Scaling for a growing phase boundary with nonlinear diffusion

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The scaling properties of a growing phase boundary with a nonlinear diffusion coefficient are considered. Scaling exponents are derived by the field renormalization-group method (dimensional regularization combined with minimal subtractions). An exact relation among these exponents is obtained, modified by the introduction of anomalous dimensionality.

The main problem in studying the dynamics of growing surfaces, either the surface of a substance precipitating on a plane or a moving phase boundary, is to find the scaling laws for the microprofile $h(x,t)$ of the surface. For instance, in interpreting the results of numerical simulation of the precipitation of a substance on a flat surface with a characteristic size L, Family and Vicsek[1] suggested the following expression:

$$\xi = <h^2 - \langle h \rangle^2>^{1/2} = L^{\chi} f(t/L^{-z}), \tag{1}$$

where χ and z are the scaling exponents, and the function $f(x)$ is determined by the following asymptotic expressions: $f(x) \to \text{const}$ as $x \to \infty$, and $f(x \to 0) \sim x^{\beta}$, with $\beta = \chi/z$. Later it was found (see the review article in Ref. 2) that many simulations behave according to Eq. (1), that the scaling relation $\chi + z = 2$ holds true, and that in the one-dimensional case the exponents $\chi = \frac{1}{2}$ and $z = \frac{3}{2}$ are of a universal nature and depend only slightly on the method of calculation. Summing up, it can be said that precipitation results in a rough surface characterized by an algebraic increase in the intensity of profile fluctuations with the size of the system.

A number of analytical models based on stochastic differential equations have been suggested for justifying scaling, starting with the ordinary diffusion equation with a random source, for which $z = 2$ and $\chi = \frac{1}{2}(2-d)$, where d is the dimensionality of the surface.[3] Some agreement with the experimental data was noted for the one-dimensional case. Not surprisingly, the simplest nonlinearity introduced by the kinematic features of growth ensured a more exact result: $z = \frac{1}{2}(2+d)$ and $\chi = \frac{1}{2}(2-d)$ (see Ref. 4). The scaling relation $\chi + z = 2$ follows from translation invariance. Note that these expressions become invalid for $d \leqslant 2$; more suitable, probably, are exponents obtained by heuristic means, $\chi = 2/(d+3)$ and $z = 2(d+2)/(d+3)$ (see Ref. 5). A generalization to the case of nonlinear diffusion was done by Nagatani,[6] who numerically solved the equation of nonlinear diffusion with a random source. Linear diffusion was ignored completely, which led to the rather paradoxical result $\beta = \frac{1}{4}(k-1)$, where k is defined as the exponent in the expression for the diffusion coefficient: $D \sim h^k$.

This paper studies the effect of nonlinear diffusion on the fluctuation dynamics of a growing phase boundary. The method employed is that of the field renormalization group, that is, dimensional regularization combined with minimal subtractions (see, e.g., Ref. 7). To this end let us start with the equation

$$\frac{\partial h}{\partial t} = \nu\, \Delta h + \alpha\, \Delta h^2 + \eta, \tag{2}$$

where $h(x,t)$ is the surface profile (the precipitation "thickness"), and ν is the diffusion coefficient or the surface tension.

It can easily be verified that Eq. (2) incorporates the nonlinearity caused by the kinematic growth of the surface. Indeed,

$$\tfrac{1}{2}\, \Delta h^2 = (\nabla h)^2 + h\, \Delta h,$$

so that the first term on the right-hand side of Eq. (2) represents the nonlinearity used by Kardar, Parisi, and Zhang,[4] the second the nonlinear contribution to surface tension, and the last term the random function describing local variations in the growth rate, with the correlation function

$$\langle \eta(x,t)\, \eta(x',t') \rangle = 2D\, \delta^d(x-x')\, \delta(t-t'). \tag{3}$$

Let us now examine the scaling properties of Eq. (2). We start by performing the transformations $x \to sx$, $t \to s^z t$, and $h \to s^{\chi} h$. Substituting into Eq. (2) yields

$$\frac{\partial h}{\partial t} = \nu s^{z-2} \Delta h + \alpha s^{\chi+z-2} \Delta h^2 + s^{z/2 - d/2 - \chi} \eta. \tag{4}$$

For the linear modification of (4), scaling invariance yields the scaling exponents $z_0 = 2$ and $\chi_0 = \frac{1}{2}(2-d)$. If these values are substituted into the scale factor of the nonlinearity, we get $\chi_0 + z_0 - 2 = \frac{1}{2}(2-d)$. Thus, $d = 2$ is the critical dimensionality; for $d > 2$ the nonlinearity is not essential, and the scaling behavior is determined by the exponents z_0 and χ_0. But nontrivial scaling should be expected for $d < 2$, when nonlinearity increases owing to the scaling transformation. Obviously, in view of continuity, corrections to ideal scaling tend to zero as $\varepsilon = 2 - d \to 0$.

