Vacuum radiation in the effective equations of quantum gravity with arbitrary form factors

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(Submitted 11 August 1993; resubmitted 8 April 1994)
Zh. Eksp. Teor. Fiz. **106**, 5–34 (July 1994)

A model-independent approach to the equations for the expectation values in the theory of gravity is considered. The action of the gravitational field in the in-vacuum state is expanded in a basis of nonlocal curvature invariants with arbitrary form factors. It is shown that for the solution of the effective equations to be asymptotically flat the form factors for small $\Box$ should behave as $\ln(-\Box)$, which coincides with their one-loop behavior in field theory. The vacuum-radiation flux is calculated in lowest order in the curvature without any assumptions about the quantum dynamics of the model. The kernel of $\ln(-\Box)$ with retarded boundary conditions is found. In the case of a spherically symmetric in-state, a simple expression for the radiated vacuum energy is obtained.

1. INTRODUCTION

In this paper we consider the model-independent approach to the quantum-gravity equations that was formulated in Ref. 1. The observed gravitational field in problems with initial data, like the problem of gravitational collapse, is regarded as an average over a certain quantum state defined in the remote past. To simplify the problem it is assumed that the initial state differs from the in-vacuum state[2] by the presence of a massive classical source, so that the action for the observed gravitational field can be represented in the form of a sum:[1)]

$$S_{\rm eff}=\frac{1}{16\pi}\int d^4x\sqrt{-g}R+S_{\rm vac}+S_{\rm source}, \tag{1}$$

where $S_{\rm source}$ is the action of the classical source, and the sum of the first two terms is the action of the gravitational field in the in-vacuum state. The action $S_{\rm vac}$ is not calculated on the basis of any quantum dynamical model. Instead, we write the most general functional allowed by the quantum state. An important property of the in-vacuum state is the fact that the action $S_{\rm vac}$ should be analytic in the curvature.[1] In this case it can be expanded in a basis of nonlocal invariants with arbitrary operator functions as the coefficients.[1] In first order in the curvature, the action can only be local and contain a single arbitrary constant, which is identified with the Newtonian constant. In second order, two arbitrary functions of one operator argument appear:

$$S_{\rm vac}=\frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g}\{R^{\mu\nu}\gamma_1(-\Box)R_{\mu\nu}+R\gamma_2(-\Box)R+O[R^3_{..}]\}, \tag{2}$$

$$\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu, \tag{3}$$

in third order, ten functions of three commuting arguments appear, and so on.[1,3–5] In the Lorentz action, the arguments of the form factors must be regarded as formal operators, behaving under variation as finite matrices. After the variation, retarded boundary conditions must be assigned to them.[2)] As a result, the equations for the expectation values have the form

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=8\pi(T^{\mu\nu}_{\rm vac}+T^{\mu\nu}_{\rm source}), \tag{4}$$

where

$$T^{\mu\nu}_{\rm vac}=\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm vac}}{\delta g_{\mu\nu}}\bigg|_{\Box\to\Box_{\rm ret}}. \tag{5}$$

In lowest order in the curvature,

$$T^{\mu\nu}_{\rm vac}=\frac{1}{(4\pi)^2}\{\nabla^\mu\nabla^\nu[\gamma_1(-\Box_{\rm ret})+2\gamma_2(-\Box_{\rm ret})]R-\tfrac{1}{2}g^{\mu\nu}\Box[\gamma_1(-\Box_{\rm ret})+4\gamma_2(-\Box_{\rm ret})]\times R-\Box\gamma_1(-\Box_{\rm ret})R^{\mu\nu}\}+O[R^2_{..}]. \tag{6}$$

The functions $\gamma(-\Box)$ and the higher-order form factors in the expression (2) for $S_{\rm vac}$ were initially defined for real, negative $\Box$. The basic assumption concerning them is that the form factors admit a spectral representation. This means that the functions γ can be analytically continued into the complex plane, while all their singularities lie at real, non-negative $\Box$. In order to restrict the choice of form factors as little as possible, we shall assume that they have an arbitrary power-law increase at large $|\Box|$:

$$\gamma(-\Box)=\Box^n\mathcal{O},\quad \mathcal{O}\to 0,\quad |\Box|\to\infty. \tag{7}$$

Then, making use of the Cauchy integral formula, we find that the corresponding spectral representation for γ has the form

$$\gamma_{1,2}(-\Box)=(\Box+\mu^2)^n\int_0^\infty\frac{dm^2}{m^2-\Box}\frac{w_{1,2}(m^2)}{(m^2+\mu^2)^n}+\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}(\Box+\mu^2)^k\left(\frac{\partial}{\partial\mu^2}\right)^k\gamma_{1,2}(\mu^2), \tag{8}$$

 1063-7761/94/070001-16$10.00

$$w_{1,2}(m^2)=\frac{1}{2\pi i}\,[\gamma_{1,2}(-m^2-i0)-\gamma_{1,2}(-m^2+i0)],\tag{9}$$

where $\mu^2>0$ is an arbitrary parameter on which $\gamma(-\Box)$ does not depend. For the form factors in the spectral form the replacement $\Box\to\Box_{\rm ret}$ reduces to the replacement of $1/(m^2-\Box)$ by the retarded massive Green function $G^{\rm ret}(m^2)$. The restriction on the behavior of $\gamma(-\Box)$ at small $|\Box|$ imposed by the representation (8),

$$\gamma(-\Box)=\frac{\mathscr{O}}{\Box},\quad \mathscr{O}\to 0,\quad |\Box|\to 0,\tag{10}$$

can be lifted, although it will be shown below that for the solution of the effective equations to be asymptotically flat, the restriction on the behavior of $\gamma(-\Box)$ at zero should be even more stringent.

For the form factors in the spectral representation, the action of the operator functions $\gamma(-\Box_{\rm ret})$ on a tensor has the form

$$\gamma(-\Box_{\rm ret})I^{\mu\ldots\nu}=(1+\mu^{-2}\Box)^n\int_0^\infty dm^2\tilde{w}(m^2)G^{\rm ret}(m^2)I^{\mu\ldots\nu}+\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\left[\left(\frac{\partial}{\partial\mu^2}\right)^k\gamma(\mu^2)\right]\times(\Box+\mu^2)^kI^{\mu\ldots\nu},\tag{11}$$

$$\tilde{w}(m^2)=\frac{w(m^2)}{(1+\mu^{-2}m^2)^n},\tag{12}$$

$$(\Box-m^2)G^{\rm ret}(m^2)=-1.\tag{13}$$

In lowest order in the curvature,

$$G^{\rm ret}(m^2)I^{\mu\ldots\nu}(\mu)=\frac{1}{4\pi}\,\int d^4x\,\sqrt{-g(x)}\Bigg[\delta(\sigma)-\theta(-\sigma)\,\frac{mJ_1(m\sqrt{-2\sigma})}{\sqrt{-2\sigma}}\Bigg]\times g^{\mu}_{\mu'}(\mathscr{M},x)\ldots g^{\nu}_{\nu'}(\mathscr{M},x)I^{\mu'\ldots\nu'}(x),\tag{14}$$

where $\mathscr{M}$ is the observation point, the symbol $\int$ denotes an integral over the region lying in the past from $\mathscr{M}$, $\sigma=\sigma(\mathscr{M},x)$ is a world function[9] of the point $\mathscr{M}$ and the integration point x, J_1 is the Bessel function of order, and $g^{\mu}_{\mu'}$ is the bivector of geodesic parallel transport.[9] An important role is played by the order of the integration in (11) and (14). The integral over space-time in (14) is taken first, and only then do we take the integral over the mass in (11).

The aim of the approach proposed in Ref. 1 is to find the dependence of the properties of the space-time that is the solution of the effective equations on the form of the unknown form factors in the action $S_{\rm vac}$. The results obtained in this way make it possible to establish important properties of the form factors that can then be used as requirements on models of the fundamental interactions.

