Small-scale spatially periodic Josephson structures

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A set of new nonlinear solutions of nonlocal Josephson electrodynamics, which describe vortex states with a characteristic spatial scale smaller than the London penetration depth, has been obtained. Two types of solutions, which correspond to both states with a nonzero mean magnetic field and states without a mean field, have been considered. A description of stationary, spatially periodic vortex states has been given. The relaxation laws in the strong-dissipation limit have been obtained. Finally, a description of periodic traveling structures has been given in the nondissipative limit.

1. It is usually assumed[1] in the electrodynamics of Josephson junctions that the characteristic scale of the structures considered is large compared with the London penetration depth λ of a magnetic field into a superconductor. Nonlocal Josephson electrodynamics do not have such a restriction.[2,3] The applicability of nonlocal electrodynamics is revealed, for example, when a Josephson junction in a strong magnetic field whose field strength vector lies in the plane of the junction is described. Here a mixed state is realized with a periodic structure having a period

$$L=\frac{\phi_0}{2\pi(\lambda_+ +\lambda_- +2d)\bar{H}}, \tag{1.1}$$

where $\phi_0=\pi\hbar c/|e|=2.05\cdot 10^{-7}$ Oe $\cdot$cm^2 is the magnetic flux quantum, $2d$ is the thickness of the tunnel junction, λ_+ and λ_- are the London penetration depths of a magnetic field into the superconductors on opposite sides of the junction, and $\bar{H}$ is the magnetic field averaged over the junction. According to Ref. 4, the scale (1.1) is smaller then the London penetration depth when the magnetic field strength is close to the value of the lower critical field.

On the other hand, in ordinary Josephson electrodynamics it is assumed that the London depths (λ_+,λ_-) are small compared with the Josephson length λ_j, which is defined by the formula

$$\lambda_j^2=\frac{\hbar c^2}{8\pi|e|j_c(\lambda_+ +\lambda_- +2d)}, \tag{1.2}$$

where j_c is the critical current density through the Josephson junction. It can be assumed that tunnel junctions with a very large critical current density can be realized, and that the unusual situation in which the London depths are larger than the Josephson length is then possible. Let us discuss this situation in the simple case of a tunnel junction between two identical superconductors. Then the unusual condition

$$\lambda>\lambda_j \tag{1.3}$$

is realized according to Eq. (1.2) when

$$j_c>\frac{\hbar c^2}{16\pi|e|\lambda^3}\equiv j_0. \tag{1.4}$$

Here it is also assumed that the thickness of the junction is small compared with the London depth. At zero temperature

$$j_c=\frac{\pi\Delta(0)}{2|e|R},\quad \Delta(0)=\frac{\hbar v_F}{\pi\xi},$$

$$\lambda^2(0)=\frac{mc^2}{4\pi e^2 n_e},\quad n_e=\frac{p_F^3}{3\pi\hbar^3}, \tag{1.5}$$

where $\Delta(0)$ is the width of the superconducting gap, R is the resistivity, v_F and p_F are the velocity and momentum on the Fermi surface, m is the mass of the electron, and ξ is the correlation length. Then condition (1.4) can be written in the form

$$\kappa=\frac{\lambda(0)}{\xi}>\frac{e^2 n_e R}{2mV_F}, \tag{1.6}$$

where κ is the parameter of the Ginzburg–Landau theory. If the resistivity is represented in the form

$$R=\frac{2d}{\sigma}=\frac{2dm}{e^2 n_e \tau_{fp}}, \tag{1.7}$$

where σ is the conductivity and τ_{fp} is the effective free-path time, which determines the value of $j_c(0)$, Eq. (1.6) takes the form

$$\kappa>\frac{d}{v_F\tau_{fp}}=\frac{d}{l_{fp}}. \tag{1.8}$$

Of course, this condition is more easily satisfied, the smaller the thickness of the tunnel junction. However, the critical tunneling current need not exceed the pair-breaking current[5]

$$j_c<j_d=\frac{\hbar c^2}{12\sqrt{3}\pi|e|\lambda^2\xi}. \tag{1.9}$$

It is not difficult to see that the condition $j_d\gg j_0$ reduces to

$$\kappa\gg 1. \tag{1.10}$$

Thus, the London depth may be smaller than the Josephson length in superconductors with a large Ginzburg–Landau parameter κ, in accordance with Eq. (1.9), and in sufficiently thin tunnel junctions, in accordance with Eq. (1.8).