The form of Eq. (4) makes it easy to introduce a scaling parameter μ. To this end we replace

$$\nu \to \nu \mu^{z-2}, \quad \alpha \to \alpha \mu^{\chi+z-2}, \quad D \to D \mu^{z-2\chi-d}.$$

Accordingly, the new parameters ν, α, and D are dimensionless in the sense that all scaling transformations are related to μ. As a result the solution to Eq. (2) has the following homogeneity property:

$$h(sx, s^z t, \nu, \alpha, D, \mu) = s^{\chi} h(x, t, \alpha, \nu, D, s\mu). \tag{5}$$

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The easiest way to verify this is to substitute (5) into (4).

As usual, when the dimensionality is critical, only logarithmic divergences can occur (if we use perturbation-theoretic methods). In a space whose dimensionality is higher than the critical value, the main contribution to the observables is provided by microscopic scales, while for a dimensionality lower than the critical value it is provided by the macroscopic scales or large-scale perturbations. Thus, logarithmic divergences indicate scale-invariant behavior at critical dimensionality, with wavenumber space having no preferred region. Such behavior can be explained by the fact that the theory is renormalizable, that is, all divergences are removed by introduction into (3) of renormalization constants defined by the essential counterterms:

$$Z\frac{\partial h}{\partial t}=Z_\nu \nu\mu^{z-2}\Delta h+Z_\alpha \alpha\mu^{\chi+z-2}\Delta h^2+Z_\eta \eta. \qquad (6)$$

Clearly, μ parametrizes the renormalization scheme. Therefore, it is natural to expect that the initial, or "bare," quantities must not depend on μ. Taking into account that for "bare" quantities Eq. (6) must coincide in form with (2), we define these quantities as

$$h_0=Zh, \quad \nu_0=Z_\nu Z^{-1}\nu\mu^{z-2},$$
$$\alpha_0=Z_\alpha Z^{-2}\alpha\mu^{z+\chi-2}, \quad D_0=Z_\eta^2 D\mu^{z-2\chi-d}. \qquad (7)$$

The roughness function[1] satisfies

$$\xi_0(x,t,\nu_0,\alpha_0,D_0)=Z\xi(x,t,\nu,\alpha,D,\mu). \qquad (8)$$

Taking the derivative of Eq. (8) with respect to μ, we arrive at the main renormalization-group equation:

$$\left(\mu\frac{\partial}{\partial\mu}+\mu\frac{d\nu}{d\mu}\frac{\partial}{\partial\nu}+\mu\frac{d\alpha}{d\mu}\frac{\partial}{\partial\alpha}+\mu\frac{dD}{d\mu}\frac{\partial}{\partial D}+\gamma\right)\xi=0, \qquad (9)$$

where, as will shortly be seen, $\gamma=\mu(d\ln Z/d\mu)$ determines the anomalous dimensionality. Recall that the derivatives in (9) are taken at fixed initial parameters. The solution to Eq. (9) has the form

$$\xi(x,t,\nu(\mu),...,\mu)=\exp\left\{-\int_{s\mu}^{\mu}\frac{d\mu'}{\mu'}\gamma'\right\}\xi(x,t,\nu(s\mu),...,s\mu). \qquad (10)$$

From Eqs. (7) and (9) we can easily see that (10) depends on the dimensionless coupling constant

$$g=\frac{\alpha^2 D}{(4\pi)^{\alpha/2}\nu^3},$$

where the numerical factor is introduced for convenience. Next, combining the "na'íve" scaling property (5) with the solution (10), we write the latter as

$$\xi(x,t,g(\mu),\mu)=s^{-\chi}\exp\left\{-\int_{s\mu}^{\mu}\frac{d\mu'}{\mu'}\gamma'\right\}\xi(sx,s^z t,g(s\mu),\mu). \qquad (11)$$

If for the effective coupling constant there exists an infrared fixed point, then for sufficiently small values of s the following scaling expression holds true:

$$\xi(x,t,g,\mu)=s^{-\chi+\gamma^*}\xi(sx,s^z t,g^*,\mu). \qquad (12)$$

We see that the divergences manifest themselves in the anomalous dimensionality.