In this paper we consider the equations for the mean field in lowest order in the curvature and show that the requirement of asymptotic flatness of the solution fixes the behavior of the functions $\gamma(-\Box)$ for small $\Box$ to within two unknown constants:

$$\gamma_1(-\Box)=-w_1(0)\ln(-\Box)+\mathscr{O}\ln(-\Box),\tag{15}$$

$$\gamma_2(-\Box)=-w_2(0)\ln(-\Box)+\mathscr{O}\ln(-\Box),\quad \mathscr{O}\to 0,\quad \Box\to -0.\tag{16}$$

Here, $w_1(0)$ and $w_2(0)$ are the $m^2=0$ limits of the spectral densities (9). For the solution of the equations to be asymptotically flat it is necessary that these limits be finite. The combination $w_1(0)+3w_2(0)$ determines the asymptotic behavior of the scalar curvature at light infinity, while the combination $w_1(0)+2w_2(0)$ serves as a measure of the radiated vacuum energy. The vacuum-energy flux across the future light infinity is calculated below as a nonlocal functional of the curvature. Generally speaking, the energy-flux density at a given moment of retarded time depends on the curvature in the entire region inside the light cone of the past of the observation point, but below it will be shown that in reality only the surface of the cone contributes to the flux.

The expressions (15) and (16) coincide with the one-loop behavior of the form factors in field theory. The difference is that the one-loop form factors have logarithmic behavior at other than small $\Box$, and at intermediate values of $\Box$ they are undetermined because of ultraviolet divergences. However, the divergent terms are local and disappear at infinity. One of the conclusions of the present work is that in the space-time region near light infinity, the results of the loop expansion in field theory can be believed. This is explained by the fact that the emission of real particles from the vacuum, detected in this region, is simultaneously a quantum effect and a long-range effect. The fact that the energy flux is proportional to $w(0)$ implies that only massless vacuum particles are emitted across the light infinity of the future. If only massive particles were present in the spectrum, the cut in the complex plane due to the singularities of the form factors would start from a certain nonzero value m^2, and we would have $w(0)=0$ in (9). The constants $w(0)$ contain definite information on the spectrum of the massless particles, since in the field theory they are sums of contributions from different spins. For example, the spin-zero contribution is

$$w_1(0)=\tfrac{1}{60},\quad w_2(0)=-\tfrac{1}{180}.\tag{17}$$

2. GENERAL EXPRESSION FOR THE ENERGY FLUX ACROSS THE FUTURE LIGHT INFINITY

Assuming that the solution of the equations for the expectation values is asymptotically flat, we shall consider a congruence $[u(x)=\text{const}, (\nabla u)^2\equiv 0]$ of light rays reaching the future light infinity ($\mathscr{I}^+$). Let r be the brightness distance along the rays, so that the area of the orthogonal section is $4\pi r^2$, and let $M(u)$ be the Bondi mass on $\mathscr{I}^+$ (Ref. 10). Then it follows from the dynamical equations

(4) that the rate of emission of energy with respect to the retarded time u, with the latter normalized by the condition

$$(\nabla u,\nabla r)|_{\mathscr{I}^+}=-1, \tag{18}$$

is equal to

$$\frac{dM(u)}{du}=-\frac{1}{4\pi}\int d^2\mathscr{S}\left[\left(\frac{\partial}{\partial u}C_1\right)^2+\left(\frac{\partial}{\partial u}C_2\right)^2\right]-\int d^2\mathscr{S}\left(\frac{1}{4}r^2T^{\mu\upsilon}_{\text{source}}\nabla_\mu\upsilon\nabla_\upsilon\upsilon+\frac{1}{4}r^2T^{\mu\nu}_{\text{vac}}\nabla_\mu\upsilon\nabla_\upsilon\upsilon\right)\bigg|_{\mathscr{I}^+}, \tag{19}$$

$$(\nabla\upsilon)^2=0,\quad (\nabla u,\nabla\upsilon)|_{\mathscr{I}^+}=-2, \tag{20}$$

where the integrals are taken over the unit 2-sphere $\mathscr{S}$ formed by the intersection of the surfaces $u=\text{const}$ and $\mathscr{I}^+$; $\partial C_1/\partial u$ and $\partial C_2/\partial u$ are the Bondi–Sachs information functions[11] that determine the energy of the gravitational radiation, and the last term in (19) is the vacuum-energy flux across $\mathscr{I}^+$. The finiteness of this flux is a necessary condition for asymptotic flatness.

3. BEHAVIOR OF SPECTRAL INTEGRALS WITH THE RETARDED GREEN FUNCTION

The calculation of the vacuum-energy flux density reduces to consideration of the integral

$$\int_0^\infty dm^2\widetilde{w}(m^2)G^{\text{ret}}(m^2)I(\mathscr{M}) \tag{21}$$

for a certain class of trial functions I and determination of its asymptotic behavior as the observation point $\mathscr{M}$ tends toward $\mathscr{I}^+$: $r(\mathscr{M})\to\infty$ for a fixed u and a fixed point on the sphere $\mathscr{S}$. It follows from Eqs. (4) and (6) that the trial functions under consideration should possess those properties of $T^{\mu\nu}_{\text{source}}$ that will be inherited by the curvature of the solution. For us, the following properties of I that restrict the class of trial functions are the most important: 1) analyticity of I along timelike world lines, including timelike past infinity (i^-); 2) decrease of I at spatial infinity and light infinity as $O(1/r^3)$.

To parametrize the space-time integral (14) we choose a timelike world line L: $x=\widetilde{x}(\tau)$, passing through the observation point $\mathscr{M}$, where τ is the proper time and the tangent vector $d\widetilde{x}/d\tau$ points into the future. We shall consider the equation

$$\sigma(\widetilde{x}(\tau),x)=0, \tag{22}$$

which determines τ as a function of x. This equation has two solutions. We choose one, $\tau=\vartheta(x)$, with

$$-\frac{d}{d\tau}\sigma(\widetilde{x}(\tau),x)\bigg|_{\tau=\vartheta(x)}>0. \tag{23}$$

Then

$$\left[\frac{d}{d\tau}\sigma(\widetilde{x}(\tau),x)\nabla_\alpha\vartheta(x)+\nabla^x_\alpha\sigma(\widetilde{x}(\tau),x)\right]_{\tau=\vartheta(x)}\equiv 0, \tag{24}$$

where the derivative ∇^x_α operates on the second argument of the world function. It follows from (22)–(24) that $(\nabla\vartheta)^2\equiv 0$, and $\vartheta(x)=\text{const}$ is the family of past light cones with apices on L. On a cone $\vartheta(x)=\text{const}$ we introduce the brightness distance ρ and parametrize the light rays by means of the points of a two-dimensional sphere: $\phi\in\mathscr{S}$. We use the arbitrariness in the choice of ϕ to impose the conditions $(\nabla\phi,\nabla\vartheta)\equiv 0$. The functions $\vartheta(x)$, $\rho(x)$, and $\phi(x)$ form the Bondi–Sachs coordinates associated with the observation point, and it is convenient to use them as the coordinates of the integration point x in the integral (14).