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The equation used in nonlocal Josephson electrodynamics for the phase difference φ between Cooper pairs on opposite sides of an infinitely long junction is[3]

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2}=\lambda_0^3\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}dz'Q(z-z')\frac{\partial\varphi(z',t)}{\partial z'},\qquad(1.11)$$

where $\lambda_0^3=\lambda_j^2(\lambda_++\lambda_-+2d)$, ω_j is the Josephson frequency, β characterizes the dissipation properties of the tunnel junction, and

$$Q(z)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}K(k)e^{ikz},\qquad(1.12)$$

$$K(k)=\frac{1}{\lambda_+\sqrt{k^2\lambda_+^2+1}+\lambda_-\sqrt{k^2\lambda_-^2+1}+2d}.\qquad(1.13)$$

The magnetic field strength within the junction ($-d\leq z\leq +d$) is given by

$$H_y(z,t)=-\frac{\hbar c}{2|e|}\int_{-\infty}^{+\infty}dz'Q(z-z')\frac{\partial\varphi(z',t)}{\partial z'},\qquad(1.14)$$

and the field in the superconductors is given by

$$H_y(x,z,t)=-\frac{\hbar c}{2|e|}\int_{-\infty}^{+\infty}dz'Q_\pm(z-z',\pm x-d)\frac{\partial\varphi(z',t)}{dz'},\qquad(1.15)$$

where

$$Q_\pm(z,x)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}K(k)\exp\{ikz-x(k^2+\lambda_\pm^{-2})^{1/2}\}.\qquad(1.16)$$

The plus and minus signs in Eq. (1.15) correspond, respectively, to the regions in space occupied by the superconductors $x>d$ and $x<-d$.

The energy density of a tunnel junction is given by[1)]

$$\mathscr{E}(z,t)=\frac{\hbar j_c}{2|e|}\left\{\frac{1}{2\omega_j^2}\left(\frac{\partial\varphi}{\partial t}\right)^2+1-\cos\varphi+\frac{1}{2}\lambda_0^3\frac{\partial\varphi(z,t)}{\partial t}\times\int_{-\infty}^{+\infty}dz'Q(z-z')\frac{\partial\varphi(z',t)}{\partial z'}\right\}.\qquad(1.17)$$

In this report we present results pertaining to the solution of Eq. (1.11) in the asymptotic limit, in which φ varies sharply over a length smaller than λ_+ and λ_- and Eq. (1.11) has the form

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{l}{\pi}\fint_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'},\qquad(1.18)$$

where $l=\lambda_0^3(\lambda_+^2+\lambda_-^2)^{-1}$. Several exact solutions of this equation are presently known. However, they can still be counted on the fingers of one hand: the stationary 2π kink in Ref. 6, the traveling 4π kink in Refs. 3, and 7, and the vortices relaxing with strong dissipation in Ref. 8. The results in Refs. 2 and 4 can be attributed to approximate consequences of nonlocal Josephson electrodynamics. Below we examine some new exact solutions of Eq. (1.18), which describe the vortex state both in the presence of a mean magnetic field and in the absence of such a field. We present the results for stationary states, for the strongly dissipative relaxation of such vortex states, and for traveling nonlinear vortex structures. The stability of the nonlinear structures studied will be addressed in the Appendix.

It should be stated that the ensuing results are based mathematically on the following Hilbert transform pair ($\alpha>0$, $A>0$):

$$\frac{1}{\pi}\fint_{-\infty}^{+\infty}\frac{dx}{x-y}\frac{\sinh\alpha}{\cosh\alpha-\cos Ax}=\frac{\sin Ay}{\cos Ay-\cosh\alpha},\qquad(1.19)$$

$$\frac{1}{\pi}\fint_{-\infty}^{+\infty}\frac{dx}{x-y}\frac{\sinh\alpha\cos Ax}{\cosh 2\alpha-\cos 2Ax}=\frac{\cosh\alpha\sin Ay}{\cos 2Ay-\cosh 2\alpha},\qquad(1.20)$$

where the integral is taken in the sense of the Cauchy principal value. These formulas were established in Ref. 9 in the theory of Peierls–Nabarro dislocations during the solution of the Peierls equation,[10] which coincides with stationary limit (1.18) when the sign of its right-hand side is reversed.

2. We first present the stationary solution of Eq. (1.18) corresponding to a vortex state in the presence of a constant mean magnetic field. Such a state is described by a phase difference:

$$\varphi=\pi+2\arctan\left[\left(\sqrt{\frac{L^2}{l^2}+1}+\frac{L}{l}\right)\tan\frac{z}{2L}\right].\qquad(2.1)$$

This expression is distinguished from the known solution of Peierls equation[9] by the presence of the term π. The Hilbert transform (1.19) is essential for obtaining such a solution, as well as other solutions with a nonzero mean magnetic field.

The magnetic field corresponding to solution (2.1) in the superconductors can be represented in the following form:

$$H_y(x,z)=-\frac{\phi_0}{2\pi L}\left\{\frac{\exp[-(\pm x-d)\lambda_\mp^{-1}]}{\lambda_++\lambda_-+2d}+\frac{L}{\lambda_+^2+\lambda_-^2}\left[\frac{\pm x-d}{L}+\operatorname{arcsinh}\frac{l}{L}-\ln\left(2\left[\sqrt{1+\frac{l^2}{L^2}}\cosh\left(\frac{\pm x-d}{L}\right)+\frac{l}{L}\sinh\left(\frac{\pm x-d}{L}\right)-\cos\frac{z}{L}\right]\right)\right]\right\}.\qquad(2.2)$$