Following standard procedure, after performing the necessary calculations we arrive at the following set of renormalization constants (see Appendix A):

$$Z=1+\frac{8g}{\varepsilon}, \quad Z_\nu=1-\frac{4g}{\varepsilon},$$
$$Z_\eta^2=1+\frac{8g}{\varepsilon}, \quad Z_\alpha=1+\frac{8g}{\varepsilon}, \qquad (13)$$

where $\varepsilon=2-d$. As usual, in the method of minimal subtractions all counterterms are proportional to the poles in ε. Note that $Z=Z_\alpha$ follows from the invariance of Eq. (2) under the transformations $x\to x+2\alpha ta\nabla$ as $h\to h+a$. The equality $Z=Z_\eta^2$ is most likely a coincidence. It is exact for unbroken supersymmetry,[8] which exists if the right-hand side of (2) can be represented as a functional derivative of a positive-definite functional.

Let us substitute (13) into (7) and find the derivative with respect to μ. The result is the following system of renormalization-group equations:

$$\mu\frac{d\ln\nu}{d\mu}=2-z-12g, \quad \mu\frac{d\ln\alpha}{d\mu}=2-z-\chi-8g,$$
$$\mu\frac{d\ln D}{d\mu}=d-z+2\chi+8g, \quad \mu\frac{d\ln g}{d\mu}=-\varepsilon+28g. \qquad (14)$$

The last equation in (14) yields the value of the effective coupling constant in the infrared fixed point, $g^*=\varepsilon/28$. Assuming that the rate of variation of parameters at the fixed point is zero, we arrive at the following set of scaling parameters:

$$z=\frac{14-3\varepsilon}{7}, \quad \chi=\frac{\varepsilon}{7}, \quad \gamma^*=-8g^*=-\frac{2\varepsilon}{7}.$$

Clearly, the relation $Z=Z_\alpha$ yields the exact scaling relation $z+\chi-\gamma^*=2$. Apparently, this relation specifies a class of models that do describe growing surfaces. In our case this is even more remarkable since it required introducing anomalous dimensionality, which represents singularities related to long-range correlations. Thus, in the one-dimensional case, Eq. (1) modified by the contribution of anomalous dimensionality has the following form:

$$\xi(L,t)=L^{3/7}f(tL^{-11/7}). \qquad (15)$$

Accordingly, we have $\beta=\frac{3}{11}$, which clearly contradicts the conclusions reached by Nagatani,[6] who, however, studied Eq. (2) numerically without a diffusion term. No numerical results exist for the equation studied in the present paper, although the fact that the simplest nonlinearity in the surface tension leads only to a small increase in the roughness exponent (from $\frac{1}{2}$ to $\frac{3}{7}$) suggests the possibility of verifying Eq.

(15) experimentally. Yet the conclusion about the algebraic increase in the roughness in the initial stage of evolution is far from obvious. The problem of obtaining a scaling function by analytical means

has yet to be solved. Quite possibly, the solution lies in the techniques of modern turbulence theory (see, e.g., Ref. 9), especially since the ideas of cascade transport over the spectrum, used in finding the Kolmogorov spectrum, enabled Hentschel and Family[10] to derive qualitatively all the exponents obtained earlier. However, when this method is applied to Eq. (2), the results are erroneous: the exponents correspond to the problem with linear surface tension. As for ideal scaling in the event of critical dimensionality, we believe that conformal invariance of two-dimensional surfaces,[11] for which there are no acceptable analytical results, has yet to be used. It is clear, however, that two-dimensional growing surfaces become rough in the process of evolution; hence, ideal scaling is out of the question. On the other hand, for surfaces whose dimensionality is below critical the renormalization-group approach yields good results, and the minimal subtraction scheme makes it possible to obtain them with little difficulty.

APPENDIX A: RENORMALIZATION OF THE GREEN'S FUNCTION

We use Eq. (6) to determine the renormalized Green's function:

$$G_R^{-1}(\omega,\mathbf{k})=-i\omega+\nu\mathbf{k}^2-\Sigma(\mathbf{k},\omega)$$
$$-i\omega(Z-1)+\nu(Z_\nu-1)\mathbf{k}^2, \tag{A1}$$

where

$$\Sigma(\mathbf{k},\omega)=8\alpha^2D\mathbf{k}^2\int\frac{d^dq\,d\Omega}{(2\pi)^{d+1}}(\mathbf{k}-\mathbf{q})^2|G_0(q)|^2G_0(k-q), \tag{A2}$$

and $G_0^{-1}(q)=-i\Omega+\nu\mathbf{q}^2$; note that g is a dimensionless quantity (we do not write the dimensional factor explicitly for the other parameters). Integration over frequencies yields

$$\Sigma(\mathbf{k},\omega)=8g\mu^\varepsilon\nu\mathbf{k}^2\int\frac{d^dq}{\pi^{d/2}}\frac{(\mathbf{k}-\mathbf{q})^2}{2\mathbf{q}^2[\mathbf{q}^2+(\mathbf{k}-\mathbf{q})^2-i\omega/\nu]}. \tag{A3}$$