Let the observation point $\mathscr{M}$ correspond to $\tau=0$ on L. Then the light cone of the past of the point $\mathscr{M}$ is $\vartheta(x)=0$. We shall consider

$$\sigma(\mathscr{M};x)=\sigma(\mathscr{M};\vartheta,\rho,\phi). \tag{25}$$

From the equation for the world function[9] we find

$$\nabla^x_\alpha\sigma=g^{\mu\nu}(x)\nabla^x_\mu\sigma\nabla^x_\nu\nabla^x_\alpha\sigma. \tag{26}$$

Since

$$\partial_\rho\sigma|_{\vartheta=0}=\partial_\phi\sigma|_{\vartheta=0}=0, \tag{27}$$

by setting $\vartheta=0$ in (26) we obtain

$$\frac{1}{\Psi}\nabla_\rho\nabla_\vartheta\sigma\bigg|_{\vartheta=0}=\frac{1}{\Psi}\partial_\rho\partial_\vartheta\sigma\bigg|_{\vartheta=0}=1, \tag{28}$$

where

$$\frac{1}{\Psi}=(\nabla\vartheta,\nabla\rho)>0. \tag{29}$$

In addition, $\partial_\vartheta\sigma(\mathscr{M};x)|_{\vartheta=0}$ coincides with the left-hand side of (23) at $\vartheta=0$, and therefore takes only positive values. It follows from (28) that $\partial_\vartheta\sigma(\mathscr{M};x)$ at $\vartheta=0$ is the affine distance between $\mathscr{M}$ and x. Therefore, when $\mathscr{M}\to\mathscr{I}^+$ and x remains in the compact region, this quantity increases linearly with the brightness distance r to the point $\mathscr{M}$:

$$\partial_\vartheta\sigma(\mathscr{M};x)|_{\vartheta=0}=|\text{const}|\,r(\mathscr{M})(1+\mathscr{O}),\qquad \mathscr{O}\to 0,\quad r(\mathscr{M})\to\infty,\quad \mathscr{M}\to\mathscr{I}^+. \tag{30}$$

In the coordinates ϑ, ρ, $\phi\in\mathscr{S}$, the integral (14) takes the form

$$G^{\text{ret}}(m^2)I(\mathscr{M})=\frac{1}{4\pi}\int d^2\mathscr{S}\int_0^\infty d\rho\rho^2\left\{\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\bigg|_{\vartheta=0}-\int_{-\infty}^0 d\vartheta\Psi\frac{mJ_1(m\sqrt{-2\sigma})}{\sqrt{-2\sigma}}\bar{I}\right\}, \tag{31}$$

where $\sigma=\sigma(\mu;x)$ and

$$\bar{I}=g^\mu_{\mu'}(\mathscr{M};x)\ldots g^\nu_{\nu'}(\mathscr{M};x)I^{\mu'\ldots\nu'}(x). \tag{32}$$

After integration by parts over ϑ in (31), the cone contribution cancels:

$$G^{\mathrm{ret}}(m^2)I(\mathcal{M})=\frac{1}{4\pi}\int d^2\mathcal{S}\int_0^\infty d\rho\rho^2 \times\int_{-\infty}^0 d\vartheta J_0(m\sqrt{-2\sigma})\partial_\vartheta\left(\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\right). \tag{33}$$

Here, J_0 is the Bessel function of order zero.

We now investigate the behavior of the integral

$$\int_{-\infty}^0 d\vartheta J_0(m\sqrt{-2\sigma})\partial_\vartheta\left(\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\right) \tag{34}$$

as $\rho\to\infty$. Here we make use of the above assumption that I is analytic. Letting $z=\sqrt{-2\sigma}$ and applying the relation between the values of the Hankel function $H_0^{(1)}$ of order zero as we go around the branch point,

$$H_0^{(1)}(e^{\pi i}z)=H_0^{(1)}(z)-2J_0(z), \tag{35}$$

we find that

$$\int_{-\infty}^0 d\vartheta J_0(m\sqrt{-2\sigma})\partial_\vartheta\left(\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\right) =\frac{1}{2}\int_{-\infty+i0}^{\infty+i0} dzz\frac{H_0^{(1)}(zm)}{\partial_\vartheta\sigma}\partial_\vartheta\left(\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\right). \tag{36}$$

The integration contour in (36) can be closed in the upper half-plane. In the case of a static source I, a contribution to (36) is given only by the singularities of the geometrical factors, which for large ρ lie at $|z|\sim\rho$. Therefore, the integral (34) behaves as $\exp(-\rho m)$ as ρ tends to infinity. If, however, the source I is nonstatic, then by virtue of the assumed analyticity on the real axis, including the infinity in time, it has singularities in the complex-time plane, and the corresponding values of $|z|\sim|\mathrm{const}|\sqrt{\rho}$. In this case the integral (34) decreases as $\exp(-|\mathrm{const}|m\sqrt{\rho})$ as $\rho\to\infty$.

We shall assume first that the source I at each time has compact spatial support. Since in this case the ranges of integration over $\mathcal{S}$ and ρ in (33) are compact, for a nonstatic source we obtain

$$G^{\mathrm{ret}}(m^2)I(\mathcal{M})\propto\exp[-|\mathrm{const}|m\sqrt{r(\mathcal{M})}], \quad m\neq 0,\quad r(\mathcal{M})\to\infty,\quad \mathcal{M}\to\mathcal{I}^+ \tag{37}$$

and the dimensional constant in (37) depends on the positions of the singularities of the function I in the complex-time plane. It then follows that only the neighborhood of $m^2=0$ contributes to the leading term in the asymptotic form of the integral (21) as $\mathcal{M}\to\mathcal{I}^+$. To be more precise, this neighborhood includes $0<m<r^\epsilon/\sqrt{r}$, with $0<\epsilon<1/2$. We put $\xi=m\sqrt{r}$ and consider the limit

$$f(\xi)=\lim_{\mathcal{M}\to\mathcal{I}^+} r(\mathcal{M})G^{\mathrm{ret}}\left(\frac{\xi^2}{r(\mathcal{M})}\right)I(\mathcal{M})$$

$$=\lim_{\mathcal{M}\to\mathcal{I}^+}\frac{r(\mathcal{M})}{4\pi}\int d^2\mathcal{S}\int_0^\infty d\rho\rho^2 \times\int_{-\infty}^0 d\vartheta J_0\left(\xi\sqrt{\frac{-2\sigma}{r(\mathcal{M})}}\right)\times\partial_\vartheta\left(\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\right). \tag{38}$$

Since the presence of the factor $(\partial_\vartheta\sigma)^{-1}$ leads, as $\mathcal{M}\to\mathcal{I}^+$, to the appearance of an additional inverse power of $r(\mathcal{M})$, while the Bessel function J_0 is bounded, for finite values of ξ the limit (38) is a finite function of ξ. As $\xi\to\infty$, according to (37), this function falls off exponentially. After multiplication by $r^2(\mathcal{M})$ the integral (21), expressed in terms of $f(\xi)$, takes the form $(r=r(\mathcal{M})\to\infty)$

$$r^2(\mathcal{M})\int_0^\infty dm^2\tilde{w}(m^2)G^{\mathrm{ret}}(m^2)I(\mathcal{M}) =2\int_0^{r^\epsilon} d\xi\xi\tilde{w}\left(\frac{\xi^2}{r}\right)f(\xi)(1+\mathcal{O}),\quad \mathcal{O}\to 0. \tag{39}$$

For asymptotic flatness it is required that the limit of the integral (39) as $r\to\infty$ be finite, which, for the class of trial functions I, is the case if and only if $\tilde{w}(0)$ is finite. If $\tilde{w}(m^2)$ does not have a finite limit at $m^2=0$, the limit (39), even if it exists for special values of $\tilde{w}$ and I, is unstable against small variations of I induced by changes in $T^{\mu\nu}_{\mathrm{source}}$. The function $f(\xi)$ is of alternating sign, but depends on I, while the vacuum form factors are universal.

