frac{2\alpha}{3m}\omega_c^2+\frac{\alpha}{m}\omega_c^2\sum_{k=1}^{\infty}\left[\Delta_4(2\Lambda(k+t-1))+\Delta_4(2\Lambda(k-t))-2\Delta_4(2\Lambda k)\right]. \tag{63}$$

In the special case $t=1/2$ (when the charge is in the midplane of the flat cavity) Eqs. (58) ($\varphi=0$) and (63) transform into the corresponding equations in Ref. 6. We present these expressions here:

$$\frac{\delta\omega_c'}{\omega_c}=\frac{2\alpha}{ml}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[-\cos 2\Lambda n+\frac{\sin 2\Lambda n}{2\Lambda n}-\frac{1-\cos 2\Lambda n}{4n^2\Lambda^2}\right], \tag{64}$$

$$\frac{\delta\omega_c''}{\omega_c}=-\frac{2\alpha}{3m}\omega_c+\frac{2\alpha}{ml}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times\left[\sin 2\Lambda n+\frac{\cos 2\Lambda n}{2\Lambda n}-\frac{\sin 2\Lambda n}{4\pi^2\Lambda^2}\right]. \tag{65}$$

The Fourier series (64) and (65) can be summed (see Ref. 29) to establish the presence of logarithmic divergences ("resonances") in $\delta\omega_c'$ for $\Lambda=(2N+1)\pi/2$ and finite discontinuous jumps in $\delta\omega_c''$ (see Ref. 6). The relationship between our notation and the notation in Ref. 6 is as follows:

$$l\equiv 2L,\quad \alpha/m\equiv r_0,\quad 2\Lambda=\omega_c l\equiv 2\pi\xi_c\simeq 2\pi\xi, \tag{66}$$

$$\delta\omega_c'\simeq R_P(\omega_c),\quad \delta\omega_c''\simeq-\tfrac{1}{2}I_P(\omega_c). \tag{67}$$

3.2. Resonances and antiresonance points

Series (58) and (63) reduce to Fourier series, whose coefficients are represented by rational functions of k (see Appendix). The functions $\delta\omega_c'$ and $\delta\omega_c''$ have singularities when

$$\Lambda=\Lambda_r\equiv N\pi/2,\quad \omega_c=N\pi/l, \tag{68}$$

where $N=1,2,\ldots$, due to the slow ($\propto k^{-1}$) decay coefficients in some of the series. The character of these singularities and their dependence on the second parameter t can be established after calculating the trigonometric sums (58) and (63). We at once note that the singularities in $\delta\omega_c'$ and $\delta\omega_c''$ discovered in Ref. 6 do not exhaust all the possibilities, since between the cavity plates there are "antiresonance" points,[23] whose coordinate t satisfies the condition

$$1-\cos 4\Lambda_r t=0,\quad\text{or}\quad t=M/N,\quad M=\overline{1,N-1}. \tag{69}$$

The midpoint of the cavity, where Brown *et al.*[6] considered the cyclotron frequency shift, is just one of these points. The latter are identified by the fact that the coefficient vanishes when there is a singularity at these points, if (69) is satisfied.[7)]

The complete expression for $\delta\omega_c'/\omega$ can be obtained by applying equations from the Appendix to the sum (58). The result is not presented here due to its complicated nature [for the special case of $\delta\omega_c'/\omega_c$ see Eq. (A9) in the Appendix]. Let us examine the resonance part, since it alone contains singularities of interest to us. We bring Eq. (A8) for $(\delta\omega_c'/\omega_c)_{\text{res}}$ into the form

$$\left(\frac{\delta\omega_c'}{\omega_c}\right)_{\text{res}}=\frac{\alpha(2-\sin^2\varphi)}{4ml}\left[\left(\beta(t)-\beta(1-t)-\frac{1}{t}\right)\times\cos\pi t+\frac{\cos 4\Lambda t}{t}+2\ln 2+2\int_{\pi/2}^{2\Lambda}dy(1-\cos 2ty)\cot y\right],$$

$$0<t<1. \tag{70}$$

The last (integral) term in square brackets contains logarithmic singularities. In cases in which (68) is not satisfied, the integral is finite. In the limit $\Lambda\to\Lambda_r$ it can be represented in the following form:

$$\int_{\pi/2}^{2\Lambda}dy(1-\cos 2ty)\cot y=(1-\cos 4\Lambda_r t)\ln|\sin 2\Lambda|+\ldots, \tag{71}$$

where the ellipses denote terms that are finite in the limit $\Lambda\to\Lambda_r$. Thus, when $\Lambda\to\Lambda_r$, there are no singularities in (70), if t satisfies (69).

