Mechanism for the formation of a virtual cathode in open beam–waveguide systems

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In the linear approximation, the initial-value problem is solved for the evolution of an arbitrary perturbation in the density and velocity of a magnetized hollow electron beam propagating in a circular waveguide with a smoothly varying radius. The beam charge in the unperturbed state is neutralized by a fixed ion background; the dispersion of the space–charge waves is disregarded. It is shown that a virtual cathode develops at the point where the velocity of the slow space–charge mode passes through zero. An expression is derived for the perturbation growth rate at this point.

1. INTRODUCTION

It is well known that a time-independent electron beam can propagate in a drift space bounded by electrodes with a prescribed electron energy at the entrance to the drift space only for beam currents less than some maximum value. For currents above this limit the beam develops a virtual cathode, a cloud of electrons which on the average allows the maximum current to propagate and reflects "surplus" electrons back to the injection point. This phenomenon occurs both for beams with unneutralized space charge and for quasineutral beams. In the latter case there exist beam steady states with current above the limiting value, but these are unstable and the problem of interpreting the mechanism by which the virtual cathode forms reduces to explaining the reasons for the instability.

We refer to beam–waveguide systems as "open" if an electron beam passing through the waveguide does not intersect electrodes with the prescribed potentials on entering or leaving the drift space (in general the two boundary conditions have no preassigned correlations) which differs from the well-known Pierce mechanism, which occurs in the opposite case of "closed" drift space configurations.

The Pierce instability[1,2] results from positive feedback produced by the electric field of charges induced in the bounding electrodes. When they redistribute themselves in the external electric circuit connected to the bounding electrodes (the role of the circuit may be played by the waveguide walls), the induced charges allow the prescribed boundary conditions to be satisfied at the entrance and output from the drift space, but at the same time they create an additional electric field in the drift space, which assists or impedes the removal of space-charge perturbations from the drift space; the latter case corresponds to the formation of a virtual cathode. (Here and below we will be discussing only the electrostatic instability mechanism.) The growth rate of the Pierce instability is proportional to $1/L$, where L is the length of the drift space, and approaches zero in the limit $L\to\infty$.

Open systems model the situation in which a Pierce mechanism is ruled out or unimportant. For unneutralized beams this situation occurs in "long" systems, when a current above the limiting value is injected into a drift space whose longitudinal dimension is greater than the transverse dimension (in the limiting case, for injection into a semiinfinite waveguide,[3] in which case there is no feedback). As shown by numerical and laboratory experiments (e.g., in vircators, which generate microwaves and have a virtual cathode), in long systems the virtual cathode occurs either close to the injection point or in the region where the waveguide or beam radius is varying, i.e., in the nonuniform part of the system where the beam current passes from the sub- to the superlimiting state. In the present work the role of this transition region in the formation of a virtual cathode is studied in the model of an open system with a quasineutral beam.

2. FORMULATION OF THE PROBLEM

A hollow infinitely thin electron beam of radius R is propagating in a circular waveguide with smoothly varying radius $R_0(Z)$ (Fig. 1). We assume that the beam is nonrelativistic and cold; the waveguide is a perfect conductor; the beam charge in the unperturbed equilibrium state (N_0,V_0) is neutralized by a fixed ion background; in the direction of the axis of symmetry Z of the system an infinitely strong external magnetic field is imposed; we neglect the self-magnetic field of the beam; we consider only azimuthally uniform states. Such a beam is described by the following system of hydrodynamic equations:

$$N=N_0+\delta N(Z,t),\quad V=V_0+\delta V(Z,t),$$

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial Z}(NV)=0, \tag{1}$$

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial Z}=\frac{e}{m}\frac{\partial\Phi(R,Z,t)}{\partial Z}, \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial Z^2}=2e\delta N\frac{\delta(r-R)}{R}, \tag{3}$$

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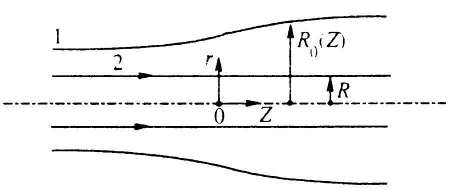

FIG. 1. Model of an open system; 1) waveguide; 2) hollow electron beam.

$$\Phi(R_0(Z),Z,t)=0,\ |\Phi(r,Z,t)|<\infty,$$

$$-\infty<Z<+\infty,\ 0\leqslant r\leqslant R_0(Z).$$

Here we take the charge to be positive, $e>0$; m is the electron mass, t is time; $N(Z,t)$ is the electron linear number density, $V(Z,t)$ is the hydrodynamic velocity; and $\Phi(R,Z,t)$ is the beam potential.

