Steady-state generation regimes in lasers with external delayed feedback

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The evolution of the nonlinear differential-difference equations modeling the operation of laser systems controlled by external delayed feedback has been reduced to the dynamics of analytically determined finite-dimensional mappings. The existence of a hierarchy of periodic multistable states of various structures has been demonstrated. Complex temporal structures, viz., metastable chaos, intermittency, and quasiperiodicity, have been found as a result of the bifurcation of many-dimensional mappings.

1. INTRODUCTION

Various delayed-feedback loops appear naturally when sources of optical coherent radiation are included in complex electronic networks and can be intentionally created for the purposeful regulation of the spatial and temporal state of a system. Some simple examples are provided by hybrid and completely optical bistable resonator devices,[1,2] an external-cavity semiconductor laser,[3] and lasers with electro-optical parameter modulators.[4–9]

The differential-difference equations describing such devices exhibit nontrivial dynamics, particularly high-dimensional dynamic chaos and multistability.[10] The latter is known to be a fundamental property of the basic structural elements of an optical computer and is therefore of great practical importance. From the general theoretical standpoint their investigation is important for understanding the relationship between the discrete and continuous approaches to describing dynamic phenomena and for revealing the general laws shaping the complex dynamics in infinite-dimensional systems with a delaying argument.

In the present study we considered some important classes of nonlinear differential-difference equations, which are basic equations in the theory of laser systems. The corresponding models were based on single-mode rate equations for lasers with electro-optical loss modulators in the cavity[11,12] and in the pumping,[13,14] which employ feedback with an optical delay loop. The investigative procedure was based on the asymptotic method developed in Refs. 15 and 16 for studying nonlocal irregular oscillations. The use of this method is allowed by the presence of a large parameter in the models considered below. Ultimately, it is possible to reduce the original problem of the dynamics of a continuous temporal flux in an infinite phase space to a problem of the dynamics of finite-dimensional mappings that can be determined analytically. It is possible to reveal a set of multistable, structurally different periodic regimes, to obtain their asymptotic forms, and to determine their domains in the parameter space and in the phase space of initial conditions.

It is important to note that the method developed in this paper is fairly effective and quite general. For example, it was used to investigate relaxational oscillations in radiophysical problems in Refs. 17–19, to study the dynamics of a model of a nuclear reactor in Ref. 20, and to explore some important problems in mathematical economics, medicine, and chemistry in Refs. 15, 16, and 21–23.

We specially stress the important role of the analytical equations for complex oscillatory regimes that can be obtained in the problems considered below. They not only provide a good approximation for real values of the physical parameters, but also make it possible to optimize the parameters of the oscillations on the same basis. In addition, owing to the highly relaxational character of the solutions, reliable numerical investigations cannot be performed even with the aid of modern computers. In this respect asymptotic formulas are irreplaceable.

We begin by studying the relaxational auto-oscillations in the system of single-mode rate equations

$$\dot{u}=vu[y-1-\alpha u(t-\tau)]+\varepsilon,\quad \dot{y}=\lambda-y(1+u). \quad (1.1)$$

Such equations model the dynamics of the output of a laser with a nonlinear element such as a Pockels cell, whose losses increase with increasing intensity of the laser radiation passing through the external feedback loop. They were first proposed in Refs. 24 and 25 for the purpose of determining the conditions for suppressing peaks of free generation with the aid of negative feedback.

The variables and parameters in (1.1) have simple physical meanings: u is the intensity of the laser radiation normalized to the saturation intensity of the radiation, the value of y is proportional to the gain of the active medium, λ is the ratio of the unsaturated gain to the loss coefficient, which does not depend on the radiation intensity, α characterizes the feedback depth, and v is the intracavity radiation damping rate in terms of the population-inversion relaxation rate. The current time t and the delay time τ due to passage of the radiation through the feedback loop are normalized to the population-inversion relaxation time, ε is proportional to the intensity of the radiation from outside sources, particularly the noise radiation of the same direction and frequency as the radiation generated. Eqs. (1.1)

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hold when we consider lasing regimes with times for variation of the output characteristics that do not significantly exceed the round trip time of the radiation in the cavity.

The regions of stability and instability of an equilibrium state were determined in Refs. 26 and 27 on the basis of linear analysis. The numerical solution in regions of instability showed that periodic pulsing of the radiation develops in the system. For some laser parameters the system displays a hierarchy of multistable periodic states (which oscillate slowly and rapidly) and a hysteretic dependence of the pulsation period and amplitude as the delay time τ increases. Under the optimal parameters for the realization of such multistability, the population-inversion relaxation rate D and the intracavity radiation damping rate $\nabla\mu k$ are of the same order, and high pumping levels and smaller intracavity losses are possible. For example, in a gas laser with an intracavity radiation damping rate $\nabla\mu k=5\times10^6$ s^{-1} and a population-inversion relaxation rate $D=1.05\times10^7$ s^{-1} we obtain $v=0.952$, $A=4.2$, $\lambda=47.6$. In a dye laser with $\nabla\mu k=D=2\times10^8$ s^{-1} the possible parameters of the system are $v=2$, $\alpha=0.5$, and $\lambda=100$. In a rare-earth laser with $\nabla\mu k=10^7$ s^{-1} and $D=1.1\times10^6$ s^{-1} these parameters can have the values $v=36.364$, $\alpha=0.045$, and $\lambda=2273.73$. Thus, the phenomenon of multistability can occur for $\lambda\gg1$.

In the case of solid-state lasers, for example, a ruby laser with a population-inversion relaxation rate $D=1800$ s^{-1} and an intracavity radiation damping rate $\nabla\mu k=10^8$ s^{-1}, the relationship between the parameters in Eqs. (1.1) is different: $v=2.22\times10^5$, $\alpha=0.004$, and $\lambda=6.67$. Here the large parameters is $v\gg1$. The numerical solution showed that only a slowly oscillating solution is realized in this case. This is also confirmed by experimental investigations of a ruby laser with negative feedback.[5] When the delay time was $\tau=100$ nsec, the laser output had the form of a sequence of regular pulses with a duration from 0.1 to 2 μs (depending on λ), a repetition time equal to 15–50 μs (i.e., the period of the auto-oscillations was $\sim$100–500 τ), and an intensity an order of magnitude greater than the intensity of the peaks of free generation.

The important role of ε, which is proportional to the intensity of the radiation from external illumination, was also revealed numerically. If a regime of periodic pulses of stimulated emission is realized, external illumination of even weak amplitude ($\varepsilon\approx10^{-5}$–$10^{-2}$, i.e., on the level of the intensity of the noise radiation of the same direction and frequency as the radiation generated) causes significant shortening of the pulsation period in comparison with the solutions when $\varepsilon=0$.

It is technically convenient to regulate the lasing in a semiconductor laser by modulating the pump current through a feedback loop:[6–10]

$$\dot{u}=vu[\gamma f(y)-1]+\varepsilon(y+d),$$
$$\dot{y}=\lambda+\alpha u(t-\tau)-f(y)(1+u). \tag{1.2}$$

Such an optoelectronic system can serve as a source of short pulses and can be employed in optical signal transmission and processing systems. Short light pulses, whose period correlates with the delay time in the feedback loop and varies abruptly as the constant component of the injection current is swept, have been obtained experimentally.[7,9]

The variables and parameters in (1.2) have the same meaning as in (1.1), $f(y)$ is a positive, monotonically increasing function, which characterizes the dependence of the gain on the carrier inversion y, and the relaxation rate of the absorption coefficient $\gamma>0$. A simple case that is frequently used in practice is a linear dependence with $\gamma=1$ and $f(y)=y$. Other alternatives, which take into account various processes in the working zones of the semiconductor, were presented, for example, in Ref. 29. Optoelectronic feedback is assumed to be infinitely high-pass, i.e., to have an unlimited bandwidth, and may be represented by the term $\alpha u(t-\tau)$.

In a real experiment the bandwidth β of the feedback loop is limited. This situation reduces the number of degrees of freedom that can be introduced by feedback from ∞ to $N=2\tau\beta$ (Refs. 8 and 29) and can significantly alter the dynamics of the system. To take into account the limited bandwidth of the feedback loop in our case, instead of (1.2) we should treat the equations[8]

$$\dot{u}=vu(y-1),$$
$$\dot{y}=\lambda+\alpha z(t-\tau)-y(1+u),$$
$$\dot{z}=\beta(-z+ru), \tag{1.3}$$

where r is the nonlinearity coefficient of the filter and $z(t)$ is proportional to the voltage in the feedback loop. In the limit $\beta\to\infty$ (1.3) takes on the form of (1.2) with a feedback depth constant $\alpha\to\alpha r$.

Many phenomena pertaining to the dynamics of injection diodes, for example, self-pulsing, have been explained on the basis of the concept of a two-component laser. Then, instead of (1.2) we should consider the system

$$\dot{u}=vu(y-1-k),$$
$$\dot{y}=\lambda+\alpha u(t-\tau)-y(1+u),$$
$$\dot{k}=\gamma(k_0-k)-rku, \tag{1.4}$$

where k_0 is the saturated value of the absorption coefficient k and γ is relaxation rate of the absorption coefficient in terms of the population-inversion relaxation rate. When different parameters are introduced, these equations can model the spatially nonuniform distribution of the injection current, the refractive index, the characteristics of the band structure, the presence of impurities with saturable absorption, etc.[30]

We note that the limited bandwidth of the feedback loop and the presence of a saturable absorber can be taken into account in a similar manner in the problem of the modulation of intracavity losses [Eqs. (1.1)].