Let us consider the imaginary part of the cyclotron frequency shift. The sum (63) is calculated in the Appendix and has the form (A7). The following properties of $\delta\omega_c''$ can easily be derived from (A7):

$$\delta\omega_c''=0, \quad N=0 \ (2\Lambda<\pi), \quad 0<t<1, \tag{72}$$

$$[\delta\omega_c''(\lambda-0)-\delta\omega_c''(\lambda+0)]_{\lambda=N}=\frac{\alpha\omega_c^2}{m}\frac{1}{N} \times(1-\cos 2\pi tN)>0. \tag{73}$$

The magnitude of the jump decreases monotonically with increasing $\lambda=2\Lambda/\pi=N$ and vanishes at the antiresonance points (69). Furthermore, a value of $\omega_c l\sim 25$ [see text after Eq. (37)] corresponds to the presence of seven or eight such points.

4. CONCLUSIONS

The main results of this work were enumerated in the introduction and in the abstract. Several qualitative remarks are in order. First of all, we stress that the mass shift, which is a classical first-order radiation correction, is determined on unperturbed world lines. The presence of boundaries, therefore, requires setting a criterion for smallness of the trajectory perturbation caused by a boundary. The latter can be obtained from a comparison of effective Hamiltonian (39) with its unperturbed part:

$$|\delta m_C|\sim\alpha/R\ll mV_\perp^2/2, \tag{74a}$$

$$\delta m\ll m. \tag{74b}$$

Condition (74b) might have been violated in the case of resonance; however, in reality it holds for nonideal plates.[8] Condition (74a) may be rewritten in the form $r_0\ll RV_\perp^2$, so that it is clearly violated for slow motions. In $g-2$ experiments the corresponding values of the parameters are such that this condition is satisfied ($R\sim 1$ cm, $V_\perp\sim 10^{-5}$, $r_0/R\sim 10^{-13}$).[9] Furthermore, there is an important difference between δm and δm_C: δm becomes δm_C only under special conditions ($\varphi=0$, $\omega_c R\ll 1$).[18,23] This has spurred on attempts to find additional "shifts" in the cyclotron frequency, magnetic moment, etc. by means of the simple replacement $m\to m+\delta m_C$ (see, for example, Ref. 31).

In conclusion, we stress that the principal "practical" results of this work [Eqs. (58) and (63) and their corollaries] have only a qualitative connection to real $g-2$ experiments. Nevertheless, they raise interest in the question of the existence and dynamic role of resonances and antiresonance points in "relativistic" models of cavities.

We thank V. I. Ritus for some stimulating discussions of the questions addressed.

APPENDIX

The main role in the calculation of trigonometric sums (58) and (63) is played by the series

$$f_s(x)=\sum_{k=0}^{\infty}\frac{\sin kx}{(k+t)^s}, \quad g_s(x)=\sum_{k=0}^{\infty}\frac{\cos kx}{(k+t)^s}, \tag{A1}$$

for which closed expressions were given in Ref. 31. In particular, for $s=1$,[8)]

$$f_1(x)=\beta(t)\sin(\pi t-tx)+\frac{1}{2}\int_x^\pi\frac{\sin[-xt+(t-1/2)y]}{\sin(y/2)}dy, \tag{A2}$$

$$g_1(x)=\beta(t)\cos(\pi t-tx)+\frac{1}{2}\int_x^\pi\frac{\cos[-xt+(t-1/2)y]}{\sin(y/2)}dy. \tag{A3}$$

However, the ranges of applicability of Eqs. (A2) and (A3) were not indicated in Ref. 31. An analysis shows that these equations hold in the range $0<x<2\pi$, the existence of a strict inequality being significant, since at the ends of the interval $(0,2\pi)$, f_1 is discontinuous (its jump is equal to π and does not depend on t), and g_1 goes to infinity. For periodic extension along the entire numerical axis, the obvious symmetry properties of expressions (A2) and (A3) [for example, $f_1(x)=-f_1(2\pi-x)$] become important. For this extension x should be replaced on the right-hand sides of Eqs. (A2) and (A3) by

$$\tilde{x}=x-2\pi[x/2\pi]\equiv x-2\pi N. \tag{A4}$$

Thus, we obtain, for example,

$$\sum_{k=0}^{\infty}\frac{\sin(k+t)x}{(k+t)}=\beta(t)\sin(\pi t+2\pi tN)+\frac{1}{2}\int_{\tilde{x}}^\pi\frac{\sin[2\pi tN+(t-1/2)y]}{\sin(y/2)}dy, \tag{A5}$$

$$\sum_{k=0}^{\infty}\frac{\cos(k+t)x}{(k+t)}=\beta(t)\cos(\pi t+2\pi tN)+\frac{1}{2}\int_{\tilde{x}}^\pi\frac{\cos[2\pi tN+(t-1/2)y]}{\sin(y/2)}dy. \tag{A6}$$

The last equations should be used together with the known results for Fourier series of the forms[29] $\Sigma k^{-s}\sin kx$ and $\Sigma k^{-s}\cos kx$ to calculate $\delta\omega_c'$ and $\delta\omega_c''$.