Equation (A3) implies that there are problems with infrared divergences. The advantage of the minimal subtraction method, however, is that there is no difference between infrared and ultraviolet divergences in the sense that all are represented in the form of poles in ε. Next, for the divergent part of (A3) we have

$$\Sigma_{\mathrm{D}}(\mathbf{k},\omega)=-4g\nu k^2\frac{\nu\mathbf{k}^2+i\omega}{(\nu\mathbf{k}^2-i\omega)\varepsilon}=\Sigma'+i\Sigma''. \tag{A4}$$

It can easily be shown that, in accordance with (A1),

$$Z-1=-\lim_{\omega\to0}\frac{\Sigma''}{\omega}=\frac{8g}{\varepsilon},\quad Z_\nu-1=\lim_{\omega\to0}\frac{\Sigma'}{\nu\mathbf{k}^2}=-\frac{4g}{\varepsilon}. \tag{A5}$$

APPENDIX B: AN EFFECTIVE NOISE CORRELATION FUNCTION

In the lowest order of perturbation theory we have

$$D(\mathbf{k},\omega)=D+4\alpha^2D^2\mathbf{k}^2\int\frac{d^dq\,d\Omega}{(2\pi)^{d+1}}|G_0$$
$$\times(k-q)|^2|G_0(q)|^2. \tag{B1}$$

This follows from the definition

$$\langle h(\mathbf{k},\omega)h(k',\omega')\rangle$$
$$=|G_0(k)|^2 2D(\mathbf{k},\omega)(2\pi)^{d+1}\delta^d(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'). \tag{B2}$$

After integration with respect to frequency, we get (in the limit of $\omega\to0$)

$$D(\mathbf{k},\omega\to0)$$
$$=D\left\{1+2g\mu^\varepsilon\mathbf{k}^4\int\frac{d^dq}{\pi^{d/2}\mathbf{q}^2(\mathbf{k}-\mathbf{q})^2[\mathbf{q}^2+(\mathbf{k}-\mathbf{q})^2]}\right\}. \tag{B3}$$

Passing to integration with respect to the parameters, we can easily integrate with respect to q:

$$D(\mathbf{k})=D\left\{1+\frac{g\mu^\varepsilon\Gamma(3-d/2)}{(\mathbf{k}^2)^{\varepsilon/2}}\right.$$
$$\left.\times\int_0^1dx\int_{x/2}^{1-x/2}dy\,[y(1-y)]^{d/2-3}\right\}. \tag{B4}$$

Clearly, the pole singularities arise only when we integrate with respect to the parameters. Let us introduce one more parametric integration via the relation

$$\frac{1}{y^\alpha(1-y)^\beta}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int\frac{dz\,z^{\alpha-1}(1-z)^{\beta-1}}{[zy+(1-z)(1-y)]^{\alpha+\beta}}.$$

After this the integral in (B4) can easily be calculated:

$$\int_0^1dx\int_{x/2}^{1-x/2}dy\,[y(1-y)]^{d/2-3}$$
$$=(-1)^{d/2-3}\frac{\Gamma(6-d)}{\Gamma^2(3-d/2)}\frac{2}{(d-5)(d-4)}$$
$$\times\left\{\Gamma\left(3-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)[1+(-1)^{d-4}]\right.$$
$$\left.\times-2\int_0^1dz\,z^{2-d/2}(1-z)^{2-d/2}(\tfrac{1}{2}-z)^{d-4}\right\}.$$

The integral on the right-hand side is finite for $\varepsilon\to0$. Finally, the pole part in (B3) is equal to $D(1-8g/\varepsilon)$, which leads to $Z_\eta^2=1+8g/\varepsilon$.

APPENDIX C: VERTEX RENORMALIZATION

The contribution to the vertex consists of three triangular diagrams with different positions of the noise correlation functions, As usual, we do the calculations in the limit of $\omega\to0$. As a result we have the following expression for the divergent part:

$$\lim_{k\to 0}\frac{\Gamma(k,k-p,p)}{-\alpha k^2}=1-4g\mu^2\frac{\Gamma(\frac{1}{2}\varepsilon)}{(\frac{1}{4}p^2)^{\varepsilon/2}}, \qquad \text{(C1)}$$

where $\Gamma_0(k,k-p,p)=-\alpha\mathbf{k}^2$. Equation (C1) implies $Z_\alpha=1+8g\varepsilon^{-1}$.

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