The magnetic field within the junction has the form

$$H_y(z)=-\bar{H}+\delta H_y(z),\qquad(2.3)$$

where for the averaged field we have

$$\bar{H}=\frac{\phi_0}{2\pi L(\lambda_++\lambda_-+2d)},\qquad(2.4)$$

and the periodic part is described by (see Ref. 11)

$$\delta H_y(z)=-\frac{\phi_0}{2\pi(\lambda_+^2+\lambda_-^2)}\left[\operatorname{arcsinh}\frac{l}{L}-\ln\left(2\left[\sqrt{1+\frac{l^2}{L^2}}-\cos\frac{z}{L}\right]\right)\right]=-\frac{\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{\frac{l^2}{L^2}+1}-\frac{l}{L}\right)^n\cos\frac{nz}{L}. \tag{2.5}$$

Solution (2.1) corresponds to the following energy density in the tunnel junction:

$$\mathscr{E}(z)=\frac{\hbar j_c}{2|e|}\left\{1-\sqrt{1+\frac{l^2}{L^2}}+\frac{(l/L)^2}{\sqrt{1+(l/L)^2}-\cos(z/L)}\times\left[1+\frac{\lambda_+^2+\lambda_-^2}{2L(\lambda_++\lambda_-+2d)}+\frac{1}{2}\ln\frac{\sqrt{l^2+L^2}+l}{2(\sqrt{l^2+L^2}-L\cos(z/L))}\right]\right\}. \tag{2.6}$$

The energy density averaged over the period of the spatial oscillations is

$$\bar{\mathscr{E}}=\frac{\hbar j_c}{2|e|}\left\{1+\frac{l}{L}-\sqrt{1+\frac{l^2}{L^2}}+\frac{l(\lambda_+^2+\lambda_-^2)}{2L^2(\lambda_++\lambda_-+2d)}-\frac{l}{L}\ln\left[\frac{2l}{L}\left(\sqrt{1+\frac{l^2}{L^2}}-\frac{l}{L}\right)\right]\right\}. \tag{2.7}$$

In the asymptotic limit $L\ll l$ we have

$$\delta H_y(z)=-\frac{\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{L}{2l}\right)^n\cos\frac{nz}{L}, \tag{2.8}$$

$$\varphi(z)\simeq\pi+\frac{z}{L}. \tag{2.9}$$

In the opposite limit $L\gg l$ it follows from (2.1) that

$$\varphi(z)=\pi+2\arctan\left[\frac{2L}{l}\tan\frac{z}{2L}\right]. \tag{2.10}$$

Hence the formal transition ($L\rightarrow\infty$) can be made to a 2π kink[6,9]:

$$\varphi(z)=\pi+2\arctan\frac{z}{l}. \tag{2.11}$$

3. We turn now to the next solution, which corresponds to strong dissipation (compare Ref. 8), in which the nonstationary evolution of a vortex state is determined by the term containing the first derivative with respect to time, and the second derivative on the left-hand side of Eq. (1.18) can be neglected:

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t}=\frac{l}{\pi}⨍_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'}. \tag{3.1}$$

Our proposed solution of this equation has the form

$$\varphi(z,t)=\pi+2\arctan\left[\frac{\tan(z/2L)}{\tanh\{\alpha(t)/2\}}\right]. \tag{3.2}$$

Here

$$\frac{\beta}{\omega_j^2}\frac{d\alpha}{dt}+\sinh\alpha=\frac{l}{L}. \tag{3.3}$$

The relaxation solution of Eq. (3.3) has the form

$$\tanh\frac{\alpha(t)}{2}=-\frac{L}{l}+\sqrt{\frac{L^2}{l^2}+1}\times\frac{\left[\tanh\frac{\alpha_0}{2}+\frac{L}{l}\right]\cosh\frac{t}{\tau}+\sqrt{1+\frac{L^2}{l^2}}\sinh\frac{t}{\tau}}{\left[\tanh\frac{\alpha_0}{2}+\frac{L}{l}\right]\sinh\frac{t}{\tau}+\sqrt{1+\frac{L^2}{l^2}}\cosh\frac{t}{\tau}}, \tag{3.4}$$

where $\tau=2\beta[\omega_j^{-2}(L^2+l^2)^{-1/2}]$. According to (3.4), as the time increases, the solution (3.2) departs from the initial value $\alpha_0=\alpha(t=0)$ and relaxes to stationary state (2.1).