In the case of semiconductor lasers, λ, τ, and α usually have values of order unity, and $v\sim10^3$. The problem of investigating the asymptotes of steady-state regimes of systems (1.2)–(1.4) for $v\gg1$ naturally arises. One distinctive feature of these equations is the fact that the "degenera-

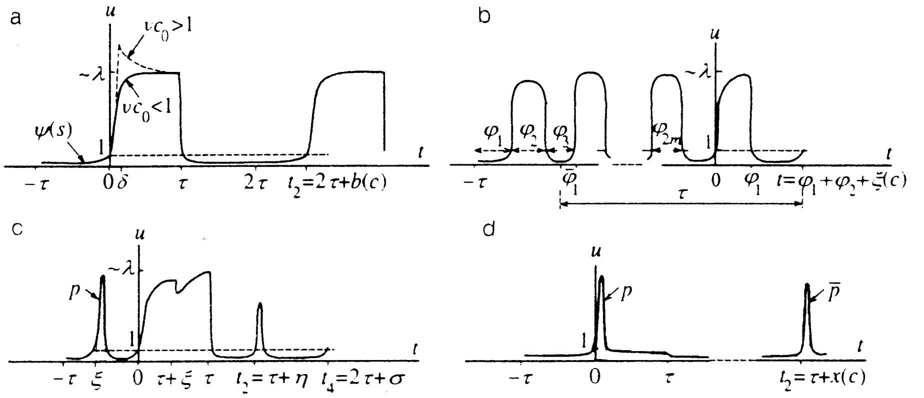

FIG. 1. Initial functions $\psi(s)$ for solutions of the system (1.1) of various structure.

cies" (which we obtain by dividing the first equation by v and treating the case of $v^{-1}=0$) do not provide information on the dynamics as $v\to\infty$. Consequently, the amplitudes of the oscillations increase without bound, and the duration of the peaks of $u(t)$ tends to zero. Therefore, when $v\gg 1$, we shall consider regimes consisting of short pulses.

Let us consider the investigative method. The phase space of the systems considered is the direct product of a Banach space $C_{[-\tau,0]}$ of functions continuous in $[-\tau,0]$ by the real line R^1, i.e., values of the functions u from $C_{[-\tau,0]}$ and the values $y\in R^1$ are assigned as initial conditions. Using arguments of a physical nature, we separate the (fairly broad) set $S(\xi)$, which depends on the vector parameter ξ, and we consider solutions with initial conditions from this set. Uniform asymptotic approximations of all such solutions can be constructed, and it can be shown that after a certain time interval these solutions again belong to S. A sequencing operator, which maps a function from $S(\xi)$ onto a function that is also from $S(\xi)$, is thereby defined. It turns out that the properties of this sequencing operator are assigned in the principal finite-dimensional mapping $\bar{\xi}=g(\xi)$. Thus, a fixed point of the mapping corresponds to a fixed point of the sequencing operator, and the latter corresponds to a periodic solution of the original system of the same stability. We note that the asymptotic integration method makes it possible to obtain uniform asymptotic formulas for steady-state regimes with arbitrary accuracy. These formulas, however, are excessively cumbersome; therefore, we restrict ourselves to a study of only the principal terms of the solution asymptotic form. The method used was described in detail in Refs. 31–33.

The material of this paper is organized in the following manner. In Secs. 2–4 we study relaxational solutions of system (1.1) for $\lambda\gg 1$. In Sec. 2 we show that the evolution of the slowly oscillating solutions of the system (1.1) in the absence of outside radiation sources ($\varepsilon=0$) is described by a one-dimensional nonlinear mapping, while the evolution of the rapidly oscillating solutions with $m\gg 1$ pulses in an interval of duration τ is described by a $(2m+1)$-dimensional mapping. In Sec. 3 we obtain finite-dimensional mappings for $\varepsilon>0$ and show that the rapidly oscillating structures are quasistable. In Sec. 4 we discuss the question of the existence of solutions of complex structure with alternating peaks (of finite duration) and spikes (whose duration is very short when λ is large) of radiation intensity. In Sec. 5 we construct periodic solutions of system (1.1) for $v\gg 1$. In Sec. 6 we present mappings which appear in the problem (1.2) for $v\gg 1$ and $\varepsilon=0$ and show that their bifurcation results in the appearance of complex temporal structures, i.e., metastable chaos and intermittency. In Sec. 7 we take into account some important physical factors affecting a real experiment: the presence of a high level of spontaneous radiation in the laser mode, the limited bandwidth of the feedback loop, and the presence of impurities with saturable absorption.

2. Periodic solutions of the system (1.1) for $\lambda\gg 1$ and $\varepsilon=0$

Let us first examine the simplest solutions of Eqs. (1.1), in which there can be no more than one peak (raised segment) in an interval with a length equal to the delay τ. Such solutions are termed slowly oscillating.

We determine the set of initial conditions under which the problem will be solved. Let $S_0\in C_{[-\tau,0]}$ be the set of functions $\psi(s)$ having the properties (Fig. 1a)

$$\psi(0)=1,\quad 0\ll\psi(s)\ll 1,$$
$$\times\int_\tau^0 \psi(s)ds\ll[\lambda v(1-e^{-\tau})]^{-1}. \qquad (2.1)$$

The level of population inversion at the initial moment $y(0)=\lambda c$, and $c\in(0,1)$.

From the standpoint of physical realization, this is the simplest initial condition corresponding to an intensity of the initial external illumination at the noise level (or its total absence) and to the selection of $t=0$ as the time when it reaches $u(0)=1$.

We solve the problem by successive integration in steps. In the interval $t\in[\delta,\tau]$, where δ is an arbitrary, fairly small constant, which does not depend on λ, Eqs. (1.1) with initial conditions (2.1) can be reduced to a system of singularly perturbed equations:

$$\dot{G}=W,$$

$$\mu W=\mu W+vW\left\{1-e^{-t}\left[\frac{W}{v}+G(1-v^{-1})-c+1\right]+o(1)\right\}, \tag{2.2}$$

where

$$\mu=\lambda^{-1}(\mu\to 0 \text{ as } \lambda\to\infty),$$

$$G=\mu\dot{F}(t),\quad F(t)=\int_0^t u(s)e^s ds.$$

The terms which are much smaller than μ are included here and below in $o(1)$. Equations (2.2) satisfy the conditions of Tikhonov's theorem;[34] therefore, their solutions at $t\in[\delta,\tau]$ will tend to the solution of a degenerate system when $\mu=0$:

$$u(t)=\lambda[1+(vc-1)e^{-vt}+o(1)],$$
$$y(t)=[1+(vc-1)e^{-vt}+o(1)]^{-1}. \tag{2.3}$$

A boundary layer which reconciles the initial conditions appears in the interval $t\in[0,\delta]$. In the second step, $t\in[\tau+\delta,2\tau]$, the asymptotic evaluations give the equalities

$$u(t)=\exp\{\lambda vA(t-\tau,c)[1+o(1)]\},$$
$$y(t)=\lambda[1-\exp(\tau-t)+o(1)], \tag{2.4}$$

which hold under the condition

$$A(t,c)<0 \quad \text{for all } t\in(0,\tau], \tag{2.5}$$

where

$$A(t,c)=(1-\alpha)t+e^{-t}-1+\alpha(c-v^{-1})(e^{-vt}-1),$$

is satisfied.

We now seek the solution of Eqs. (1.1) in the interval $t\in[2\tau,t_2(\psi,c)]$, where t_2 is the second positive root of the equation $u(t)=1$. For $t>2\tau$ the formula for $y(t)$ from (2.4) and $u(t-\tau)=o(1)$ hold as long as $u(t)$ is sufficiently small; therefore,

$$u(t)=\exp\{\lambda v[A(\tau,c)+t-2\tau+\exp(\tau-t)-\exp(-\tau)+o(1)]\}. \tag{2.6}$$

Hence we obtain $t_2=2\tau+b(c)+o(1)$, where $b(c)$ is a root of

$$1-[A(\tau,c)-b]e^{\tau}=e^{-b}. \tag{2.7}$$

The solution is plotted schematically in Fig. 1a. It follows from the asymptotic formulas obtained [(2.3)–(2.7)] that the problem of the further construction of the asymptotic forms of the solution of $u(t)$ and $y(t)$ at $t>t_2$ reduces to the original problem with the replacement of c by $\bar{c}$, where

$$\bar{c}=g(c)+o(1),\quad g(c)=1-\exp\{-[\tau+b(c)]\}. \tag{2.8}$$

The properties of the mapping $g(c)$ are the principal factor determining the character of the solution of the original infinite system. Figures 2a and 2b present plots of this mapping for the laser parameters given in the introduction. In all cases there is a stable fixed point at $c_0=g(c_0)$, $g'(c_0)<1$. We define the sequencing operator Π

$$\Pi(\psi(c),\lambda c)=(u(t_2(\psi,c)+s,c,\psi),\quad y(t_2(\psi,c),c)),$$

and we note that it maps the initial set $S(c)$ onto itself, since $u(t_2+s,c,\psi)\in S_0$ and $y(t_2,c,\psi)=\lambda c$. It follows from this and general theorems of functional analysis[35] that there exists a fixed point of Π corresponding to an attractor in the phase space of system (1.1), i.e., to a stable limit cycle with a period $T=2\tau+b(c_0)+o(1)$ and asymptotic characteristics that can be determined from (2.3)–(2.7) when $c=c_0$. Condition (2.5) demarcates the domain of this attractor in the parameter space. The solution obtained is in good agreement with the results of numerical analysis.

Figures 2c and 2e present plots of $A(\tau,c)$ for parameters characteristic of a gas laser and a dye laser. In the former case it is negative everywhere, and a slowly oscillating solution exists for all $\tau>0$. In the dye laser violation of condition (2.5) results in the appearance of one or several "spikes" of radiation intensity in an interval of duration τ, and a new set of initial conditions, which assigns new rapidly oscillating structures, thus forms. Such a set can also be assigned directly from physical arguments as the initial intensity of the external illumination in the form of several fairly powerful pulses in an interval of duration τ.

We select the initial conditions for rapidly oscillating solutions in the following manner. Let $S(\varphi)\times[\varphi=(\varphi_1,\varphi_2,...,\varphi_m)]$ be the set of $\psi(s)$ that are continuous and positive in $[-\tau,0]$ and have the following properties:

1) $\psi(-\tau+\varphi_1)=\psi(-\tau+\varphi_1+\varphi_2)=...=\psi(-\tau+\varphi_1+...+\varphi_{2m})=\psi(0)=1;$

2) $\int_{-\tau}^{-\tau+\varphi_1}\psi(s)ds+\int_{-\tau+\varphi_1+\varphi_2}^{-\tau+\varphi_1+\varphi_2+\varphi_3}\psi(s)ds+\int_{-\tau+\varphi_1+...+\varphi_{2m}}^{0}\psi(s)ds\leqslant N/\lambda;$

3) each $s\in[-\tau+\varphi_1+...+\varphi_{2j-1},-\tau+\varphi_1+...+\varphi_{2j}]$ satisfies the conditions $\lambda R-N\leqslant\psi(s)\leqslant\lambda R+N$, $R=1+(vd_j-1)\times\exp[-v(s+\tau-\varphi_1-...-\varphi_{2j-1})]$.