4. ASYMPTOTIC BEHAVIOR OF THE FORM FACTORS

If the function $\tilde{w}(m^2)$ and a few of its first derivatives have finite limits at $m^2=0$, the asymptotic expansion of (21) can be obtained by using in (33) the formula

$$J_0(m\sqrt{-2\sigma})=\left(\frac{2}{\partial_\vartheta\sigma}\frac{\partial}{\partial m^2}\frac{\partial}{\partial\vartheta}\right)J_0(m\sqrt{-2\sigma}) \tag{40}$$

and integrating by parts over ϑ and m^2. The integration by parts with the aid of (40) can be repeated the necessary number of times. At each stage there arises an extra factor of $1/\partial_\vartheta\sigma$ at $\vartheta=0$, which, according to (30), is proportional to $r^{-1}(\mathcal{M})$ for $\mathcal{M}\to\mathcal{I}^+$. The leading contribution to (21) behaves as $r^{-2}(\mathcal{M})$:

$$\int_0^\infty dm^2\tilde{w}(m^2)G^{\mathrm{ret}}(m^2)I(\mathcal{M}) =\frac{\tilde{w}(0)}{2\pi}\int d^2\mathcal{S}\int_0^\infty d\rho\rho^2 \times\left[\frac{1}{\partial_\vartheta\sigma}\frac{\partial}{\partial\vartheta}\left(\frac{\Psi}{\partial_\vartheta\sigma}\bar{I}\right)\right]_{\vartheta=0}+\frac{\mathcal{O}}{r^2},\quad \mathcal{O}\to 0,\quad \mathcal{M}\to\mathcal{I}^+, \tag{41}$$

and lies entirely on the past light cone of the point $\mathcal{M}$. Rewriting this integral in covariant form, we finally obtain

$$\int_0^\infty dm^2\tilde{w}(m^2)G^{\mathrm{ret}}(m^2)I(\mathcal{M})$$

$$= -\frac{\tilde{w}(0)}{2\pi} \int d^4x \sqrt{-g(x)} \delta'[\sigma(\mathcal{M};x)] \bar{I} + \frac{\mathcal{O}}{r^2(\mathcal{M})}, \quad (42)$$
$$\mathcal{O} \to 0, \quad \mathcal{M} \to \mathcal{I}^+,$$

where δ' is the derivative of the delta function, and the symbol $\int$ denotes, as before, an integral over the region lying in the past of the observation point.

We now take into account that $\tilde{w}(0) = w(0)$, and the local terms in (11), just like the action of local operators on (42), do not give contributions to the leading term of the asymptotic form of $\gamma(-\Box_{\text{ret}}) I(\mathcal{M})$. Then

$$\gamma(-\Box_{\text{ret}}) I(\mathcal{M}) = -\frac{w(0)}{2\pi} \int d^4x \sqrt{-g(x)} \delta'[\sigma(\mathcal{M};x)] \bar{I} + \frac{\mathcal{O}}{r^2(\mathcal{M})}, \quad \mathcal{O} \to 0, \quad \mathcal{M} \to \mathcal{I}^+. \quad (43)$$

On the other hand, $w(0)$ determines the behavior of $\gamma(-\Box)$ as a function of $\Box$ when $\Box \to -0$. In fact, it follows from (8) that

$$\gamma(-\Box) = -w(0) \ln(-\Box) + \mathcal{O} \ln(-\Box), \quad \mathcal{O} \to 0, \quad \Box \to -0, \quad (44)$$

if $w(O)$ is finite, and

$$[\ln(-\Box)]^{-1} \gamma(-\Box) \to \infty, \quad \Box \to -0, \quad (45)$$

if $w(0)$ is infinite. Thus, with the assumptions that we have made about the trial function I, the behavior that the contraction (43) must have for asymptotic flatness is ensured by the asymptotic behavior (44) of the form factor. The condition that the spatial support of I be compact will be lifted below.

5. KERNEL OF THE OPERATOR $\ln(-\Box_{\text{ret}}/\mu^2)$ IN LOWEST ORDER IN THE CURVATURE

The exact (not asymptotic) form of the kernel for $\ln(-\Box_{\text{ret}}/\mu^2)$ can be obtained as follows. We consider the contraction

$$\ln\left(\frac{-\Box_{\text{ret}}}{\mu^2}\right) I(\mathcal{M}) = \int_0^\infty dm^2 \left[\frac{1}{m^2+\mu^2} - G^{\text{ret}}(m^2)\right] I(\mathcal{M}), \quad (46)$$

where μ is a constant and the source I falls off at infinity as $O(1/r^3)$. In lowest order in the curvature, separating out the contribution from the neighborhood of the apex of the cone, we have from (33)

$$\ln\left(\frac{-\Box_{\text{ret}}}{\mu^2}\right) I(\mathcal{M}) = -\frac{1}{4\pi} \int_0^\infty dm^2 \int d^2\mathcal{S} \int_{\rho_0}^\infty d\rho \rho^2 \times \int_{-\infty}^0 d\vartheta J_0(m\sqrt{-2\sigma}) \frac{\partial}{\partial\vartheta} \left(\frac{\Psi \bar{I}}{\partial_\vartheta \sigma}\right)$$
$$+ \int_0^\infty dm^2 \left\{\frac{I(\mathcal{M})}{m^2+\mu^2} - \frac{1}{4\pi} \times \int d^2\mathcal{S} \int_0^{\rho_0} d\rho \rho^2 \times \int_{-\infty}^0 d\vartheta J_0(m\sqrt{-2\sigma}) \frac{\partial}{\partial\vartheta} \left(\frac{\Psi \bar{I}}{\partial_\vartheta \sigma}\right)\right\}, \quad (47)$$

where $\bar{I}$ is defined in (32) and ρ_0 is a small parameter.

The integrals over m^2 and ϑ in (47) are not commutative. To calculate them we need the following auxiliary assertion. We consider the integral

$$\int_0^\infty dm^2 \int_{-\infty}^{\vartheta_0} d\vartheta J_0(m\sqrt{-2\sigma}) F(\vartheta) \equiv \int_0^\infty dm^2 \int_{-\infty}^{\vartheta_0} d\vartheta \frac{2F(\vartheta)}{\partial_\vartheta \sigma} \frac{\partial}{\partial m^2} \frac{\partial}{\partial\vartheta} J_0(m\sqrt{-2\sigma}), \quad (48)$$

where $\vartheta_0 \leqslant 0$ is an arbitrary constant, and the function $F(\vartheta)/\partial_\vartheta \sigma$ is regular in the region of integration. We shall make use of the fact that the world function $\sigma \equiv \sigma(\mathcal{M};\vartheta,\rho,\phi)$ is monotonic in ϑ: $(\partial_\vartheta \sigma \geqslant 0, \vartheta \leqslant 0)$ and vanishes at $\vartheta = 0$. Then, integrating by parts over ϑ and m^2, we obtain

$$\int_0^\infty dm^2 \int_{-\infty}^{\vartheta_0} d\vartheta J_0(m\sqrt{-2\sigma}) F(\vartheta) = \begin{cases} \left.\dfrac{2F(\vartheta)}{\partial_\vartheta \sigma}\right|_{\vartheta=0}, & \vartheta_0 = 0, \\ 0, & \vartheta_0 < 0. \end{cases} \quad (49)$$

According to (49), the regions of integration in (47) that do not have a common boundary with the surface of the past cone do not contribute to the integral, and therefore,

$$\ln\left(\frac{-\Box_{\text{ret}}}{\mu^2}\right) I(\mathcal{M}) = -\frac{1}{2\pi} \int d^2\mathcal{S} \int_{\rho_0}^\infty d\rho \rho^2 \frac{1}{\partial_\vartheta \sigma} \frac{\partial}{\partial\vartheta} \left(\frac{\Psi \bar{I}}{\partial_\vartheta \sigma}\right)\Bigg|_{\vartheta=0}$$
$$+ \int_0^\infty dm^2 \left\{\frac{I(\mathcal{M})}{m^2+\mu^2} - \frac{1}{4\pi} \int d^2\mathcal{S} \int_0^{\rho_0} d\rho \rho^2 \times \int_{-\infty}^0 d\vartheta J_0(m\sqrt{-2\sigma}) \frac{\partial}{\partial\vartheta} \left(\frac{\Psi \bar{I}}{\partial_\vartheta \sigma}\right)\right\}. \quad (50)$$