Equation (63) for $\delta\omega_c''$ ($0<t<1$, $\varphi=0$) can be transformed by replacing x with $\tilde{x}$ ($\lambda=2\Lambda/\pi$, $N=[\lambda]$):

$$\frac{\delta\omega_c''}{\omega_c}=\frac{\alpha\omega_c}{2m}\left\{-\frac{1}{\lambda}\left(N+\frac{1}{2}\right)-\frac{1}{3}\lambda^{-3}N\left(N+\frac{1}{2}\right)(N+1)+\frac{1}{2}\lambda^{-3}(\lambda^2+N^2)\frac{\sin(2\pi tN+\pi t)}{\sin\pi t}-\frac{1}{2\lambda^3}\left(\frac{\sin 2\pi tN\cos\pi t}{2\sin^3\pi t}-N\frac{\cos 2\pi tN}{\sin^2\pi t}\right)\right\}. \tag{A7}$$

The corresponding expression for $\delta\omega_c'$ is fairly cumbersome, and we present here only the "resonant" part, which originates from the first terms in Δ_1, Δ_2, and Δ_3 [see Eqs. (53), (54), and (59)]:

$$\left(\frac{\delta\omega_c'}{\omega_c}\right)_{\rm res}=\frac{\alpha(2-\sin^2\varphi)}{4ml}\sum_{k=1}^{\infty}\left[\frac{\cos 4\Lambda(k-t)}{(k-t)}+\frac{\cos 4\Lambda(k-1+t)}{k-1+t}-\frac{2\cos 4\Lambda k}{k}\right]$$
$$=\frac{\alpha(2-\sin^2\varphi)}{4ml}\left[(\beta(t)-\beta(1-t))\cos(\pi t+2\pi tN)+\int_{\tilde{x}}^{\pi}\frac{\cos[2\pi tN+(t-1/2)y]}{\sin(y/2)}dy+\ln(4\sin^2(x/2))\right], \qquad (A8)$$

where $x\equiv 4\Lambda$. We also present the complete expression specifying $\delta\omega_c'/\omega_c$ for the special values $t=1/2$ and $\varphi=\pi/2$:

$$\frac{\delta\omega_c'}{\omega_c}=-\frac{\alpha}{2lm}\left[-\ln(4\cos^2\Lambda)+7\zeta(3)\frac{1+3(-1)^N}{16\Lambda^2}-\frac{1}{16\Lambda^2}\int_0^{\tilde{x}}(y+2\pi N)[3(-1)^N\ln\tan(y/4)+\ln(2\sin(y/2))]dy\right]. \qquad (A9)$$

It is easily verified that Eq. (A8) gives a singular logarithm in (A9) (the first term in the brackets) for $t=1/2$ and $\varphi=\pi/2$. Oddly enough, the remaining terms in the square brackets in (A9) represent the continuous function $x=4\Lambda$ ($N=[x/2\pi]$). This is due to their origin: unlike the sum (A8) the trigonometric series representing them, like f_s and g_s ($s=2,3$), converge uniformly [see (58) and Ref. 29].

1)We omit the analysis of the inaccuracies and errors accompanying the solution of the problem of the instrumental contributions in quantum electrodynamics. For information on this subject, see Refs. 4, 8, and 14.

2)A system of units in which $c=1$, $\hbar=1$, and $\alpha=e^2/4\pi\hbar c$ and a metric in which $x_\alpha=(x,ix_0)$, $\tilde{F}_{\mu\nu}=(i/2)\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ ($F_{\mu\nu}$ is the electromagnetic field tensor), etc. is used.

3)Such invariance would be absent, for example, in the case of a lattice boundary: a charge moving without acceleration perpendicularly to conducting strips radiates (the Smith–Purcell effect). In our case the uniformly accelerated electron does not radiate: Re $\Delta m=-\alpha/2R$, Im $\Delta m=0$, see Eq. (17).

4)We thank V. I. Ritus, who also turned our attention to Eqs. (19) and (20), for this remark.

5)We should also point out Ref. 28, in which such resonances were predicted qualitatively for an arbitrary cavity. The model of coupled oscillators used in Ref. 28 is in many ways similar to the rigorous treatment of the problem of the cyclotron frequency shift in Ref. 6.

6)The correspondence between $\delta\omega_c$ and δm specified by Eq. (40) follows from expression (39) for the energy.

7)A similar effect in the real part of the spin precession frequency shift $\delta\omega_s$ was noted in Ref. 8.

8)$\beta(t)=\frac{1}{2}[\psi(1/2+t/2)-\psi(t/2)]$.

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