Such a set assigns the initial radiation intensity in the form of m pulses of intensity $\sim d_j$ and duration φ_{2j} separated by the intervals φ_{2j+1} $(j=1,...,m)$, as shown in Fig. 1b. More detailed evaluations for d_j were presented in Ref. 11.

We construct the asymptotic (for $\lambda\gg 1$) solutions of system (1.1) with initial conditions from $S(\varphi)$ and $y(0)=\lambda c(\varphi)$. In the interval $t\in[\delta,\varphi_1]$ we have

$$u(t,\psi)=\lambda[1+(vc-1)e^{-vt}+o(1)],$$

$$y(t,\psi)=\lambda[1+(vc-1)e^{-vt}+o(1)]^{-1},$$

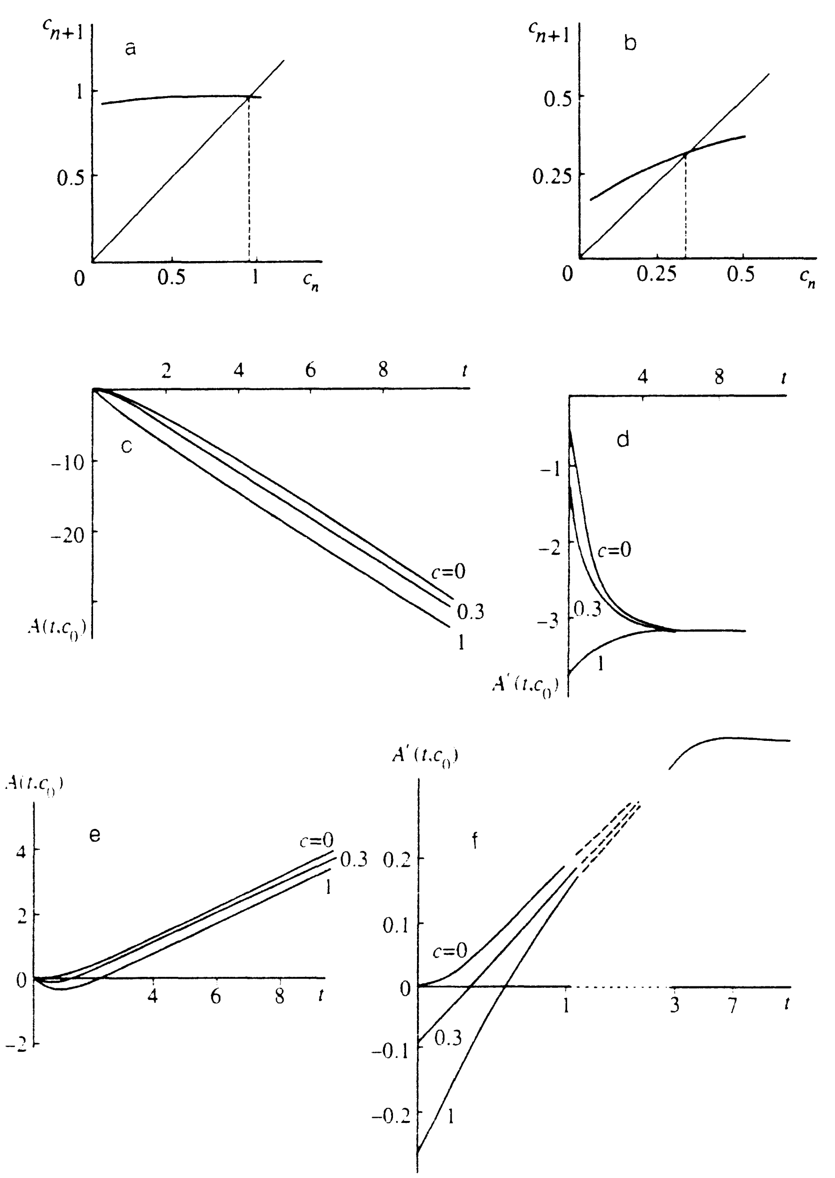

FIG. 2. The mapping (2.8) (a,b) and the functions $A(t,c)$ and $A'(t,c)$ for $v=0.952$ (a,c,d), 2 (b,e,f) and $\alpha=0.5$.

where $c(\varphi)=1-\exp(-\tau+\varphi_1+\varphi_2)$. We shall assume below that a condition of the form (2.5) is satisfied for all $t\in[0,\varphi_2]$. Then for $t\in(\varphi_1,\varphi_1+\varphi_2]$ we have

$$u(t,\psi)=\exp\{\lambda v[A(t-\varphi_1,c)+o(1)\},$$

$$y(t,\psi)=\lambda\{1-\exp(\varphi_1-t)+o(1)\}$$

and for $t\in[\varphi_1+\varphi_2,t_2]$ [t_2 is the second positive root of the equation $u(t)=1$]

$$u(t,\psi)=\exp\{\lambda v\rho(t,\varphi,c)[1+o(1)]\},$$

where

$$\rho(t,c,\varphi)=A(\varphi_2,c)+t-\varphi_2-\varphi_1+e^{\varphi_1-t}-e^{-\varphi_1}.$$

Hence we determine $t_2+\varphi_1+\varphi_2+b(c)$ as a root of the equation $\rho(t_2,s,\varphi)=0$.

When the conditions (for each iteration)

$$A(t,c)<0,\quad t\in(0,\varphi_2],\quad b(c)<\varphi_3 \tag{2.9}$$

are satisfied, the sequencing operator Π transforms $(\psi(s),\lambda c)$ into an element of the same type

$$u(t_2+s,\psi,s)\in S_0(\bar{\varphi},\bar{c}),\quad y(t_2,\psi,s)=\lambda\bar{c},$$

and to determine $\bar{\varphi}=(\bar{\varphi}_1,\bar{\varphi}_2,...,\bar{\varphi}_{2m})$ and $\bar{c}$ to $o(1)$ we obtain the relations

$$\begin{aligned}
&\bar{\varphi}_1=\varphi_3-b(\varphi,c),\\
&\bar{\varphi}_j=\varphi_{j+2},\quad j=2,...,2m-2,\\
&\bar{\varphi}_{2m-1}=\tau-\varphi_1-\varphi_2-\cdots-\varphi_{2m},\\
&\bar{\varphi}_{2m}=\varphi_1,\\
&\bar{c}=1-\exp[-\varphi_2-b(\varphi,c)].
\end{aligned}\tag{2.10}$$

The dynamics of this $(2m+1)$-dimensional mapping specifies the structure of the rapidly oscillating attractors of Eqs. (1.1) for $\varepsilon=0$ and λ fairly large.

3. SPECIAL FEATURES OF LASING UNDER A WEAK EXTERNAL INFLUENCE

If there is weak external illumination or noise of the same direction and frequency as the radiation generated in the system, then $\varepsilon>0$ holds, and the slowly oscillating solution takes the following asymptotic form. In the first step, $t\in(0,\tau]$, Eqs. (2.3) hold. In the second step, $t\in(\tau,2\tau)$, we obtain

$$u(t)=-\varepsilon[\lambda A(t-\tau,c)]^{-1}[1+o(1)],$$

$$y(t)=\lambda[1-\exp(\tau-t)+o(1)]$$

when the condition

$$A(t,c)<0 \tag{3.1}$$

is satisfied for all $t\in(0,\tau]$. Integration in the interval $t\in(2\tau,t_2)$ reveals that $t_2(\psi,c)\to 2\tau$ as $\lambda\to\infty$. Hence, the system has a stable periodic solution with a period $T=2\tau+o(1)$. We note that the pulsing period is shortened in comparison to the case of $\varepsilon=0$ and that the value of the minimum radiation intensity increases from $\sim\exp(-\lambda N)$ to $\sim\varepsilon\lambda^{-1}$.

The condition $\dot{A}(t,c)<0$ is stronger than $A(t,c)<0$ (Figs. 2d and 2f); therefore, rapidly oscillating regimes are more characteristic of Eqs. (1.1) with $\varepsilon>0$. When they are constructed, we obtain the mappings

$$\bar{\varphi}_j=\varphi_{j+2}+o(1),\quad j=1,...,2m-1,$$

$$\bar{\varphi}_{2m-1}=\tau-\varphi_1-\varphi_2-...-\varphi_m+o(1), \tag{3.2}$$

$$\bar{\varphi}_{2m}=\varphi_1+o(1).$$

The mappings (3.2) can be written for the most part [neglecting terms $o(1)$] in the form of a single difference equation of order $2m$ for $z_n=\varphi_{1,n}$, which has the solution

$$z_n+\frac{\tau}{2m+1}=\sum_{j=1}^{m}(\nu_j\rho_j^n+\text{const}),$$

where the ν_j are complex constants specified by the initial conditions and the ρ_j are the roots of the characteristic equation $\rho^m+\rho^{m-1}+...+\rho+1=0$. Hence $\rho_j=\exp[2\pi_{ij}/(2m+1)]$, and after n iterations of the operator Π, we have

$$\varphi_{j,n}=\frac{\tau}{2m+1}+\sum_{k=1}^{m}\gamma_{jk}\cos\left(\frac{2\pi nk}{2m+1}+\kappa_{jk}\right),$$

where the κ_{jk} differ from one another by an integral multiple defined by $2\pi/(2m+1)$.

For $\varepsilon>0$ the rapidly oscillating solution of (1.1) has the form of a sequence of $2m+1$ pulses with a period $T\approx 2\tau$, a duration $\approx\tau/(2m+1)$ (when γ_{jk} and κ_{jk}) are small, and an interval between pulses $\approx\tau/(2m+1)$. The intensity of these pulses depends weakly on the intensity of the initial pulses and, after several iterations, is determined from (2.3), in which $c=c_{0m}=1-\exp[-\tau/(2m+1)]$; therefore, the basin of attraction of such solutions is fairly broad. It is important to note that since $|\rho_j|=1$, the mapping for z_m is not coarse. This means that the terms of order $o(1)$ appearing in (3.2) can accumulate as the number of iterations increases and the solutions of such a structure can vanish at (asymptotically) large times, small τ, and large m.