Solution (3.1) corresponds to the following dependence of the magnetic field strength in the superconductors on the coordinates and time:

$$H_y(x,z,t)=-\frac{\phi_0}{2\pi L}\left\{\frac{\exp[-(\pm x-d)\lambda_\mp^{-1}]}{\lambda_++\lambda_-+2d}+\frac{L}{\lambda_+^2+\lambda_-^2}\left[\alpha(t)+\frac{(\pm x-d)}{L}-\ln\left(2\left[\cosh\left(\frac{\pm x-d}{L}+\alpha(t)\right)-\cos\frac{z}{L}\right]\right)\right]\right\}. \tag{3.5}$$

As in Eq. (2.3), the magnetic field without the junction is the sum of the constant averaged field $\bar{H}$ defined by Eq. (24) and the oscillating field[11]

$$\delta H_y(z,t)=-\frac{\phi_0}{2\pi(\lambda_+^2+\lambda_-^2)}\times\left[\alpha(t)-\ln\left(2\left\{\cosh\alpha(t)-\cos\frac{z}{L}\right\}\right)\right]=-\frac{\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\sum_{n=1}^{\infty}\frac{1}{n}\exp[-n\alpha(t)]\cos\frac{nz}{L}. \tag{3.6}$$

The nonstationary energy density of the tunnel junction corresponding to solution (3.2) has the form

$$\mathscr{E}(z,t)=\frac{\hbar j_c}{2|e|}\left\{1-\cosh\alpha(t)+\frac{\sinh^2\alpha(t)}{\cosh\alpha(t)-\cos(z/L)}+\frac{l}{2L}\frac{\sinh\alpha(t)}{\cosh\alpha(t)-\cos(z/L)}\left\{\frac{\lambda_+^2+\lambda_-^2}{L(\lambda_++\lambda_-+2d)}+\alpha(t)-\ln\left(2\left[\cosh\alpha(t)-\cos\frac{z}{L}\right]\right)\right\}\right\}. \tag{3.7}$$

The energy density averaged over a spatial period is

$$\bar{\mathscr{E}}=\frac{\hbar j_c}{2|e|}\left\{1-e^{-\alpha(t)}+\frac{l(\lambda_+^2+\lambda_-^2)}{2L^2(\lambda_++\lambda_-+2d)}-\frac{l}{L}\ln(1-e^{-2\alpha(t)})\right\}. \qquad (3.8)$$

4. The third solution of (1.18) corresponds to a nonstationary state in the nondissipative limit ($\beta=0$), where Eq. (1.18) reduces to the following:

$$\sin\varphi+\frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{l}{\pi}\fint_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'}. \qquad (4.1)$$

The solution of this equation in the form of a traveling wave has the form

$$\varphi(z,t)=4\arctan\left[\sqrt{\frac{l\omega_j+V}{l\omega_j-V}}\tan\left(\frac{z-Vt}{2L}\right)\right]. \qquad (4.2)$$

The velocity of the traveling wave is related to L by

$$L=\frac{V^2}{\omega_j\sqrt{l^2\omega_j^2-V^2}}. \qquad (4.3)$$

This velocity is clearly restricted:

$$V<l\omega_j. \qquad (4.4)$$

Equation (4.3) can be represented in the form

$$V^2=\frac{1}{2}\omega_j^2L(\sqrt{L^2+4l^2}-L). \qquad (4.5)$$

This formula makes it possible, in particular, to trace how the velocity of a standing wave approaches the limiting value $l\omega_j$ as L increases. The limiting value corresponds to the velocity of a 4π kink,[3,7] into which solution (4.2) transforms as $L\to\infty$.

Equation (4.2) corresponds to the following expression for the magnetic field in the superconductors:

$$H_y(x,z,t)=-\frac{\phi_0}{\pi L}\left\{\frac{\exp[-(\pm x-d)\lambda_\mp^{-1}]}{\lambda_++\lambda_-+2d}+\frac{L}{\lambda_+^2+\lambda_-^2}\left[\frac{\pm x-d}{L}+\ln\left(\frac{l\omega_j}{V}+\sqrt{\frac{l^2\omega_j^2}{v^2}-1}\right)-\ln\left(2\left[\frac{l\omega_j}{V}\cosh\left(\frac{\pm x-d}{L}\right)+\frac{V}{\omega_jL}\sinh\left(\frac{\pm x-d}{L}\right)-\cos\left(\frac{z-vt}{L}\right)\right]\right)\right]\right\}. \qquad (4.6)$$

Within the tunnel junction

$$H_y(z,t)=-\frac{\phi_0}{\pi L(\lambda_++\lambda_-+2d)}+\delta H_y(z,t). \qquad (4.7)$$

Here the constant magnetic field is twice the magnetic field (2.4), and for the oscillating part we have:[11]

$$\delta H_y(z,t)=-\frac{\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\ln\frac{l\omega_j+\sqrt{l^2\omega_j^2-V^2}}{2\left(l\omega_j-V\cos\frac{z-Vt}{L}\right)}=-\frac{2\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\sum_{n=1}^{\infty}\left[\frac{l\omega_j}{V}-\sqrt{\frac{l^2\omega_j^2}{V^2}-1}\right]^n\cos\left[\frac{n(z-Vt)}{L}\right]. \qquad (4.8)$$

Traveling wave (4.2) corresponds to the following expression for the energy density:

$$\mathscr{E}(z,t)=\frac{\hbar j_cVl}{|e|\omega_jL^2}\frac{1}{(l\omega_j/V)-\cos[(z-Vt)/L]}\times\left\{1+\frac{V}{\omega_jl}\cos\frac{z-Vt}{L}+\frac{\lambda_+^2+\lambda_-^2}{L(\lambda_++\lambda_-+2d)}+\ln\left(\frac{l\omega_j}{V}+\frac{V}{L\omega_j}\right)-\ln\left(2\left[\frac{l\omega_j}{V}-\cos\left(\frac{z-Vt}{L}\right)\right]\right)\right\}. \qquad (4.9)$$