The beam current J_0 in the initial unperturbed state is chosen so that at the point $Z=0$ it is equal to the limiting value (for $L\to\infty$) of the Pierce current for a quasineutral beam of radius R in a waveguide with constant radius equal to $R_0(0)$:

$$J_0\equiv eN_0V_0=\frac{mV_0^3}{2e\ln[R_0(0)/R]}. \qquad (4)$$

[Note that the same expression (4) determines the value of the maximum possible steady current for a nonrelativistic unneutralized beam in the same geometry.[8]]

In what follows we will describe the longitudinal variation of the system given by the function $R_0(Z)$ in a more general form, using the local margin $\Delta(Z)$ above the threshold:

$$\Delta(Z)\equiv\frac{J_0-J_*(V_0,Z)}{J_*(V_0,Z)},\quad \frac{R_0(Z)}{R_0(0)}=\left(\frac{R_0(0)}{R}\right)^{\Delta(Z)}, \qquad (5)$$

$$-1<\Delta(Z)<\infty,$$

where the current $J_*(V_0,Z)$ is defined in Eq. (4) with $R_0(Z)$ instead of $R_0(0)$.

3. SOLUTION OF THE INITIAL-VALUE PROBLEM AND CONCLUSIONS

First we solve the Poisson equation (3). For a slowly varying function $R_0(Z)$ and for long perturbations $\delta N(Z)$, the second term on the left-hand side of the equation is small in comparison with the first. Under these conditions the solution of Eq. (3) can be found to any desired accuracy by means of iterations in the form of a perturbation series. We limit ourselves to the leading term of this series:

$$\Phi(r=R,Z,t)$$

$$=-2e\ln[R_0(0)/R]\delta N(Z,t)[1+\Delta(Z)]. \qquad (6)$$

The conditions under which the corrections to this approximation can be ignored reduce to a system of inequalities:

$$\frac{k^2}{k_{\perp 1}^2}\ll 1,\ \Delta_0\frac{k\kappa}{k_{\perp 2}^2}\ll 1,\ |\Delta(Z)|\frac{\kappa^2}{k_{\perp 2}^2}\ll 1; \qquad (7)$$

$$k_{\perp 1}^2=\frac{4\ln(R_0/R)}{R_0^2-R^2[1+2\ln(R_0/R)]},\ k_{\perp 2}^2=\frac{2}{R_0^2-R^2}. \qquad (8)$$

Here k and $\varkappa$ are the reciprocals of the length scales for the variation of $\delta N(Z)$ and $\Delta(Z)$, Δ_0 is the amplitude of $\Delta(Z)$, and we have written $R_0=R_0(Z)$ [for $\Delta_0\ll 1$ we can use $R_0=R_0(0)$ in estimates].

Substituting Eq. (6) for the beam potential into the equation of motion (2), we can go over to "long-wavelength" hydrodynamics, in which higher derivatives with respect to Z of $\delta N(Z,t)$ are disregarded, beginning with the second. Physically this implies that diffusion of the space-charge waves associated with the terms of the form $\sim[d\Delta(Z)/dZ]\partial^2\delta N(Z,t)/k_{\perp 2}^2\partial Z^2$, is neglected, and so is their dispersion, the contribution of which begins with the term $\sim\partial^3\delta N(Z,t)/k_{\perp 1}^2\partial Z^3$. (Note that the Pierce instability cannot grow in beams with bounded transverse dimensions unless there is dispersion, since in such beams if we neglect dispersion there is no slow space-charge wave with $k\neq 0$ and with phase velocity equal to zero for beam currents above the limiting value.)

After substituting expression (6) for the potential into the right-hand side of Eq. (2), we go over in Eqs. (1) and (2) to the dimensionless variable

$$n=\frac{\delta N}{N_0},\quad v=\frac{\delta V}{V_0},\quad z=\frac{Z}{R_0(0)},\quad \tau=\frac{V_0t}{R_0(0)},\quad q=\kappa R_0(0)$$

and linearize these equations with respect to n and v. The system of equations thus obtained can be transformed into the characteristic form

$$\pm[1+\Delta(z)]^{1/2}\frac{dn}{d\tau}+\frac{dv}{d\tau}+\Delta'(z)n=0, \qquad (9)$$

$$\frac{dz}{d\tau}=1\pm[1+\Delta(z)]^{1/2}; \qquad (10)$$

here the prime denotes differentiation with respect to z, and the plus and minus signs refer to fast and slow space-charge waves respectively.

Even at this stage the physical significance of the virtual cathode formation is obvious. As can be seen from the characteristic equation (10) for the slow space-charge waves, at the point $z=0$ where the current of the unperturbed beam is equal to the local threshold [$\Delta(0)=0$], the velocity of the slow wave vanishes. Throughout the remainder of the beam space the velocity vector of this wave is directed toward $z=0$. That is, the slow space-charge wave accumulates the perturbations it transports at the point where its velocity goes to zero. But that still leaves the fast wave, for which the point $z=0$ is not singular and which carries off its share of the perturbations in the direction of the beam motion. In a nonuniform system the linear modes are not independent, so the question remains: how is an arbitrary initial beam perturbation divided between the fast and slow space-charge waves, and what is the result of their subsequent competition?