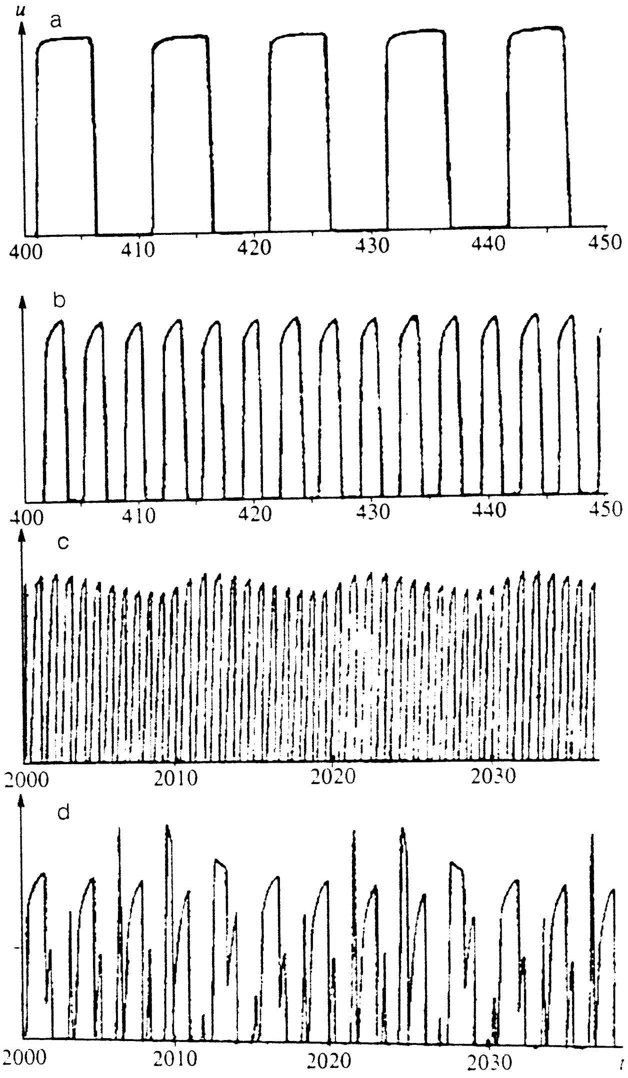

FIG. 3. Numerical integration of Eqs. (1.1): $\upsilon=0.952$, $\alpha=4.2$, $\tau=5$, $\lambda=47$, $\varepsilon=0.1$ (a–c) and $\upsilon=2$, $\alpha=0.5$, $\tau=1.5$, $\lambda=100$, $\varepsilon=0$ (d).

Rapidly oscillating solutions exist when $\varepsilon>0$ holds, if the following conditions are satisfied:

$$\dot{A}(t,c0m)<0,$$

$$t\in(0,\tau/(2m+1)],$$

$$c_{0m}=1-\exp[-\tau/(2m+1)]. \tag{3.3}$$

Hence it follows that if (3.3) is true for $m=0$, it certainly holds for $m=1, 2,...$. Therefore, the number of limit cycles with periods $T_m\approx 2\tau/(2m+1)$ in the phase space of the system increases without bound as $\lambda\to\infty$, i.e., multistability is realized. Figures 3a–3c present examples of numerical solutions of system (1.1) for parameters that are characteristic of a gas laser under various initial conditions. When the delay time in the feedback loop $\tau=5$, five steady-state lasing regimes with pulse repetition periods $2\tau/(2m+1)$, where $m=0, 1,...,4$, are discovered.

The solutions obtained as a result of the nonlinear analysis exist at each value of τ, if λ is fairly large. The more closely τ approaches 0 or ∞, the higher is the minimal value of $\lambda=\lambda_0(\tau)$ at which such cycles exist. As τ increases at a fixed $\lambda \gg 1$, cycles with a period of 2τ are born first, then cycles with a period of $2\tau/3$ appear, and so forth. Similarly, the cycles with a period of 2τ die first, then the cycles with a period of $2\tau/3$ die, and so forth. As has been shown, the cycles can also die when conditions (3.3) are violated. While a cycle which has died can be restored under the first scenario when λ increases, this does not occur under the second scenario.

4. DYNAMICS OF THE SOLUTIONS OF SYSTEM (1.1) WITH "SPIKES"

When the system (1.1) was investigated numerically, regimes with a qualitatively different structure were discovered along with the slowly and rapidly oscillating attractors described above. An example of such a solution is presented in Fig. 3d. It exists in the phase space simultaneously with the slowly oscillating solution. Besides the radiation peaks with a finite duration (which does not depend on λ), it has spikes, which are characterized by the fact that the values of u vary from unity to asymptotically large values and drop back down to unity during an asymptotically short (as $\lambda\to\infty$) time interval. We shall show below that the dynamics of such complex attractors is described by a finite-dimensional mapping.

Let us first consider the simplest structure for $\varepsilon=0$, in which the peaks and spikes alternate in a strict pattern.

We define the set of initial conditions such that the corresponding solution would begin a peak at $t=0$ and that a spike of radiation intensity would appear before it at a certain $s=\xi\in(-\tau,0)$ (see Fig. 1c). We use $S_0(\xi,p)$ to denote the set of continuous positive functions $\psi(s)$, where $s\in[-\tau,0]$, which have one spike at ξ of duration Δ and energy p and satisfy the conditions (Fig. 1c)

$$1)\ \psi(\xi)=\psi(\xi+\Delta)=\psi(0)=1,\Delta<\lambda^{-1/2},$$

$$2)\ \int_{\xi}^{\xi+\Delta}\psi(s)ds=p, \qquad (4.1)$$

$$3)\ \int_{-\tau}^{\xi}\psi(s)ds+\int_{\xi+\Delta}^{0}\psi(s)ds<\lambda^{-1/2}.$$

We present the asymptotic forms of the solutions as $\lambda\to\infty$ with the initial conditions $u(s)\in S_0(\xi,p)$ and $y(0)=\lambda c$, where $c\in(0,1)$. In the interval $t\in(0,\tau]$ we have

$$u(t,\psi)=\lambda[Q(t)+o(1)]^{-1},$$

$$Q(t,\xi,c,p)=\begin{cases}1+(vc-1)e^{-vt}, & 0<t<\tau+\xi,\\ q(t), & \tau+\xi<t<\tau,\end{cases}$$

$$q(t)=\{1+(vc-1)\exp[-v(\xi+\tau)]\}\exp[-v(\alpha p+t-\xi-\tau)]+1-\exp[v(\xi+\tau-t)].$$

In the interval $t\in(\tau,\tau+\eta]$ we obtain

$$u(t,\psi)=\exp\{\lambda v[R(t-\tau,\xi,c,p)+o(1)]\},$$

$$y(t,\psi)=\lambda[1-\exp(\tau-t)+o(1)].$$

Here

$$R(t,\xi,c,p)=\int_0^t[1-e^{-s}-\alpha Q(s,\xi,c,p)]ds,$$

and η is determined as the first positive root of the equation $R(\eta,\xi,c,p)=0$. If no such root exists, we set $\eta=0$. The energy z of the new short pulse at the time $\tau+\eta$ is found from the equation

$$(1-e^{-\eta})e^{-z}=1-e^{-\eta}-\alpha zQ(\eta,\xi,c,p)+o(1).$$

Assuming that $u(t)$ is asymptotically small in the interval $t\in[\tau+\eta+\delta,2\tau]$, we obtain

$$u(t,\psi)=\exp\{\lambda v[R_1(t,\xi,c,p)+o(1)]\},$$

$$y(t,\psi)=\lambda[Y(t,\xi,c,p)+o(1)],$$

$$Y(t,\xi,c,p)=1+(1-e^{-\eta})(1-e^{\eta-t})e^{-z},$$

$$R_1(t,\xi,c,p)=\int_{\tau+\eta}^{t}[Y(s,\xi,c,p)-\alpha Q(s,\xi,c,p)]ds.$$

Our assumption is valid, if we require that

$$R_1(t,\xi,c,p)<0,\ \text{when}\ t\in(\tau+\eta,2\tau]. \qquad (4.2)$$

Condition (4.2) is similar to condition (2.8) for slowly oscillating attractors and demarcates regions in parameter space where regimes with successively alternating intensity peaks and spikes can occur.

In the interval $t\in[2\tau,2\tau+\sigma]$ we have

$$u(t)=\exp\{\lambda v\rho_1(t,\xi,c,p)[1+o(1)]\},$$

where

$$\rho_1(t,\xi,c,p)=-\alpha\int_{\tau+\eta}^{2\tau}Q(s,\xi,c,p)ds+\int_{\tau+\eta}^{t}Y(s,\xi,c,p)ds.$$

Therefore, σ will asymptotically coincide with a root of the equation $\rho_1(2\tau+\sigma,\xi,c,p)=0$. We introduce a sequencing operator according to the rule

$$\Pi(\sigma(s),\lambda c)=(u(2\tau+\sigma+s,\psi,c),y(2\tau+\sigma,\psi,c)).$$

Under condition (4.2) we have

$$u(2\tau+\sigma+s,\psi,c)\in S_0(\bar{\xi},\bar{p}),y(2\tau+\sigma,\psi,c)=\lambda\bar{c},$$

where

$$\bar{\xi}=\begin{cases}\eta-\tau-\sigma, & \eta-\sigma>0,\\ 0, & \eta-\sigma<0,\end{cases}$$

$$\bar{p}=z, \qquad (4.3)$$

$$\bar{c}=Y(2\tau+\sigma,\xi,c,p).$$

Thus, the problem of constructing the asymptotic forms of the solutions of Eqs. (1.1) with initial conditions (4.1) reduces to the original problem with ξ, p, and c replaced by $\bar{\xi}$, $\bar{p}$, and $\bar{c}$, respectively. The evolution of these solutions is determined by iterations of Π, and the latter are given by the three-dimensional mapping (4.3). When conditions (4.2) are satisfied and λ is sufficiently large, the steady-state trajectories of this mapping correspond to a rough steady-state regime of system (1.1) of corresponding structure.

When there is external illumination ($\varepsilon>0$), the three-dimensional mapping (4.3) reduces to the two-dimensional mapping

$$\bar{\xi}=\eta(\xi,F(\xi,p),p)^{-\tau},$$

$$\bar{p}=z(\xi,F(\xi,p),p),$$

where

$$F(\xi,p)=1+(1-e^{-\xi-\tau})(1-e^{\xi-\tau})e^{-z},$$

η is a root of the equation $R'(\eta,\xi,c,p)=0$, $\sigma\to 0$, and $c=F(\xi,p)$.