We cannot apply the lemma (49) to the last integral in (50), since at the apex of the cone we have $\partial_\vartheta \sigma = 0$. However, we can use this lemma to replace the region of space-time integration in the last term of (50) with any small neighborhood of the apex whose boundary intersects the cone at $\rho = \rho_0$. It is convenient to choose this neighborhood in the form of a region $\mathcal{D}$ bounded by the past cone of the point $\mathcal{M}$ and a future cone of a point lying in the past of $\mathcal{M}$. We then find

$$\frac{1}{4\pi} \int_{\mathcal{D}} d^2\mathcal{S} d\rho d\vartheta J_0(m\sqrt{-2\sigma}) \frac{\partial}{\partial\vartheta} \left(\frac{\rho^2 \Psi \bar{I}}{\partial_\vartheta \sigma}\right)$$

$$= -[(\nabla\rho)^2\bar{I}]\Big|_{\mathscr{M}}\frac{1}{m^2}$$

$$\times\{J_0(2m\rho_0[(\nabla\rho)^2\big|_{\mathscr{M}}]^{-1/2})-1\},\quad \rho_0\to 0, \tag{51}$$

in which, for regularity of the curvature at the point $\mathscr{M}$, it is necessary that

$$(\nabla\rho)^2|_{\mathscr{M}}=1. \tag{52}$$

Integration of the expression (51) over m^2 leads to a logarithmic divergence that is cancelled in (50). As a whole, the integral over m^2 in (50) is finite. Thus, we find

$$\ln\left(\frac{-\Box_{\rm ret}}{\mu^2}\right)I(\mathscr{M})$$

$$=\lim_{\rho_0\to 0}\left\{-\frac{1}{2\pi}\int d^2\mathscr{S}\int_{\rho_0}^{\infty}d\rho\rho^2\frac{1}{\partial_\vartheta\sigma}\frac{\partial}{\partial\vartheta}\left(\frac{\Psi\bar{I}}{\partial_\vartheta\sigma}\right)\bigg|_{\vartheta=0}\right.$$

$$\left.-2I(\mathscr{M})(\ln(\mu\rho_0)+\mathbb{C})\right\}+O[R_{..}], \tag{53}$$

where $\mathbb{C}$ is the Euler constant. Rewriting (53) in covariant form, we finally find

$$\ln\left(\frac{-\Box_{\rm ret}}{\mu^2}\right)I(\mathscr{M})$$

$$=\lim_{\rho_0\to 0}\left\{\frac{1}{2\pi}\int\!\!\!\!\!\!\!\smallsmile_{\rho>\rho_0}d^4x\sqrt{-g(x)}\delta'[\sigma(\mathscr{M};x)]\bar{I}(x)\right.$$

$$\left.-2I(\mathscr{M})[\ln(\mu\rho_0)+\mathbb{C}]\right\}+O[R_{..}], \tag{54}$$

where, in the integral $\int\!\!\!\!\smallsmile$ over the past light cone, a neighborhood of the apex, specified by the inequality $\rho<\rho_0$, is excised. The limit $\rho_0\to 0$ is finite.

The above assumption that the spatial support of the source I is compact ensures the convergence of the integrals (41)–(43) at the apex of the cone, since as the observation point tends to $\mathscr{I}^+$, the apex $\mathscr{M}$ moves outside the support of I. The expression (54), which is valid in the general case, permits us to replace the assumption of compactness of the support with the condition that I falls off at infinity as $O(1/r^3)$. Since the contribution of the apex of the cone, calculated in (54), is local and falls off as $O(1/r^3)$, the leading term in the asymptotic form factor is determined by a cone integral with a small neighborhood of the apex excised:

$$\gamma(-\Box_{\rm ret})I(\mathscr{M})=-\frac{w(0)}{2\pi}\int\!\!\!\!\!\!\!\smallsmile_{\rho>\rho_0}d^4x\sqrt{-g(x)}$$

$$\times\delta'[\sigma(\mathscr{M};x)]\bar{I}+\frac{\mathscr{O}}{r^2(\mathscr{M})},$$

$$\mathscr{O}\to 0,\quad \mathscr{M}\to\mathscr{I}^+. \tag{55}$$

Here, ρ_0 is a small parameter. The leading contribution to the asymptotic form of $\gamma(-\Box_{\rm ret})I(\mathscr{M})$ does not depend on the quantity ρ_0.

6. LOWEST-ORDER FORM FACTORS IN THE EFFECTIVE EQUATIONS

We now apply the results obtained above to the form factors of lowest order in the effective equations. Consider the trace of the field equation (4). According to (6),

$$g_{\mu\nu}T^{\mu\nu}_{\rm vac}=-\frac{2}{(4\pi)^2}\Box[\gamma_1(-\Box_{\rm ret})+3\gamma_2(-\Box_{\rm ret})]R$$

$$+O[R^2_{..}]. \tag{56}$$

Since by virtue of the asymptotic flatness the scalar curvature R of the solution should fall off as $O(1/r^3)$ on $\mathscr{I}^+$, we obtain

$$[\gamma_1(-\Box_{\rm ret})+3\gamma_2(-\Box_{\rm ret})]R=O\left(\frac{1}{r^2}\right). \tag{57}$$

Then, as proved above,

$$\gamma_1(-\Box)+3\gamma_2(-\Box)=\text{const}\cdot\ln(-\Box)(1+\mathscr{O}),$$
$$\mathscr{O}\to 0,\quad \Box\to -0. \tag{58}$$

We now consider the energy flux density in (19). Because of the presence of the local operator $\Box$ in the term with the Ricci tensor in (6), the contribution to the radiation is due entirely to the terms with the scalar curvature R:

$$T^{\mu\nu}_{\rm vac}\nabla_\mu v\nabla_\nu v=\frac{1}{(4\pi)^2}\nabla_\mu v\nabla_\nu v\{\nabla^\mu\nabla^\nu[\gamma_1(-\Box_{\rm ret})$$

$$+2\gamma_2(-\Box_{\rm ret})]R\}(1+\mathscr{O})+O[R^2_{..}], \tag{59}$$

where $\mathscr{O}\to 0$ on $\mathscr{I}^+$. Since the expression (59) should behave as $O(1/r^2)$ on $\mathscr{I}^+$, we have

$$\gamma_1(-\Box)+2\gamma_2(-\Box)=\text{const}\cdot\ln(-\Box)(1+\mathscr{O}),$$
$$\mathscr{O}\to 0,\quad \Box\to -0. \tag{60}$$

From (58) and (60) we obtain the results (15) and (16).

The expressions (59) and (55) lead to the following result for the vacuum-energy flux across $\mathscr{I}^+$:

$$-\frac{dM(u)}{du}=-\frac{1}{32\pi^3}[w_1(0)+2w_2(0)]$$

$$\times\partial^y_u\partial^y_u\int d^2\mathscr{S}(y)\lim_{r(y)\to\infty}r^2(y)$$

$$\times\int\!\!\!\!\!\!\smallsmile d^4x\sqrt{-g}R(x)\delta'[\sigma(x,y)]+O[R^2_{..}], \tag{61}$$

where we have omitted other contributions to the derivative (19) of the Bondi mass.[3)] In (61), u is the retarded time, normalized by the condition (18), and it is indicated explicitly that the differentiation with respect to u and the integration over the coordinates on the sphere pertain to the observation point $y\to\mathscr{I}^+$.

7. VACUUM RADIATION IN A SPHERICALLY SYMMETRIC STATE

The kernel (54) takes an especially simple form in the case of spherical symmetry. A general spherically symmetric metric has the form

$$ds^2 = d\Gamma^2 + r^2 d\Omega^2, \quad r = r(\Gamma) > 0, \tag{62}$$

$$d\Gamma^2 = r^2 d\tilde{\Gamma}^2, \tag{63}$$

where Ω is the unit 2-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi, \tag{64}$$

Γ is a certain two-dimensional Lorentz space, and r is a function on Γ. In light coordinates

$$(\nabla v)^2 = 0, \quad (\nabla u)^2 = 0, \quad (\nabla v, \nabla u)^{-1} \equiv \kappa < 0 \tag{65}$$

the metric of the space Γ has the form

$$d\Gamma^2 = 2\kappa du dv, \tag{66}$$

and $\tilde{\Gamma}$ is a space conformal to Γ. To denote points in Γ we shall use the letters A, B, C, and for points in Ω we shall use the letters a, b, c. Four-dimensional points are combinations of these: $\mathcal{M} = (A,a)$, $x = (\bar{A},\bar{a})$.