For the energy density averaged over an oscillation period we have

$$\bar{\mathscr{E}}=\frac{\hbar j_cl}{|e|L}\left\{2-\frac{V^2}{\omega_j^2l}+\frac{\lambda_+^2+\lambda_-^2}{L(\lambda_++\lambda_-+2d)}-2\ln\left(\frac{2l}{L}-\frac{2V^2}{L^2\omega_j^2}\right)\right\}. \qquad (4.10)$$

This presently exhausts the solutions of Eq. (1.18) with nonzero magnetic field averaged over the oscillations that we have obtained.

5. Let us now turn to the periodic solutions of Eq. (1.18), which correspond to states with zero mean mag-netic field. The corresponding stationary solution has the form

$$\varphi(z)=\pi+2\arctan\left(\sqrt{\frac{L^2}{l^2}-1}\sin\frac{z}{L}\right). \qquad (5.1)$$

Transformation (1.20) is essential for obtaining this and the ensuing solutions. Unlike (2.1), solution (5.1) is possible only at $L>l$. Accordingly, for the magnetic field in the superconductors we have

$$H_y(x,z)=-\frac{\phi_0}{2\pi(\lambda_+^2+\lambda_-^2)}\ln\frac{L\cosh\left(\frac{\pm x-d}{L}\right)+l\sinh\left(\frac{\pm x-d}{L}\right)+\sqrt{L^2-l^2}\cos\frac{z}{L}}{L\cosh\left(\frac{\pm x-d}{L}\right)+l\sinh\left(\frac{\pm x-d}{L}\right)-\sqrt{L^2-l^2}\cos\frac{z}{L}}. \qquad (5.2)$$

Within the tunnel junction the magnetic field is

$$H_y(z)=-\frac{\phi_0}{2\pi(\lambda_+^2+\lambda_-^2)}\ln\frac{L+\sqrt{L^2-l^2}\cos(z/L)}{L-\sqrt{L^2-l^2}\cos(z/L)}. \qquad (5.3)$$

The energy density of the tunnel junction for solution (5.1) has the form

$$\mathscr{E}(z)=\frac{\hbar j_c}{|e|}\frac{l^2}{L^2+l^2-(L^2-l^2)\cos(2z/L)}\times\left\{2+\sqrt{1-\frac{l^2}{L^2}}\cos\frac{z}{L}\ln\frac{L+\sqrt{L^2-l^2}\cos(z/L)}{L-\sqrt{L^2-l^2}\cos(z/L)}\right\}. \tag{5.4}$$

Finally the energy density averaged over a period is

$$\bar{\mathscr{E}}=\frac{\hbar j_c l}{|e|L}\left\{1+\ln\frac{L}{l}\right\}. \tag{5.5}$$

We note that in the limit $L\rightarrow\infty$ solution (5.1), like solution (2.1), becomes a 2π kink [Eq. (2.11)].

6. Here we present a solution of Eq. (3.1) corresponding to the strong-dissipation limit, which describes the relaxation process for vortex states without a mean magnetic field. This solution is

$$\varphi(z,t)=\pi+2\arctan\frac{\sin(z/L)}{\sinh\alpha(t)}, \tag{6.1}$$

where

$$\frac{\beta}{\omega_j^2}\frac{d\alpha}{dt}+\tanh\alpha=\frac{l}{L}. \tag{6.2}$$

The solution of Eq. (6.2) corresponding to the initial problem has the form

$$\exp\left[\frac{l}{L}(\alpha-\alpha_0)\right]\frac{\sinh(\alpha-\mathrm{arctanh}(l/L))}{\sinh(\alpha_0-\mathrm{arctanh}(l/L))}=\exp\left[-\left(1-\frac{l^2}{L^2}\right)\frac{\omega_j^2 t}{\beta}\right], \tag{6.3}$$

where $\alpha_0=\alpha(t=0)$. Hence it follows, in particular, that $\alpha(t)$ increases without bound as the time increases at $L<l$. This corresponds to disappearance of the vortex structure. In the opposite case of $L>l$, we have $\alpha(t\rightarrow\infty)=\arctan(l/L)$. This corresponds to the relaxation of solution (6.1) to stationary solution (5.1).