Synthesizing the approaches to constructing the set $S_0(\xi,p)$ and the set of initial conditions for the rapidly oscillating solutions, we can obtain the conditions for the existence of attractors which assign solutions with several peaks and spikes during an interval of duration τ, and we can write down the corresponding finite-dimensional mapping corresponding to the dynamics of these attractors.

The nonlinear many-dimensional mappings obtained above can be fairly complicated and have rich dynamics.[36] For example, in the case of the existence of several attractors for a mapping, multistability of a different type will be observed. The coexistence of different attractors (differing with respect to the amplitude and duration of the peaks and spikes) is then possible, but they have the same structure (i.e., the same number of peaks and spikes in a period).

5. PERIODIC SOLUTIONS OF THE SYSTEM (1.1) FOR $v\gg 1$, $\varepsilon=0$

As was pointed out in the introduction, in solid-state lasers, for example, a ruby laser, the intracavity radiation damping rate greatly exceeds the population-inversion relaxation rate. In this case, which is important for practical applications, the large parameter in Eqs. (1.1) is $\tau v\gg 1$, with $\tau\ll 1$. Let us consider the slowly oscillating solutions in such a situation with the initial conditions

$$u(s)=\psi(s)=S_0,\quad y(0)=c.$$

The functions $\psi(s)$ satisfy the conditions (see Fig. 1d)

$$\psi(0)=1,\quad 0<\psi(s)<1,$$

$$s\in[-\tau,0],\ \int_{-\tau}^{0}\psi(s)ds<(\tau v)^{-1}.$$

Under the initial conditions selected there is an abrupt rise in the radiation intensity $[u(t)\gg 1]$ in the interval $t\in[0,t_1]$, and the pulse duration satisfies $t_1\to 0$ as $v\to\infty$. We determine the energy in the pulse p from the equation

$$p=c(1-e^{-p}),\quad p=\int_0^{t_1}u(s)ds. \tag{5.1}$$

We next integrate system (1.1), noting that, owing to the negative character of the feedback, $u(t)$ is asymptotically small in the interval $t\in(t_1,t_2)$. Therefore only slowly oscillating attractors are realized, and their dynamics is determined in the basic approximation by iterations of the one-dimensional mapping

$$\bar{c}=\lambda+(c-p-\lambda)e^{-x(c)}, \tag{5.2}$$

where $p(c)$ and $x(c)$ are roots, respectively, of Eq. (5.1) and

$$\alpha p=(\lambda-1)x+(c-p-\lambda)(e^{-x}-1).$$

When the values of the parameters correspond to the parameters of the laser in the figure, the mapping (5.2) has a stable fixed point c_0, which corresponds to a regime in which stable periodic pulses of radiation are generated with an energy in each peak $p=p(c_0)$ and a period $T=\tau+x(c_0)$, which reaches values hundreds of times greater than τ, in good agreement with the experimental data.

When external illumination is present, Eqs. (1.1) do not allow such solutions, since the conditions for constructing the sequencing operator Π are not satisfied. A numerical solution reveals that complicated irregular lasing regimes are possible here. We also note that the structure of the solutions excludes rapidly oscillating attractors when there is negative feedback.

6. RELAXATIONAL AUTO-OSCILLATIONS IN THE MODEL (1.2) WITH $v\gg 1$ AND $\varepsilon=0$

In this section we present data on the existence, structure, and asymptotic forms (for $v\gg 1$) of a rich set of steady-state periodic temporal structures with different dynamic properties in the absence of external radiation ($\varepsilon=0$).

We first evaluate the set of initial conditions for the slowly oscillating solutions. We assign the value of the radiation intensity in the interval $s\in[-\tau,0]$ from the set S_0 of nonnegative functions $\varphi(s)\in C_{[-\tau,0]}$ satisfying the conditions

$$\varphi(0)=1,\varphi(s)<1\quad\text{for } s=0,\quad \int_{-\tau}^{0}\varphi(s)ds<v^{-1/2}.$$

The level of population inversion at the initial time is $y(0)=c$, with $\gamma f(c)>1$. Physically, this corresponds to a radiation intensity at the noise level in the interval $s\in[-\tau,0]$ and the choice of $t=0$ at the onset of the emission pulse, with $u(0)=1$ (Fig. 4a). We use t_1, t_2, ... to denote the successive positive roots of the equation $u(t,c,\varphi)=1$.

Under the initial conditions chosen there is an abrupt rise in the radiation intensity to $u(t)\gg 1$ in the interval $t\in[0,t_1]$, and the pulse duration satisfies $t_1(\varphi,c)\to 0$ as $v\to\infty$. We apply the theory for constructing solutions described above to the simple case of a linear dependence of the gain on the population inversion, which is often employed in practice: $f(y)=y$, $\gamma=1$. Then we find that the

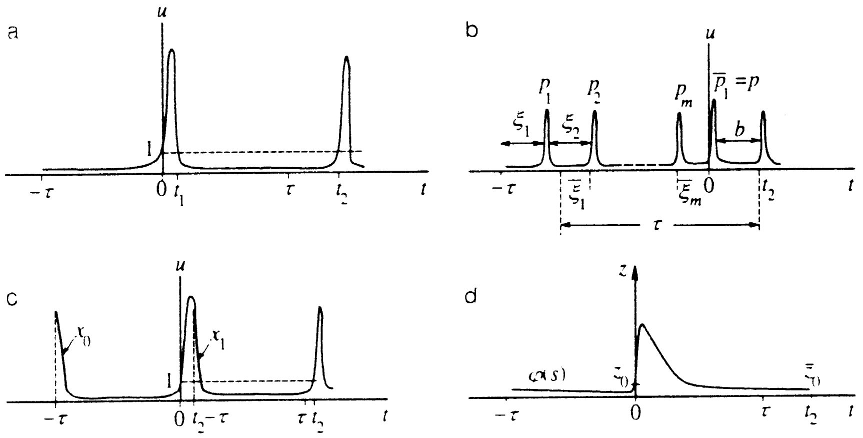

FIG. 4. Initial functions $\varphi(s)$ and solutions of Eqs. (1.2) and (1.3) of various structure.

dynamics of the slowly oscillating solutions of system (1.2) are determined by iterations of the one-dimensional mapping

$$\bar{c}=\lambda+[\alpha p+(c-p-\lambda)e^{-\tau}]e^{\tau-b}, \qquad (6.1)$$

where $p(c)$ and $b(c)$ are roots of the equations

$$p=c(1-e^{-p}), \qquad (6.2)$$

$$a(\tau,c)+(\lambda-1)(b-\tau)+[\alpha p+(c-p-\lambda)e^{-\tau}][1-e^{\tau-b}]=0. \qquad (6.3)$$

Here

$$a(x,c)=(\lambda-1)x+(c-p-\lambda)(1-e^{-x}), \qquad (6.4)$$

and it is assumed that

$$a(\tau,c)<0. \qquad (6.5)$$

Figure 5a presents plots of mapping (6.1), which were constructed for typical values of the parameters of semiconductor lasers. For $\alpha>0$ it has a stable fixed point c_0, which corresponds under condition (6.5) to a stable periodic solution of system (1.2) with a period $T=b(c_0)+o(1)$, $T>\tau$, a generation peak energy $p_0=p(c_0)$, and a maximum amplitude $u_0=v(c_0-1-\ln c_0)+1+o(1)$. The asymptotic values obtained closely correspond to the data from the numerical integration of Eqs. (1.2). As expected, when there is positive feedback, the oscillation period T is close to the delay time τ.

As the delay time in the feedback loop increases, the portion of the mapping (6.1) in which the inequality (6.5) is not satisfied expands (it is denoted in Fig. 5 by a dashed line). Therefore, slowly oscillating solutions are realized here under severe conditions, i.e., when a high initial level of population inversion $y(0)=c$, is created where $c>c^*$, or there is a quasistatic increase in τ.

The upper bound of the region for the realization of a slowly oscillating attractor with respect to τ in the parameter space is determined from the condition $a(\tau,c_0)=0$. The lower bound corresponds to small (asymptotically) values of $v\tau$. More precisely, it can be found near the boundary for disruption of the stability of the equilibrium state.[8]

The requirement $a(\tau,c_0)<0$ implies reproduction of the assigned initial set of functions S_0, i.e., the absence of radiation pulses during an interval of duration τ. Its violation results in the appearance of one or several pulses during such an interval and the emergence of new periodic structures, i.e., rapidly oscillating solutions. Such a situation arises when there is positive feedback ($\alpha>0$) and can also be realized directly in the physical problem by assigning initial external illumination in the form of several pulses during an interval of duration τ.

The set of initial conditions for such regimes consists of a level of population inversion $y(0)=c$, where $c>1$, and the set $S(\xi,P)$ of the functions $\varphi(s)$, where $s\in[-\tau,0]$, which assign the values of the radiation intensity in the form of m peaks in this interval (Fig. 4b). The parameters $\xi=(\xi_1,\xi_2,...,\xi_m)$ characterize the distance between pulses, so that $0<\xi_1+\xi_2+...+\xi_m<\tau$, and when $s=\tau+\xi_1+...+\xi_j$ holds, the j-th peak of $\varphi(s)$ of duration $\delta_j<v^{-1/2}$ begins; $P=(p_1,p_2,...,p_m)$ characterizes the areas under the peaks (the energy of the pulses):

$$p_j=\int_{-\tau+\xi_1+...+\xi_j}^{-\tau+\xi_1+...+\xi_j+\delta_j}\varphi(s)ds, \varphi(0)=1.$$

The values of $\varphi(s)$ for the remaining $s\in[-\tau,0)$ are asymptotically small.