Let $I^{\mu'}\dots{}^{\nu'}$ be a spherically symmetric tensor. By virtue of the spherical symmetry, the world function and the parallel-transported tensor

$$\bar{I}(x) = g^{\mu}_{\mu'}(\mathcal{M},x)\dots g^{\nu}_{\nu'}(\mathcal{M},x) I^{\mu'\dots\nu'}(x) \tag{67}$$

depend on the angles only in the combination

$$\alpha = \cos\theta\cos\bar{\theta} + \sin\theta\sin\bar{\theta}\cos(\varphi - \bar{\varphi}) = \cos\sqrt{2\sigma_\Omega}, \tag{68}$$

where σ_Ω is the world function on the sphere Ω.

To realize the δ-function in (54) it is necessary to solve the equation of the past light cone in the metric (62). The equation

$$\sigma(\mathcal{M};x) = 0 \tag{69}$$

is equivalent to

$$\sigma_{\tilde{\Gamma}}(A,\bar{A}) + \sigma_\Omega(a,\bar{a}) = 0. \tag{70}$$

Here, $\sigma_{\tilde{\Gamma}}(A,\bar{A})$ is the world function in $\tilde{\Gamma}$. We shall fix the observation point and solve the equation of the cone for the point $(\bar{A},\bar{a})$. The region $P_1 \cup P_2$ in Fig. 1 is the interior of the past cone of the point A in the two-dimensional space Γ:

$$\sigma_\Gamma(A,\bar{A}) < 0. \tag{71}$$

Here, σ_Γ is the world function in Γ. If $\bar{A} \in P_2$, then for all values of $\bar{a}$ the points $(\bar{A},\bar{a})$ lie inside the four-dimensional past cone of the observation point. Now let $\bar{A} \in P_1$. The lines 2 and 3 in Fig. 1 depict radial light rays arriving at the observation point and are specified by the equations

$$\sigma_\Omega(a,\bar{a}) = 0, \quad \sigma_{\tilde{\Gamma}}(A,\bar{A}) = 0. \tag{72}$$

Line 1 depicts the continuation of one of these rays into the past through $r=0$. Its equation has the form

$$\sigma_\Omega(a,\bar{a}) = \tfrac{1}{2}\pi^2, \quad \sigma_{\tilde{\Gamma}}(A,\bar{A}) = -\tfrac{1}{2}\pi^2. \tag{73}$$

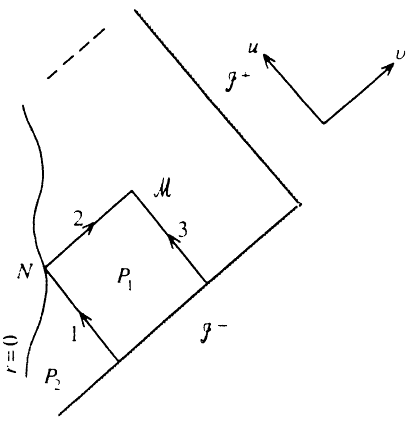

FIG. 1. Penrose diagram for a spherically symmetric asymptotically flat space-time. Light coordinates u, v are used; $\mathcal{J}^-$ and $\mathcal{J}^+$ are light infinities. The solid timelike line is $r=0$. The dashed line depicts the lightlike (or almost lightlike) part of the visibility horizon. The lines $1\cup 2$ and 3 are the world lines of two radial light rays arriving at the observation point $\mathcal{M}$.

If the point $\bar{A}$ lies in the region P_1, then

$$-\frac{\pi^2}{2} \leqslant \sigma_{\tilde{\Gamma}}(A,\bar{A}) \leqslant 0. \tag{74}$$

For each such point one can find values of $\bar{a}$ for which the point $(\bar{A},\bar{a})$ lies inside the four-dimensional past cone of the observation point, on this cone, or outside it. The surface of the past cone of the point $\mathcal{M}$ is

$$\bar{A} \in P_1, \quad \sigma_\Omega(a,\bar{a}) = -\sigma_{\tilde{\Gamma}}(A,\bar{A}). \tag{75}$$

We shall choose the angular coordinates θ, φ so that the value $\theta = 0$ corresponds to the observation point $\mathcal{M}$. Then $\sqrt{2\sigma_\Omega} = \bar{\theta}$ and $\alpha = \cos\bar{\theta}$, and the expression (54) for the kernel takes the form

$$\ln\left(\frac{-\Box_{\rm ret}}{\mu^2}\right) I(\mathcal{M}) = \lim_{\rho_0\to 0}\left\{-2\int_\Gamma \bar{\kappa}\bar{r}^2 d\bar{u}d\bar{v}\int_{-1}^{1} d\alpha \bar{I}\theta \times(\rho-\rho_0)\delta'(\sigma) - 2I(\mathcal{M}) \times(\ln(\mu\rho_0) + \mathbb{C})\right\} + O[R_{..}]. \tag{76}$$

Here and below, a bar on a symbol signifies that the given quantity is taken at the integration point. Solving the equation $\sigma = 0$ for α and integrating by parts over α, we find

$$\ln\frac{(-\Box_{\rm ret}}{\mu^2} I(\mathcal{M})$$

$$= \lim_{\rho_0\to 0}\left\{-2\int_{P_1} \bar{\kappa}\bar{r}^2 d\bar{u}d\bar{v}\frac{1}{\partial_\alpha\sigma}\frac{d}{d\alpha}\left(\frac{\bar{I}}{\partial_\alpha\sigma}\right)\right|_{\alpha=\cos\sqrt{-2\sigma_{\tilde{\Gamma}}}}$$

$$-2\int \bar{\kappa}\bar{r}^2 d\bar{u}d\bar{v}\frac{\bar{I}}{\partial_\alpha\sigma}\theta(\rho-\rho_0)\delta(\sigma)\Big|_{\alpha=1}$$

$$+2\int \kappa \bar{r}^2 d\bar{u}d\bar{v}\frac{\bar{I}}{\partial_\alpha\sigma}\theta(\rho-\rho_0)\delta(\sigma)\bigg|_{\alpha=-1}$$

$$-2I(\mathscr{M})\quad(\ln(\mu\rho_0)+\mathbb{C})\Bigg\}+O[R_{..}], \qquad (77)$$

where we have taken into account that $\partial_\alpha\sigma<0$.

In (77) the terms with $\delta(\sigma)$ at $\alpha=\pm 1$ are integrals over the radial light geodesics shown in Fig. 1. We have

$$\delta(\sigma)|_{\alpha=1}=\frac{\delta(\bar{u}-u)}{\partial_{\bar{u}}\sigma}\bigg|_{\alpha=1}+\frac{\delta(\bar{v}-v)}{\partial_{\bar{v}}\sigma}\bigg|_{\alpha=1},$$
$$\delta(\sigma)|_{\alpha=-1}=\frac{\delta(\bar{v}-v_N)}{\partial_{\bar{v}}\sigma}\bigg|_{\alpha=-1}, \qquad (78)$$

where (v_N,u) are the coordinates of the point N in the space Γ (see Fig. 1). We note that

$$\partial_{\bar{v}}\sigma|_{\alpha=1}^{\bar{u}=u}=0,\quad \partial_{\bar{u}}\sigma|_{\alpha=1}^{\bar{v}=v}=0,$$
$$\partial_{\bar{u}}\sigma|_{\alpha=-1}^{\bar{v}=v_N}=0. \qquad (79)$$

These inequalities and the expression (80) given below make it possible to transform the measure in the integrals over the light geodesics by expressing it entirely in terms of the function r on the rays.