Assuming that the margin Δ_0 by which the threshold is exceeded is a small quantity, we can restrict ourselves in

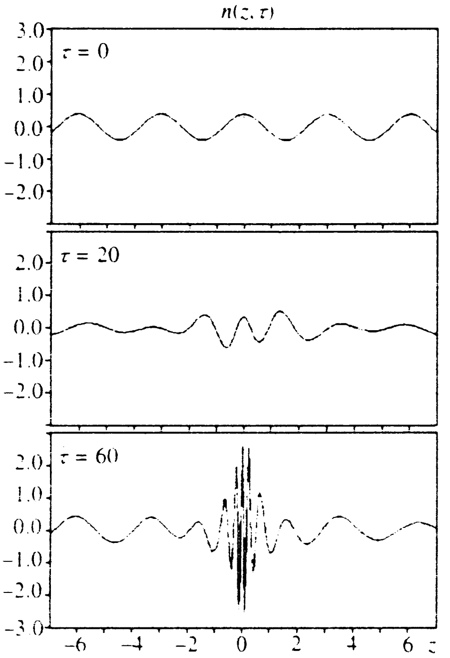

FIG. 2. Evolution of an initial density perturbation with $\Delta(z)$ of the form (12) for $\Delta_0=0.3$, $q=0.3$.

Eqs. (9) and (10) to quantities that are first order in $\Delta(z)$. By means of the transformation of the independent variables given by

$$r^{\pm}(z,\tau)=[1+\Delta(z)/2]n(z,\tau)\pm v(z,\tau),\quad |\Delta(z)|\leqslant\Delta_0\ll 1, \tag{11}$$

we can convert Eqs. (9) and (10) to the Riemann form

$$\frac{dr^+}{d\tau}=0[\Delta_0^2] \text{ on the characteristics } \left(\frac{dz}{d\tau}\right)_+=2+\frac{\Delta(z)}{2},$$

$$\frac{dr^-}{d\tau}=\frac{1}{2}[r^-+r^+]\Delta'(z) \text{ on the characteristics } \left(\frac{dz}{d\tau}\right)_- =-\frac{\Delta(z)}{2}. \tag{12}$$

To first order in Δ_0 the variable r^+ is the Riemann invariant. Because of this we can solve Eqs. (12) exactly without further simplifications.

In order to find the solution explicitly we use for the profile $\Delta(z)$ the function

$$\Delta(z)=\Delta_0\tanh(qz). \tag{13}$$

The solution of the initial-value problem for Eqs. (12) with $\Delta(z)$ is given by Eq. (13) for arbitrary initial perturbations of the density $n_0(z)$ and velocity $v_0(z)$ and is

$$r^+(z,\tau)=r_0^+[\xi_+(z,\tau)],$$

$$r^-(z,\tau)=r_0^-[\xi_-(z,\tau)]\frac{\cosh(qz)}{\cosh(q\xi_-)}\exp(\Delta_0 q\tau/2)+q\coth(qz)\int_z^{\xi_-}\frac{r_0^+[\xi_+(x,\tau_-(x))]}{\cosh^2(qx)}dx,$$

$$\xi_+\equiv\xi_+(z,\tau)=z-2\tau+\frac{\Delta_0}{4q}\ln\left\{\frac{\cosh[q(z-2\tau)]}{\cosh(qz)}\right\},$$

$$\xi_-\equiv\xi_-(z,\tau)=\frac{1}{q}\sinh^{-1}[\sinh(qz)\exp(\Delta_0 q\tau/2)]; \tag{14}$$

$$\tau_-(x)=-\frac{2}{\Delta_0 q}\ln\left|\frac{\sinh(qx)}{\sinh(q\xi_-)}\right|.$$

Here z and τ are the coordinates of the point of observation in the (z,τ) plane; $(z,\tau);\xi_\pm(z,\tau)$ are the points at which the $(\pm)$ characteristics intersect the z axis as they pass through the point of observation; $r_0^\pm(\xi)$ are the initial values of $r^\pm$ evaluated using Eqs. (11) with $n_0(\xi),v_0(\xi)$ in place of $n(z,\tau),v(z,\tau)$ respectively. Using the transformation (11) with respect to these values of $r^\pm(z,\tau)$ we can reconstruct $n(z,\tau)$ and $v(z,\tau)$.

The growth rate γ of the perturbations vanishes in the uniform part of the tube and is a maximum at the point $z=0$, where it equals

$$\gamma(z=0)=V_0\Delta'(0)/2. \tag{15}$$

(We recall that the growth rate of the Pierce instability for a quasineutral beam in a uniform tube of length L with a small margin $0<\Delta\ll 1$ above the threshold is equal to $\gamma_p=V_0\Delta/L$.) To illustrate this solution, in Fig. 2 we display step-by-step the stages through which an initial perturbation of the form $n_0(z)=a\cos(kz),v_0(z)\equiv 0$ evolves in accordance with Eqs. (14).

This mechanism for the formation of a virtual cathode is a special case of a universal phenomenon which takes place in nonuniform material flows of any sort, namely cumulation of perturbations at a point (on a line or surface) where the characteristic velocity with which perturbations propagate for the given medium goes through zero. This includes such phenomena as the formation of shock waves in gas flows at the sonic point, traffic jams[9], etc.

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