Integrating (1.2), as before we note that the situation repeats after the time interval $t_2=b(c,\xi,p_1)$ with c replaced by $\bar{c}$ and (ξ,P) by $(\bar{\xi},\bar{P})$; therefore, the dynamics of the rapidly oscillating solutions of such a class is determined by attractors of the $(2m+1)$-dimensional nonlinear mapping:

$$\bar{c}=\lambda+[\alpha p_1+(c-p-\lambda)e^{-\xi_1}]e^{\xi_1-b},$$

$$\bar{\xi}_1=\xi_1+\xi_2-b,\bar{\xi}_2=\xi_3,...\bar{\xi}_{m-1}=\xi_m,\bar{\xi}_m=\tau-\sum_1^m\xi_j, \qquad (6.6)$$

$$\bar{p}_1=p_2,\bar{p}_2=p_3,...,\bar{p}_{m-1}=p_m,\bar{p}_m=p,$$

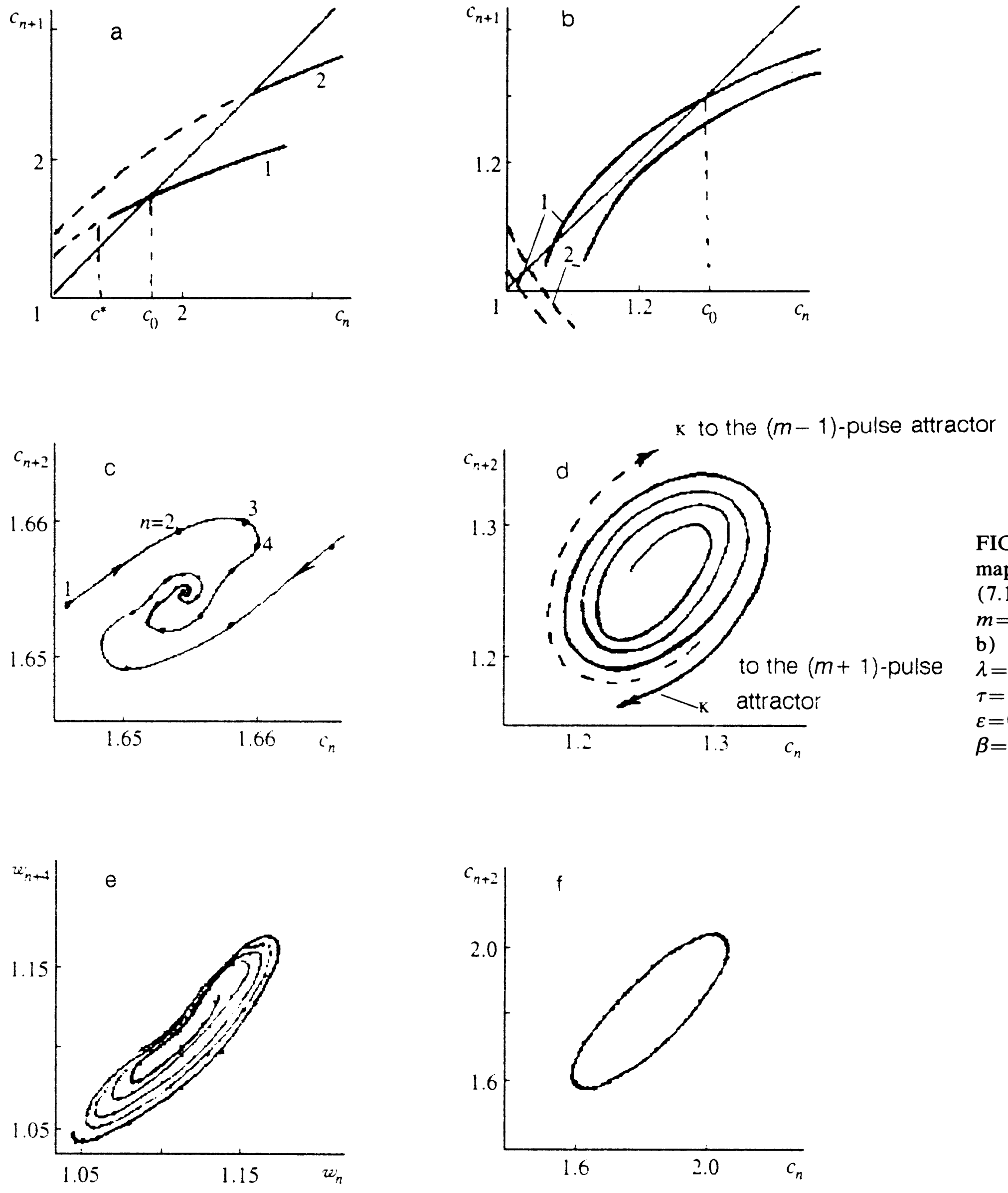

FIG. 5. Mapping (6.1) for $\lambda=1.5$ (a,b), mapping (6.6) for $m=1$ (c,d), mapping (7.1) for $m=3$ (d), and mapping (7.7) for $m=1$ (f): a) $\alpha=0.5$, $\tau=0.9$ (1), 2.2 (2); b) $\alpha=-0.5$, $\tau=0.2$ (*1*), 0.3 (*2*); c) $\lambda=1.35$, $\alpha=0.5$, $\tau=3$; d) $\lambda=1.35$, $\alpha=0.5$, $\tau=1.2$; e) $\lambda=1.5$, $\alpha=0.5$, $\tau=-1$, $\varepsilon=0.001$; f) $\lambda=1.5$, $ar=0.7$, $\tau=1.4$, $\beta=20$.

where $p=p(c)$ and $b=b(c,\xi_1,p_1)$ are roots of the equations

$$p=c(1-e^{-p}),$$

$$a(\xi_1,c)+(\lambda-1)(b-\xi_1)+[\alpha p_1+(c-p-\lambda)e^{-\xi_1}][1-e^{\xi_1-b}]=0.$$

A central point here is the assumption that each iteration satisfies the conditions

$$a(\xi_1,c)<0, \qquad (6.7)$$

$$\xi_1+\xi_2-b>0. \qquad (6.8)$$

The domain of the mapping (6.6) generally includes a stable fixed point (c_0,ξ_0,P_0), which corresponds to the stable periodic solution of the original system with period $T_m=\tau/(m+1)+x(c_0,\xi_0,p_0)+o(1)$.

The transient regime preceding the periodic solution is extremely complicated and prolonged. The convergence properties of the many-dimensional mapping (6.6) are revealed more easily by constructing the m-fold projections $c_{n+m+1}(c_n)$ (Fig. 5c), from which it follows that the motion of the trajectories of the mapping may be regarded as motion of the trajectories of m nonlinearly bound particles (radiation pulses) to a state that is homogeneous in time and space. As in the case of the slowly oscillating regimes, the region of initial conditions under which such convergence is possible is restricted.

The upper bound with respect to τ for the realization of an assigned rapidly oscillating regime in the parameter space is specified by violation of the inequality $a(\xi_0,c_0)<0$; i.e., in an interval of duration τ one more pulse appears, then transition to the next rapidly oscillating solution occurs.

The lower bound of a rapidly oscillating structure is associated with bifurcation of the fixed point of mapping (6.6) in the form of a subcritical Hopf bifurcation. When τ decreases, the region of initial conditions for an assigned

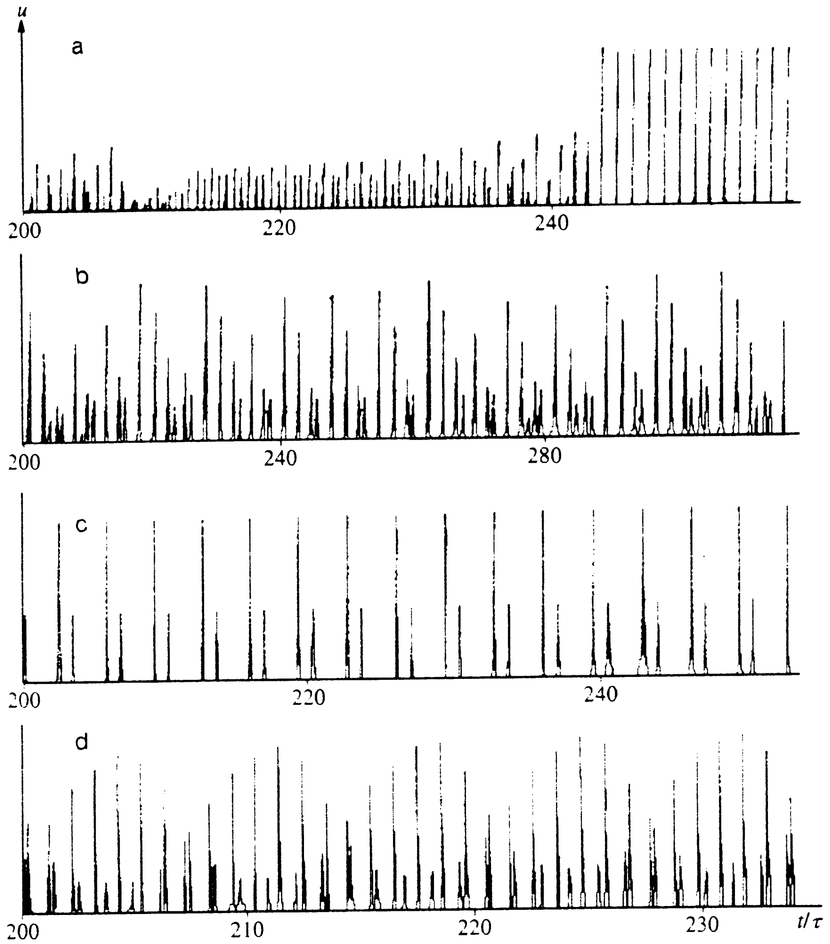

FIG. 6. Numerical integration of systems (1.2) (a–c) and (1.3) (d): $v=10^3$, $\lambda=1.5$, $\alpha=0.5$ (a), -0.65 (b,c), $\alpha r=0.7$ (d), $\beta=20$ (d), $\tau=0.9$ (a), 0.45 (b), 0.95 (c), 1.4 (d).

regime narrows sharply, and the convergence of the mapping deteriorates, so that the fixed point is achieved only after 10^3 or more iterations (their number also increases with the order of the mapping, i.e., the number of pulses in an interval of duration τ). When τ decreases further, the fixed point loses its stability (Fig. 5d), and after a certain number of iterations the system departs from the region where the particular attractor is defined due to the violation of the condition (6.7) (an additional pulse appears) or (6.8) (one pulse disappears). This results in the formation of a specific set of initial conditions, which lead the system to a neighboring attractor and then back, since the basin of attraction of the latter is restricted with respect to c. This dynamic process continues indefinitely until the system comes into the vicinity of the $(m-1)$-pulse attractor. Upon numerical integration the dynamics of the original system (1.2) near the lower bound for the existence of a rapidly oscillating solution appears to be fully chaotic, and nothing predicts the sudden ordering of the structure (Fig. 6a). Similar metastable irregular regimes were observed in Ref. 37 for an N-dimensional mapping modeling the dynamics of a chain of N elements of a neutral net, and it was suggested that such processes may be regarded as an alternative to a steady-state chaotic regime, since the time to reach an attractor in transient spatial chaos increases hyperexponentially as a function of the number of elements.