In lowest order in the curvature we have

$$\partial_\alpha\sigma=-r\bar{r}+O[R_{..}],$$

$$(\bar{\nabla}\bar{r},\bar{\nabla}\sigma)\equiv\frac{1}{\bar{\kappa}}(\partial_{\bar{u}}\bar{r}\partial_{\bar{v}}\sigma+\partial_{\bar{v}}\bar{r}\partial_{\bar{u}}\sigma)=\bar{r}-r\alpha+O[R_{..}], \quad (80)$$

$$\rho=\sqrt{r^2+\bar{r}^2-2r\bar{r}\alpha}+O[R_{..}].$$

As a result, we obtain the following expression for the kernel $\ln(-\Box_{\rm ret}/\mu^2)$:

$$\ln\left(\frac{-\Box_{\rm ret}}{\mu^2}\right)I(\mathscr{M})$$
$$=\lim_{\rho_0\to 0}\Bigg\{-\frac{1}{r^2}\int_{P_1}\bar{\kappa}d\bar{u}d\bar{v}\frac{\partial}{\partial\alpha}\bar{I}\bigg|_{\alpha=\cos\sqrt{-2\sigma_\Gamma}}$$
$$-\frac{1}{r}\int_\infty^0 d\bar{r}\frac{\bar{r}}{\bar{r}+r}\bar{I}\bigg|_{\substack{\alpha=-1\\ \text{ray1}}}+\frac{1}{r}\int_0^{r-\rho_0}d\bar{r}\frac{\bar{r}}{\bar{r}-r}\bar{I}\bigg|_{\substack{\alpha=1\\ \text{ray2}}}$$
$$+\frac{1}{r}\int_\infty^{r+\rho_0}d\bar{r}\frac{\bar{r}}{\bar{r}-r}\bar{I}\bigg|_{\substack{\alpha=1\\ \text{ray3}}}-2I(\mathscr{M})(\ln(\mu\rho_0)+\mathbb{C})\Bigg\}$$
$$+O[R_{..}], \qquad (81)$$

which contains integrals over the region P_1 and over the light rays 1, 2, and 3 in the space Γ (see Fig. 1). As $\rho_0\to 0$ in the integrals over the rays a logarithmic divergence appears, which is cancelled exactly by the local contribution proportional to $I(\mathscr{M})$. The terms containing $\ln\rho_0$ can be separated out by integration by parts, after which the limit $\rho_0\to 0$ can be calculated explicitly. Finally, we obtain

$$\ln\left(\frac{-\Box_{\rm ret}}{\mu^2}\right)I(\mathscr{M})=-\frac{1}{r^2}\int_{P_1}\bar{\kappa}d\bar{u}d\bar{v}\frac{\partial}{\partial\alpha}\bar{I}\bigg|_{\alpha=\cos\sqrt{-2\sigma_\Gamma}}$$
$$-\frac{1}{r}\int_\infty^0 d\bar{r}\frac{\bar{r}}{r+\bar{r}}\bar{I}\bigg|_{\substack{\alpha=-1\\ \text{ray1}}}$$
$$-\frac{1}{r}\int_0^r d\bar{r}\ln[\mu(r-\bar{r})]\frac{d}{d\bar{r}}\left(\bar{r}\bigg|\bar{I}_{\text{ray2}}^{\alpha=1}\right)$$
$$-\frac{1}{r}\int_\infty^r d\bar{r}\ln[\mu(\bar{r}-r)]\frac{d}{d\bar{r}}$$
$$\times\left(\bar{r}\bar{I}\bigg|_{\text{ray3}}^{\alpha=1}\right)-2I(\mathscr{M})\mathbb{C}+O[R_{..}]. \quad (82)$$

With the function $\bar{I}$ defined in (67), the expression (82) is valid for any tensor $I^{\mu\ldots\nu}(x)$. But if I is a spherically symmetric scalar, this expression is further simplified, since $\bar{I}=I$ does not depend on α. In (82) there then remain only integrals over radial light rays and a local contribution. In the expression (59) for the vacuum-energy flux density the role of the source I is played by the curvature scalar.

As can be seen from the expression (81), in the limit $r=r(\mathscr{M})\to\infty$, $\mathscr{M}\to\mathscr{I}^+$, under the condition that the scalar source I behave as $O(1/r^3)$, the local contribution and the contribution of ray 3 vanish. As a result, the leading term of the asymptotic form of the kernel $\ln(-\Box_{\rm ret}/\mu^2)$ is determined entirely by the contribution of ray $1\cup 2$. We obtain

$$\ln\left(\frac{-\Box_{\rm ret}}{\mu^2}\right)I(\mathscr{M})|_{\mathscr{M}\to\mathscr{I}^+}=-r^{-2}(\mathscr{M})\int_{\mathscr{I}^-}^{\mathscr{I}^+}d\bar{r}\bar{r}\bar{I}$$
$$+O\left(\frac{1}{r^3(\mathscr{M})}\right)+O[R_{..}], \qquad (83)$$

where the integral is taken along the entire world line of a radial light ray emerging from $\mathscr{I}^-$ and arriving at a given point of $\mathscr{I}^+$. The final expression for the vacuum-radiation flux in the spherically symmetrical in-state has the form

$$\frac{dM(u)}{du}=-\frac{1}{4\pi}[w_1(0)+2w_2(0)]$$
$$\times\frac{d^2}{du^2}\int_{\mathscr{I}^-}^{\mathscr{I}^+}d\bar{r}\bar{r}\bar{R}+O[R_{..}^2] \qquad (84)$$

and, in lowest order, contains only the curvature scalar. Here the retarded time u is normalized by the condition (18).

The expression (84) is fully analogous to the contribution linear in the curvature to the vacuum-radiation flux in two dimensions.[12] As in the case of two dimensions, the first-order radiation attenuates as the horizon is approached.[4)] In fact, in the reference frame comoving with the source, the derivatives of the curvature are finite on the horizon. As a consequence of the normalization

(18), the retarded time u in (84) and the time u_- in the frame comoving with the source are related by the condition

$$\left.\frac{du_-}{du}\right|_{\text{horizon}}=0. \tag{85}$$

Therefore, in those expressions in which the derivative d/du acts on the curvature as in (84), the result vanishes on the horizon. As in the two-dimensional case,[12] the stable component, i.e., the Hawking radiation, is contained in the terms $O[R^2]$. To find them it is necessary to take into account the third-order form factors in the action S_{vac}.

The vacuum radiation in first order in the curvature possesses a number of general properties that are discussed in the following section. To illustrate these we shall consider the concrete example of gravitational collapse and calculate the energy of the radiation accompanying it in lowest order in the Planck constant, when in the right-hand side of the expression (84) we can substitute the curvature of the corresponding classical solution.

As the classical source ($T^{\mu\nu}_{\text{source}}$) we shall take a dust sphere with uniform energy density, collapsing from a state of rest at infinity. The corresponding classical solution is a Friedmann metric of zero spatial curvature, joined to a Schwarzschild metric. The joining occurs over a shell of radial timelike geodesics with zero velocities at infinity. The classical solution is characterized by a single parameter—the ADM mass M_0. It can be shown that this coincides with the ADM mass of the self-consistent solution of the equations for the mean field.