The magnetic field in the superconductors corresponding to solution (6.1) has the form

$$H_y(x,z,t)=-\frac{\phi_0}{2\pi(\lambda_+^2+\lambda_-^2)}\times\ln\frac{\cosh[\alpha(t)+(\pm x-d)/L]+\cos(z/L)}{\cosh[\alpha(t)+(\pm x-d)/L]-\cos(z/L)}. \tag{6.4}$$

For the energy density we have

$$\mathscr{E}(z,t)=\frac{\hbar jc}{|e|}\frac{\sinh\alpha(t)}{\cosh 2\alpha(t)-\cos(2z/L)}\left\{2\sinh\alpha(t)+\frac{l}{L}\cos\frac{z}{L}\ln\frac{\cosh\alpha(t)+\cos(z/L)}{\cosh\alpha(t)-\cos(z/L)}\right\}. \tag{6.5}$$

The energy density averaged over a period is given by

$$\bar{\mathscr{E}}(t)=\frac{\hbar j_c}{|e|}\left\{\tanh\alpha(t)-\frac{l}{L}\ln[\tanh\alpha(t)]\right\}. \tag{6.6}$$

7. The last solution that we present here corresponds to the nondissipative limit [see (4.1)]. In this case a traveling vortex structure without a magnetic field can be described by

$$\varphi(z,t)=4\arctan\left[\frac{\sin\{(z-Vt)/L\}}{\sqrt{V/\omega_j l-1}}\right], \tag{7.1}$$

where

$$L=\frac{(V/\omega_j)}{\sqrt{1-(l\omega_j/V}}. \tag{7.2}$$

It is clear from the last equation that the velocity V of the traveling structure should be greater than $\omega_j l$:

$$V>l\omega_j. \tag{7.3}$$

Conversely, inequality (4.4) holds for traveling structure (4.2) with a mean magnetic field. In the limit $V\rightarrow l\omega_j$, at which $L\rightarrow\infty$ according to (7.2), Eq. (7.1) becomes a 4π kink.[3,7] Thus, Eqs. (7.1) and (4.2), which yield a 4π-kink structure when $V=l\omega_j$, point out a natural procedure, which can be effectively utilized to numerically construct more complicated kink structures as the limits of periodic solutions.

We note here that according to (7.2), as the velocity V of a traveling vortex structure increases, the scale L at first decreases, reaching the minimum value $L_{\min}=3^{3/2}2^{-1}l$ when $V=(3/2)\omega_j l$, and then increases again.

The magnetic field in the superconductors corresponding to traveling wave (7.1) has the form

$$H_y(x,z,t)=-\frac{\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\times\ln\frac{\cosh(\gamma+[\pm x-d]/L)+\cos([z-Vt]/L)}{\cosh(\gamma+[\pm x-d]/L)-\cos([z-Vt]/L)}, \tag{7.4}$$

where

$$\gamma=\ln\left(\sqrt{\frac{V}{\omega_j l}}+\sqrt{\frac{V}{\omega_j l}-1}\right). \tag{7.5}$$

Within the tunnel junction we have

$$H_y(z,t)=-\frac{\phi_0}{\pi(\lambda_+^2+\lambda_-^2)}\ln\frac{\sqrt{V}+\sqrt{\omega_j l}\cos([z-Vt]/L)}{\sqrt{V}-\sqrt{\omega_j l}\cos([z-Vt]/L)}. \tag{7.6}$$

The energy density of the tunnel junction corresponding to solution (7.1) has the form

$$\mathscr{E}(z,t)=\frac{2\hbar j_c}{|e|}\frac{V-\omega_j l}{V-\omega_j l\cos^2([z-Vt]/L)}\left\{\frac{2\omega_j l}{V}+\left(\frac{\omega_j l}{V}\right)^{3/2}\times\cos\frac{z-Vt}{L}\ln\frac{\sqrt{V}+\sqrt{\omega_j l}\cos([z-Vt]/L)}{\sqrt{V}-\sqrt{\omega_j l}\cos([z-Vt]/L)}\right\}. \tag{7.7}$$

Finally, the mean value of the energy density over one period equals

$$\bar{\mathscr{E}}=\frac{4\hbar j_c l\omega_j}{|e|V}\sqrt{1-\frac{\omega_j l}{V}}\left\{1-\frac{1}{2}\ln\left(1-\frac{\omega_j l}{V}\right)\right\}. \tag{7.8}$$

8. In this communication we have presented a set of solutions of asymptotic equation (1.18), which describes the phase difference between Cooper pairs on opposite sides of an infinitely long Josephson junction for vortex states whose spatial scale is small compared with the London depths. In publicizing such new nonlinear solutions, we understand that their appearance immediately raises numerous new problems for nonlocal Josephson electrodynamics. One such problem is the stability of the solutions that we obtained. Here it can already be noted that the previously published work in Ref. 4 corresponds to a certain degree to the positive relative stability of states with a mean field. The instability of vortex structures is treated in the appendices. At the same time, it should be noted that the unstable solutions are also of definite interest in the study of nonlinear evolution processes, which should be considered in the future. Without predicting any possible experimental observations of the nonlinear states described above, we can still focus attention on the specific dependence of the velocity of a traveling wave with mean vortex magnetic field (4.2) on the period of such vortex chains, which might be manifested to a significant degree when they are excited. In the opposite limit of strong dissipation, experiments devised to study the temporal relaxation of prepared vortex chains might make it possible to experimentally reveal specific details of the small-scale structures under discussion, for example, on the basis of the dependence of the relaxation process on the period of the chains.