We note that the inequality (6.7) is not as strong as (6.5), so that rapidly oscillating structures are realized over broad regions of parameters, which overlap with one another and with the region of slowly oscillating regimes. This multistability permits the observation of hysteretic effects in the system.

In the case of negative feedback, the dynamics of the slowly oscillating solutions reduces to the dynamics of the one-dimensional mapping (6.1) considered above for a negative value of α. The conditions for the existence of such a temporal structure include the fulfillment of inequality (6.5) and the existence of an attractor for the mapping. The period of the oscillations appearing in this case greatly exceeds the delay time τ, just as in lasers where the negative feedback controls the additional losses introduced into the cavity, rather than the pumping.

The region for the realization of slowly oscillating regimes in the parameter space is narrower than in the case of $\alpha>0$, even though (6.5) provides a broader region for the existence of these regimes. The upper bound of this region with respect to τ results from the disappearance of the stable fixed point as a consequence of an inverse tangential bifurcation of the mapping, a typical plot of which is presented in Fig. 5b. As before, the dashed line denotes the portions of the mapping where the condition $a(\tau,c)<0$ is violated. After the laminar phase corresponding to motion near the former attractor, the system enters a region where more than one radiation pulse is generated in an interval of duration τ, and after a certain time it returns to the slowly oscillating attractor owing to the negative character of the feedback. Therefore, as τ increases, irregular dynamic regimes with intermittency of the first type[38] (Fig. 6b), as well as regular cycles with a large period, may be observed. The latter differ from one another with respect to the number of alternating slowly and rapidly oscillating pulses. The dynamics of each such cycle is determined by iterations of the corresponding nonlinear one-dimensional mapping. For example, for a periodic solution with two radiation pulses separated by a time interval greater or smaller than the delay time τ, we obtain (Fig. 6c)

$$\bar{c}=\lambda+(c-p-\lambda)e^{-b}+\alpha pe^{\tau-b}+\alpha p_1e^{\tau-b-\xi}-p_1e^{\xi-b}, \tag{6.9}$$

where $p=p(c)$, $\xi=\xi(c)$, $p_1=p_1(c)$, and $b=b(c)$, respectively, are roots of the equations

$$p=c(1-e^{-p}),$$

$$\xi(\lambda-1)+(c-p-\lambda)(1-e^{-\xi})=0,$$

$$p_1=[\lambda+(c-p-\lambda)e^{-\xi}](1-e^{-p_1}),$$

$$b(\lambda-1)+(c-p-\lambda)(1-e^{-b})-\alpha p(1-e^{\tau-b})-p_1(1-e^{\xi-b})-\alpha p_1(1-e^{\tau+\xi-b})=0.$$

Under the conditions

$$\xi(c)<\tau,$$
$$b(\lambda-1)+(c-p-\lambda)(1-e^{-b})-\alpha p(1-e^{-\xi})-p_1(1-e^{-\tau})<0 \tag{6.10}$$

and the presence of a stable fixed point c_0 for mapping (6.9), we reconstruct the main characteristics of such a lasing regime: the period $T=b(c_0)+o(1)$, the energies of the pulses generated $p(c_0)$ and $p_1(c_0)$, the amplitudes

$$u_0=v(c_0-1-\ln c_0)+1+o(1),$$

$$u_1=v(c_1-1-\ln c_1)+1+o(1),$$

$$c_1=\lambda-p_1(c_0)+[c_0-p(c_0)-\lambda]\exp[-\xi(c_0),$$

and the distance between the pulses $\xi(c_0)$.

The lower bound of such a temporal structure with respect to τ results from violation of conditions (6.10), and the mapping (6.9) experiences an inverse tangential bifurcation on the upper bound. This results in the appearance of complex irregular lasing regimes of the intermittency type. Here the laminar phase is represented by a pair of pulses following one another almost periodically, rather than single, almost periodic pulses. The correlational dimensionality of solutions of such a class increases from ≈ 2 to ≈ 3.8 as τ increases from 0.5 to 1.5 ($\alpha=-0.65$, $\lambda=1.5$).

No rapidly oscillating solutions like those studied earlier exist when there is negative feedback. Complicated auto-oscillations of smooth form (not of a relaxational character) are realized at small values of $|\alpha|$.

7. STEADY-STATE GENERATION REGIMES OF A LASER DIODE IN THE PRESENCE OF A HIGH LEVEL OF SPONTANEOUS RADIATION, LOW-PASS FEEDBACK, AND IMPURITIES WITH SATURABLE ABSORPTION

Semiconductor lasers have a high level of scattering of spontaneous radiation into the laser mode. This process can be taken into account in a simple manner with the aid of an additive term in the equation for the radiation intensity. Then ε is a small positive quantity in (1.2).

Despite the similarity between systems (1.2) for $\varepsilon=0$ and when ε is small, their dynamic properties differ. When there is positive feedback, the main difference between the solutions considered below and those which were studied earlier is that even when ε is small, an increase in the values of $u(t)$ begins before the peak of $u(t-\tau)$ has been completed. Therefore, instead of S_0, the set of initial functions taken for the slowly oscillating solutions should be the more complicated set $S_0(x,h)$, where x specifies the residual area of the peak of $\varphi(s)\in C_{[-\tau,0]}$ adjacent to $s=-\tau$:

$$x=\int_{-\tau}^{-\tau+\delta}\varphi(s)ds, \delta<v^{-1/2},$$

and h determines the level of population inversion y at the time of completion of the radiation pulse. As before, the value of y at the onset of the pulse is denoted by c (Fig. 4c).

The dynamics of the slowly oscillating solutions with a period $T=\tau+o(1)$ and initial functions from $S_0(x,h)$ are described by iterations of the mapping for $w_0=c_n+\alpha x_n$:

$$w_{n+1}=F(h_{n+1})+\alpha w_n-\alpha h_n+o(1), \tag{7.1}$$

where

$$F(h)=\lambda\tau+h-\int_0^\tau y_0(s,h)ds,$$

$$y_0(s,h)=\lambda+(h-\lambda)e^{-s},$$

under the condition

$$y_0(\tau,h_n)<1. \tag{7.2}$$

For h_n we have the nonlinear mapping $\Phi(h_n,h_{n+1})=0$, where

$$\Phi(h,g)=\alpha\int_0^\tau\left[\frac{y_0(s,h)+d}{1-y_0(s,h)}+\int_0^s e^{r-s}\frac{y_0(r,h)+d}{1-y_0(r,h)}dr\right]ds+\int_0^\tau\left[y_0(s,g)+d-\frac{y_0(s,g)+d}{1-y_0(s,g)}+\int_0^s y_0(r,g)\frac{y_0(r,g)+d}{1-y_0(r,g)}e^{r-s}dr\right]ds.$$

As $h_n\to h_0(n\to\infty)$ we obtain $w_n\to w_0$, where $w_0=(1-\alpha)^{-1}[F(h_0)-\alpha h_0]$. On this basis we conclude that in the phase space of Eqs. (1.2) there is an attractor [under condition (7.2)], in which the first coordinate has the structure of functions from $S_0(x,h)$, where $\alpha x+y(0)=w_0+o(1)$ and $h=h_0+o(1)$. The dynamics of c_n and x_n cannot be traced separately. Numerical investigations showed that solutions of this class can be discovered in the domain of the slowly oscillating regimes for $\varepsilon=0$ by selecting appropriate values of $\varepsilon>\varepsilon_0(v,\lambda,\tau)$, where ε_0 decreases with increasing $v\tau$.

Condition (7.2) is stronger than condition (6.5) for system (1.2). Therefore, when $\varepsilon>0$ holds, rapidly oscillating regimes are more typical. When their dynamics is examined, the set of initial conditions is obtained on the basis of the procedures described above. The finite-dimensional mappings appearing along this route for the ξ_n, which characterize the distance between pulses in an interval of duration τ, are linear, i.e.,

$$\xi_1=\xi_2,\ \xi_2=\xi_3,\dots,\xi_{m-1}=\xi_m,\quad \xi_m=\tau-\sum_1^m \xi_k+\mathbf{o}(1),$$

and have the solution

$$\xi_{j,n}=\frac{\tau}{m+1}+\sum_{k=1}^m \gamma_{jk}\cos\left(\frac{2\pi nk}{m+1}+\kappa_{jk}\right),\quad j=1,\dots m, \tag{7.3}$$

where the $\boldsymbol{\kappa}_{jk}$ differ from one another by an integral multiple defined by $2\pi/(2m+1)$.

The mapping (7.3) is not coarse, since all of its multipliers satisfy $|\rho_j|=1$. This means, in particular, that the terms of order $o(1)$ appearing in (7.2) can accumulate, i.e., can be summed as the number of iterations increases. This makes it possible to construct solutions only in a finite (asymptotically large as $v\to\infty$) time interval, which decreases as the number of pulses in an interval of duration τ increases.

Numerical integration reveals that the dynamics of the rapidly oscillating solutions depends sensitively on the value of ε (at a finite value of v). The solutions exhibit quasiperiodicity. In the general case, the mapping for w_n is a set of m segments of unwound spirals, which break apart when a radiation pulse appears or disappears in an interval of duration τ, and then the process repeats with undefined initial conditions (Fig. 5d).

It is important to note that the noncoarseness of mapping (7.2), which causes irregular dynamics, results from the presence of an additive term in the equation for the radiation intensity modeling spontaneous emission into the lasing mode. A similar result is obtained in models of lasers with modulation of the losses in the presence of weak external illumination.

Asymptotic solutions (as $v\to\infty$) of Eqs. (1.2) with $\varepsilon>0$ cannot be constructed for negative feedback. Numerical integration of the original system reveals that complex regular and irregular lasing regimes having a predominantly relaxational character (a smooth form) are possible here.

The special features of lasing with low-pass feedback, i.e., with a limited bandwidth β, can be investigated on the basis of Eqs. (1.3).

The set of initial conditions for the slowly oscillating solutions of (1.3) include the set of functions $\varphi(s)$, which assign the values of the voltage z in the interval $[-\tau,0]$, viz., $\varphi(s)=z_0\exp(-\beta s)$, the initial levels of population inversion $y(0)=c\,(c>1)$, and the intensity $u(t)=1$ (see Fig. 4a).