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APPENDIX A

Here we shall briefly describe the simplest results of the theory of the stability of the solutions of Eq. (1.18). We shall deal only with the linear approach, which not only establishes the existence of instability, but also points out the beginning of the way to find new solutions.

We assume that $\varphi(z,t)=\varphi_0(z)+\delta\varphi(z)e^{-i\omega t}$ differs only slightly from the exact nonlinear solution φ_0. According to the linear theory of stability, we are dealing with the equation

$$\delta\varphi(z)\cos\varphi_0(z)-\frac{l}{\pi}\fint_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{d\delta\varphi(z')}{dz'}$$
$$=\frac{\omega^2+i\omega\beta}{\omega_j^2}\,\delta\varphi(z)\equiv\varepsilon\,\delta\varphi(z). \tag{AI.1}$$

In the case of perturbed states corresponding to eigenvalues ω with Im $\omega>0$, the perturbations grow with time, attesting to the instability of $\varphi_0(z)$.

Discussing the solutions of Eq. (AI.1), we first of all note that when $\varepsilon\gg1$, cos φ_0 can be neglected in this equation in an approximation. Then using the relation

$$\frac{1}{\pi}\fint_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{d}{dz'}\sin(|q|z'+\eta)$$
$$=-|q|\sin(|q|z+\eta), \tag{AI.2}$$

we obtain the following asymptotic solution

$$\delta\varphi(z)=\sin(|q|z+\eta),\quad \varepsilon=|q|l\gg1.$$

On the other hand, since cos $\varphi_0\geqslant-1$, we can use (AI.2) to easily prove that

$$-1\leqslant\varepsilon<+\infty. \tag{AI.3}$$

Here the region of instability Im $\omega>0$ corresponds to negative eigenvalues ε. Therefore, for example, in a numerical study of the stability problem it is sufficient to restrict ourselves to the region

$$-1\leqslant\varepsilon<0. \tag{AI.4}$$

Below we show how such a problem can be solved analytically. In accordance with Eq. (AI.1) we have

$$\omega=i[\pm\sqrt{-\varepsilon\omega_j^2+(\beta/2)^2}-(\beta/2)]. \tag{AI.5}$$

When ε is negative, the plus sign in this equation corresponds to a perturbation growing with time.

Let $\varphi_0(z)$ correspond to the stationary solution of (5.1). Then Equation (AI.1) takes the form

$$\left[1-\frac{2l^2}{l^2+(L^2-l^2)\sin^2(z/L)}\right]\delta\varphi(z)$$
$$-\frac{l}{\pi}\fint_{-\infty}^{+\infty}\frac{dz'}{z'-z}\frac{d\delta\varphi(z')}{dz'}=\varepsilon\,\delta\varphi(z). \tag{AI.6}$$

It is not difficult to see that one solution of this equation is

$$\delta\varphi(z)=\frac{\text{const}}{l^2+(L^2-l^2)\sin^2(z/L)}. \tag{AI.7}$$

This solution corresponds to a negative eigenvalue

$$\varepsilon=-\frac{l^2}{L^2}, \tag{AI.8}$$

which, according to (AI.5), corresponds to the following expression for the instability growth rate:

$$\gamma=\sqrt{(l/L)^2\omega_j^2+(\beta/2)^2}-(\beta/2). \tag{AI.9}$$

Thus, periodic solution (5.1) with zero mean magnetic field strength during a period is unstable. A similar solution of the sine-Gordon equation of local Josephson electrodynamics is also unstable. In our case it is extremely simple to demonstrate the instability of (AI.6)–(AI.9).

We note that in the limit $\beta\gg(l/L)\omega_j$, the instability growth rate is $\gamma=(l\omega_j/L)^2\beta^{-1}$. Therefore, instability can be manifested only over sufficiently long time intervals

$$t_{\rm inst} \gtrsim \beta(L/l\omega_j)^2 \gg (L/l\omega_j).$$

The development of instability can be compared with relaxation process (6.3). For example, if $l \sim L$, the relaxation time is

$$t_{\rm rel} \sim \frac{\beta}{\omega_j} \sim \frac{1}{\gamma} \sim t_{\rm inst}.$$

In other words, in this case the time for the development of the instability of a stationary periodic structure with $L>l$ coincides with the decay time of a nonstationary vortex structure with $l \gtrsim L$.

A different situation is observed when $l \ll L$, in which case

$$t_{\rm rel} \sim \frac{\beta}{\omega_j^2} = \frac{1}{\gamma}\frac{l^2}{L^2} \ll t_{\rm inst}.$$

This inequality means that the development of instability with perturbation (AI.7) can be neglected during the characteristic time for the establishment of a stationary vortex structure with a period much greater than l according to (6.3).

Since it must be assumed for perturbation (AI.7) that $l<L$, as in ground state (5.1), the eigenvalue of (AI.8) tends to the left-hand edge of the range of eigenvalues ε corresponding to instability only in the limit $L \to l$. At the same time, solutions (AI.7) for different values of (l/L) correspond to the entire range of instability on the axis, $-1<\varepsilon<0$.