As before, integrating (1.3) in the interval $t\in(0,t_1]$, we obtain equations for determining the pulse energy and the level of carrier inversion at the time the radiation pulse is completed:

$$p=c(1-e^{-p}),\quad y(t_1)=c-p+o(1).$$

From the third equation in (1.3) we find $z(t_1)=z_0+\beta rp+o(1)$. In the interval $t\in[t_1,\tau]$ we require $u\ll 1$; hence

$$z(t)=(z_0+rp)e^{-\beta t},$$

$$y(t)=\lambda+Ae^{-t}-Be^{-\beta t},$$

$$A=A(c,z_0)=c-p-\lambda+\alpha z_0(\beta-1)^{-1}e^{\beta\tau},$$

$$B=B(z_0)=\alpha z_0(\beta-1)^{-1}e^{\beta\tau}.$$

From these and the first equation in (1.3) we find the condition for fulfillment of this requirement, which restricts the domain for slowly oscillating solutions:

$$(\lambda-1)\tau+A-(c-p-\lambda)e^{-\tau}+B(e^{-\beta\tau}-\beta e^{-\tau}-1)\beta^{-1}<0. \tag{7.4}$$

In the interval $t\in[\tau,t_1]$, as long as $u(t)$ remains asymptotically small, we have

$$z(t)=(z_0+r\beta p)e^{-\beta t},$$

$$y(t)=\lambda+Ae^{-t}-Be^{-\beta t}(1+r\beta pz_0^{-1})+\alpha\beta rp(\beta-1)^{-1}e^{\tau-t}+o(1).$$

Taking the time of the beginning of a new radiation pulse $t_2=b(c,z_0)$ as the initial time, we arrive at a problem similar to the preceding problem with c and z_0 replaced by $\bar{c}$ and $\bar{z}_0$:

$$\bar{c}=y(t_2)+o(1),\quad \bar{z}_0=z(t_2), \tag{7.5}$$

and $t_2=b(c,z_0)$ is a root of the equation

$$(\lambda-1)b+A(1-e^{-b})-B\beta^{-1}(1-e^{-\beta b})+\alpha rp(\beta-1)^{-1}(\beta-1-\beta e^{\tau-b}+e^{\beta(\tau-b)})=0.$$

Provided the inequality is satisfied, the two-dimensional mapping (7.5) describes the dynamics of the slowly oscillating solutions of system (1.3) with a period $T=b(c,z_0)$, where $T>\tau$.

In contrast to the case $\beta\to\infty$, the upper bound with respect to τ for the existence of slowly oscillating attractors in (1.3) is specified by a saddle-node bifurcation of the nonlinear mapping (7.4), rather than by violation of condition (7.5). Hence complicated irregular solutions are possible after the transition from slowly to rapidly oscillating structures.

Proceeding as in Sec. 6, for rapidly oscillating solutions with $m\geqslant 1$ intensity peaks in an interval of duration τ (for

$\alpha>0$) we obtain a $(2m+2)$-dimensional nonlinear mapping. As an example, here we present the case of $m=1$:

$$\bar{c}=\lambda+e^{-b}[A+B(z_1z_0^{-1}-1)e^{\xi-\beta\xi}]-Bz_1z_0^{-1}e^{-\beta b}+o(1),$$

$$\bar{z}_0=z_1e^{-\beta b},\quad \bar{z}_1=(z_1+r\beta p)e^{-\beta b}, \tag{7.6}$$

$$\bar{\xi}=\tau-b+o(1),$$

where $p(c)$ and $b(c,\xi,z_0,z_1)$ are, respectively, roots of the equations

$$p=c(1-e^{-p}),$$

$$(\lambda-1)b+A(1-e^{-b})-B\beta^{-1}[z_1z_0^{-1}(e^{-\beta\xi}-e^{-\beta b})-e^{-\beta\xi}+1]+B(z_1z_0^{-1}-1)e^{-\beta\xi}(1-e^{\xi-b})=0,$$

and the following inequality is assumed to hold for each iteration:

$$(\lambda-1)\xi+A-(c-p-\lambda)e^{-\xi}+B\beta^{-1}(e^{-\beta\xi}-1-\beta e^{-\xi})<0. \tag{7.7}$$

The attractors of the mapping (7.6) correspond to steady-state lasing regimes of the original dynamic system. This investigation is a separate complex problem. Here we present some particular results.

One special feature of mapping (7.6) is the narrow range of initial conditions leading to the attractor. This is the reason for the prolonged transient regimes, which differ structurally from the steady-state solution due to the violation of (7.7).

Unlike the case of an unlimited bandwidth, where the lower bound of the rapidly oscillating regime is specified by a subcritical Hopf bifurcation of the mapping (6.6) with the resultant formation of metastable chaotic structures, the mapping (7.6) exhibits a supercritical Hopf bifurcation with the formation of a limit cycle as τ decreases. One of the projections of (7.6), $c_{n+2}(c_n)$, is shown in Fig. 5f. This attractor of the mapping corresponds to the motion of phase trajectories on a two-dimensional torus (Fig. 6d). As the parameters vary, the cycle of the mapping (7.6) extends beyond the region where (7.7) is defined, and severe destruction of the two-dimensional torus occurs. We note that such a bifurcation sequence was observed experimentally in a CO_2 laser with an electro-optical intracavity modulator providing low-pass delayed feedback.[29]

The steady-state lasing regimes in a two-component laser diode with optoelectronic feedback can be examined on the basis of Eqs. (1.4). We present some results regarding the existence, structure, and asymptotic behavior of the relaxational solutions of this system without dwelling on the details of the derivation, which are similar to those presented above.

The slowly oscillating solutions with initial conditions from S_0 (Fig. 4a), $y(0)=c$, and $k(0)=h(h>0,c>1+h)$ are specified by attractors of the two-dimensional nonlinear mapping

$$\bar{c}=\lambda+(\alpha p+Ce^{-\tau})e^{\tau-b},\quad \bar{h}=k_0+He^{-\gamma b}, \tag{7.8}$$

where

$$C=ce^{-p}-\lambda,\quad H=he^{-rp}-k_0,$$

provided the equality

$$a^*(\tau,c,h)<0 \tag{7.9}$$

where

$$a^*(x,c,h)=(\lambda-1-k_0)x+C(1-e^{-x})+\gamma^{-1}H(1-e^{-\gamma x}),$$

is satisfied and $p=p(c,h)$ and $b=b(c,h)$, respectively, are roots of the equations

$$p=c(1-e^{-p})-r^{-1}h(1-e^{-rp}), \tag{7.10}$$

$$a^*(b,c,h)+\alpha p(1-e^{\tau-b})=0. \tag{7.11}$$

When there is positive feedback, rapidly oscillating structures with $m\geqslant1$ intensity peaks in an interval of duration τ are possible in Eqs. (1.4). The dynamics of such structures (with pulse energies p_j and distances between pulses ξ_j) is specified by iterations of the $(2m+2)$-dimensional nonlinear mapping

$$\bar{c}=\lambda+(\alpha p_1+Ce^{-\xi_1})e^{\xi_1-b},\quad \bar{h}=k_0+He^{-\gamma b},$$

$$\bar{p}_1=p_2,\quad \bar{p}_2=p_3,\ldots,\bar{p}_{m-1}=p_m,\quad \bar{p}_m=p, \tag{7.12}$$

$$\bar{\xi}_1=\xi_1+\xi_2-b,\quad \bar{\xi}_2=\xi_3,\ldots,\bar{\xi}_{m-1}=\xi_m,$$

$$\bar{\xi}_m=\tau-\xi_1-\xi_2-\ldots-\xi_m,$$

if the inequalities

$$\xi_1+\xi_2-b>0,\quad a^*(\xi_1,c,h)<0 \tag{7.13}$$

are satisfied for each iteration and $p(c,h)$ and $b(c,h,\xi_1,p_1)$, respectively, are roots of Eq. (7.10) and

$$a^*(b,c,h)+\alpha p_1(1-e^{\xi_1-b})=0.$$

The mappings (7.8) and (7.12) generally have a stable fixed point in the regions where they are defined. The boundaries of these regions are specified by conditions (7.9) and (7.13) and by the bifurcations of the mappings, which require additional investigation.

8. CONCLUSIONS

We have considered several simple models of lasers with optoelectronic delayed feedback. The problem of the dynamics of infinite-dimensional differential-difference systems has been reduced to the problem of the dynamics of finite-dimensional mappings on the basis of special asymptotic integration methods.

It has been shown that the steady-state solutions have a complicated relaxational structure for real laser parameters. Multistability has been described analytically in the case of negative feedback and a large pump parameter λ. This phenomenon is caused by the coexistence of attractors of different types (slowly and rapidly oscillating attractors, which create regimes with alternating intensity "peaks" and "spikes").

In the case of positive feedback with $v \gg 1$, there may be a hierarchy of multistable periodic lasing regimes with different structures (slowly and rapidly oscillating) and hysteretic transitions between them as the feedback parameters vary. The birth and death of rapidly oscillating attractors are caused by different bifurcations of many-dimensional mappings. A subcritical Hopf bifurcation (feedback with an unlimited bandwidth) leads to metastable chaotic regimes, and a supercritical Hopf bifurcation (low-pass feedback) leads to the organization of phase trajectories on a two-dimensional torus.

In the case of negative feedback with $v \gg 1$, multistability is not realized. The slowly oscillating structures are destroyed as a consequence of a saddle-node bifurcation, which leads to irregular intermittency regimes and the formation of cycles of large period and complex structure. The dynamics of the latter are specified by iterations of the corresponding one-dimensional mappings.

The important role of external influences acting on the system, for example, the level of spontaneous emission into the lasing mode, which cause irregularity in steady-state lasing regimes, has been disclosed. Such irregularity is a consequence of the neutral stability of the emerging linear mappings.

The asymptotic formulas obtained make it possible to easily reconstruct the main characteristics of the oscillations of the original system, to determine the regions of parameters and initial conditions under which the realization of a given regime is possible, and to reveal tendencies in the development of the dynamics as the parameters evolve.

The results may be extended to more complicated systems with delay that appear when problems concerning the dynamics of laser devices containing several coupled lasers are simulated. Descriptions of complex and diverse dynamic effects can be obtained in this way analytically.

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