Perturbation (AI.7) does not have a zero in the period

$$0 \leqslant z \leqslant \pi L. \qquad \text{(AI.10)}$$

The next solution that vanishes once in this period is a solution of Eq. (AI.6):

$$\delta\varphi(z) = \frac{\cos(z/L)}{l^2+(L^2-l^2)\sin^2(z/L)}. \qquad \text{(AI.11)}$$

The corresponding eigenvalue is

$$\varepsilon = 0, \qquad \text{(AI.12)}$$

which, according to (AI.5), corresponds to both a temporally decaying perturbation with a damping rate β and a time-independent marginal perturbation representing a displacement mode.

The following solution of (AI.6) has two zeros in period (AI.10):

$$\delta\varphi(z) = \frac{\sin(2z/L)}{l^2+(L^2-l^2)\sin^2(z/L)}. \qquad \text{(AI.13)}$$

The eigenvalue corresponding to it is $\varepsilon=1$, which corresponds to perturbations that decay with time.

The next solution of Eq. (AI.6) has the form

$$\delta\varphi(z) = \frac{\cos(z/L)[(L+l)^2\sin^2(z/L)-l^2]}{l^2+(L^2-l^2)\sin^2(z/L)}. \qquad \text{(AI.14)}$$

This solution has three zeros in period (AI.10): $z=(\pi L/2)$, $z=\arcsin(l/[L+l])$, and $z=\pi-\arcsin(l/[L+l])$. The eigenvalue $\varepsilon=1+(l/L)$ corresponds to solution (AI.14), and the eigenvalue $\varepsilon=1+(2l/L)$ corresponds to the solution

$$\delta\varphi(z) = \frac{\sin(2z/L)[(L+l)\sin^2(z/L)-l]}{l^2+(L^2-l^2)\sin^2(z/L)}, \qquad \text{(AI.15)}$$

which has four zeros in period (AI.10). The construction of the subsequent solutions of Eq. (AI.6) is obvious.

Now let $\varphi_0(z)$ correspond to stationary solution (2.1). Then Eq. (AI.1) takes the form

$$\left[\sqrt{1+\frac{l^2}{L^2}} - \frac{(l/L)^2}{\sqrt{1+(l/L)^2}-\cos(z/L)}\right]\delta\varphi(z) - \frac{l}{\pi}\,⨍_{-\infty}^{+\infty} \frac{dz'}{z'-z}\frac{d\delta\varphi(z')}{dz'} = \varepsilon\,\delta\varphi. \qquad \text{(AI.16)}$$

The following is the analog of solution (AI.7) for this case:

$$\delta\varphi(z) = \frac{1}{\sqrt{1+(l/L)^2}-\cos(z/L)}. \qquad \text{(AI.17)}$$

This solution corresponds to $\varepsilon=0$; therefore, it corresponds to a perturbation that does not grow with time. At the same time, since solution (AI.17) does not have zeros, it can be assumed to correspond to the smallest eigenvalue. Therefore, it can be concluded that stationary solution (2.1) has marginal stability. Unlike Eq. (AI.6), Eq. (AI.16) presently under consideration has periodicity in a period

$$0 \leqslant z \leqslant 2\pi L. \qquad \text{(AI.18)}$$

Therefore, the spatially periodic solution of (A16) having one zero in period (AI.18) has the form

$$\delta\varphi(z) = \frac{\cos(z/2L)}{\sqrt{1+(l/L)^2}-\cos(z/L)}. \qquad \text{(AI.19)}$$

The corresponding eigenvalue is

$$\varepsilon = \frac{1}{2}\left(\sqrt{1+\frac{l^2}{L^2}}-1\right) > 0, \qquad \text{(AI.20)}$$

which also corresponds to temporal decay of the perturbation with spatial structure (AI.19).

Thus, several solutions of the linearized equations of nonlocal Josephson electrodynamics have been presented here in the context of the linear theory of small perturbations, and the stability of stationary, spatially periodic nonlinear vortex states has thereby been studied.

APPENDIX B

In the limit $L \to \infty$, stationary spatially periodic structures (2.1) and (5.1) transform into a 2π kink (2.11). Such a limiting transition permits the use of the results in Appendix A to analyze the stability of a 2π kink. In fact, making the transition to this limit in solutions (AI.7), (AI.11), (AI.17), and (AI.19), we obtain the same solution of Eq. (AI.1)

$$\delta\varphi(z) = \frac{1}{l^2+z^2}. \qquad \text{(BII.1)}$$

This limiting solution does not have any zeros. The limiting eigenvalues of solutions (AI.7), (AI.17), and (AI.19) coincide and are equal to $\varepsilon=0$. Thus, it can be concluded that 2π kink (2.11) has marginal stability. The solutions in Appendix

I also make it possible to write the solution of Eq. (AI.1) in the case of a 2π kink for the eigenvalue $\varepsilon=1$.

[1)]According to Ref. 2, expression (1.17) describes the energy density per unit of length in the direction of the z axis.

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