The classical solution is singular. Let τ be the proper time on the line $r=0$, reckoned so that $-\infty<\tau<0$ and the singularity is encountered at $\tau=0$. The radial light rays in this geometry can be labeled by the parameter u_-, which is defined by the condition that a ray with given u_- intersect the line $r=0$ at $\tau=u_-$. It is convenient to introduce

$$a=\left(\frac{6u_-}{M_0}\right)^{1/3}. \tag{86}$$

The shell of radial rays with $u_-=-9M_0/2$ forms the event horizon. Light rays emerging on to $\mathscr{I}^+$ have $-\infty<a<-3$. The relationship of u_- to the external retarded time u, normalized by the condition (18), is determined by the equation

$$\frac{da}{du}=\frac{2}{M_0}\frac{(3+a)}{(1+a)^3}, \tag{87}$$

which enables us to verify that Eq. (85) is fulfilled. The limit $a\to-\infty$ corresponds to $u\to-\infty$, while the limit $a\to-3$ corresponds to $u\to+\infty$.

Calculation of the integral in (84) for this model leads to the following result:

$$\frac{d}{du}\int_{\mathscr{I}^-}^{\mathscr{I}^+} drrR=\frac{3\cdot 2^7}{M_0}\frac{(3+a)}{(1-a)^3(1+a)^6}, \tag{88}$$

and the solution of the equation for the Bondi mass with the initial condition

$$M(-\infty)=M_0$$

has the form

$$M(u)=M_0-\frac{1}{4\pi}[w_1(0)+2w_2(0)]\frac{d}{du}\int_{\mathscr{I}^-}^{\mathscr{I}^+} drrR, \tag{89}$$

$$\frac{dM(u)}{du}=-\frac{3\cdot 2^9}{\pi M_0^2}[w_1(0)+2w_2(0)]\frac{(a^2+3a-1)(3+a)}{(1-a)^4(1+a)^{10}}. \tag{90}$$

From this it can be seen that the radiation attenuates as $u\to-\infty$ and $u\to+\infty$. The contribution (84) to the energy flux can be either positive or negative, and the total energy of the radiation from this contribution is equal to zero (see Sec. 8). In the given example the function (90) executes one oscillation. It is positive in the past, negative in the future, and has one zero at $a=-(3+\sqrt{13})/2$, and the integral of it is equal to zero:

$$M(+\infty)=M(-\infty)=M_0 \tag{91}$$

We note also that the contribution linear in the curvature to the radiation energy depends in an essential way on the model of the collapsing source. For example, for collapse of a luminal fluid this contribution is entirely absent, since in this case the Ricci scalar is equal to zero.

8. COMPARISON WITH TRADITIONAL METHODS

The expansion of the effective action in powers of the curvature in (2) is a covariant perturbation-theory form, in which the metric can be written as

$$g_{\mu\nu}=\tilde{g}_{\mu\nu}+h_{\mu\nu}, \tag{92}$$

where $\tilde{g}_{\mu\nu}$ is a flat metric and the perturbation $h_{\mu\nu}$ is assumed to be uniformly small. In comparison with quantum field theory, this approach[1] makes it possible to move away from the loop expansion, but not from the expansion (92).[5)] In (92), however, the perturbation $h_{\mu\nu}$ is still not fully defined and cannot be expressed directly in terms of the curvature, since two metrics on one manifold are defined to within a relative diffeomorphism. Here, two routes are possible. One is to fix only this relative diffeomorphism by a suitable gauge condition and to leave the coordinates of both metrics arbitrary. Then the perturbation can be expressed uniquely in terms of the curvature, and the conditions for integrability are the Bianchi identities.[8] When an invariant functional (e.g., the effective action in field theory) is expanded, the auxiliary metric $\tilde{g}_{\mu\nu}$ is "banished" in each order in the curvature by means of the relation (92), after which the result is invariant and does not depend on the choice of gauge condition. This is covariant perturbation theory.[8,3–5]

The more traditional route is to fix the coordinate system of the metric $\tilde{g}_{\mu\nu}$. (When solving the dynamical equations one must fix the coordinate system of the metric $g_{\mu\nu}$ as well.) In this approach the restoration of a covariant form for the answer requires summation of an infinite series in powers of $h_{\mu\nu}$, since the curvature is represented by such a series. The technical difficulties that arise here already begin to dominate in third order in the curvature.[6)] In addition, with this method of calculation in field theory, ultraviolet divergences are present in all orders in $h_{\mu\nu}$, while in terms of the curvature they do not extend beyond the second order.[9,5] For the coordinates of a flat metric one usually takes the Minkowski coordinates, but this is not always convenient (spherical coordinates, for example, could be more convenient), and, in one case, when we are speaking of light infinity,[10,11] is incorrect. The reason is that on $\mathscr{I}^+$ the Minkowski time t tends to $\pm\infty$ simultaneously with the spatial coordinate. With these provisos, our results should, of course, be obtained in the framework of traditional perturbation theory as well.

As an example we shall consider the following model of the action S_{vac}:

$$S_{\text{vac}}=\frac{a}{2(4\pi)^2}\int d^4x\sqrt{-g}\left[R_{\alpha\beta\gamma\delta}\left(\frac{1}{-\Box}\right)^{\lambda}R^{\alpha\beta\gamma\delta}\right]+O[R^3], \tag{93}$$

where a and λ are constants and $0<\lambda<1$. To within terms $O[R^3]$ the expression (93) is the particular case of the action (2) with

$$\gamma_1(-\Box)=4a(-\Box)^{-\lambda},\quad \gamma_2(-\Box)=-a(-\Box)^{-\lambda}. \tag{94}$$

We have

$$T^{\mu\nu}_{\text{vac}}=\frac{a}{(4\pi)^2}\{2\nabla^{\mu}\nabla^{\nu}[(-\Box_{\text{ret}})^{-\lambda}R]-4\Box[(-\Box_{\text{ret}})^{-\lambda}R^{\mu\nu}]\}+O[R^2], \tag{95}$$

$$(-\Box_{\text{ret}})^{-\lambda}I=\frac{\sin(\pi\lambda)}{\pi}\int_0^{\infty}\frac{dm^2}{m^{2\lambda}}G^{\text{ret}}(m^2)I. \tag{96}$$

The classical source $T^{\mu\nu}_{\text{source}}$ will be taken to be nonstatic and to have compact spatial support and a nonzero trace. We shall show that the solution of the effective equations in this model is asymptotically flat at spatial infinity (i^0), but is not asymptotically flat on light infinity ($\mathscr{I}^+$). This means that the solution is not asymptotically flat,[10] not only because of the different behavior of an asymptotically flat metric at i^0 and on $\mathscr{I}^+$ (nonconservation of the mass on $\mathscr{I}^+$) but also because of the different behavior of the retarded solution of d'Alembert's equation with a nonstatic source. In the limit i^0 ($r\to\infty$ for fixed t) this solution can be represented by the series

$$G^{\text{ret}}(m^2)I(x)=\exp(-mr)\left(\frac{c_1}{r}+\frac{c_2}{r^2}+\frac{c_3(t)}{r^3}+\dots\right),\quad x\to i^0, \tag{97}$$

the first term of which is the static Yukawa potential. (The coefficients c are time-dependent only in the third and higher terms of the expansion.) If the source I is static, the solution behaves in the same way on $\mathscr{I}^+$ as well ($r\to\infty$ for fixed $u=t-r$), but if the source is nonstatic, then, as shown in Sec. 3,

$$G^{\text{ret}}(m^2)I(x)=\exp(-|\text{const}|m\sqrt{r})\left(\frac{c(u)}{r}+\dots\right),\quad x\to\mathscr{I}^+. \tag{98}$$

As a result, from (96) we find

$$(-\Box_{\text{ret}})^{-\lambda}I(x)=r^{-3+2\lambda}\left(\bar{c}_1+\frac{\bar{c}_2}{r}+\frac{\bar{c}_3(t)}{r^2}+\dots\right),\quad x\to i^0, \tag{99}$$

$$(-\Box_{\text{ret}})^{-\lambda}I(x)=r^{-2+\lambda}\left[\bar{c}(u)+O\left(\frac{1}{r}\right)\right],\quad x\to\mathscr{I}^+. \tag{100}$$

We shall seek the solution of the effective equations by perturbation theory, by representing the perturbation of the metric in the form

$$h_{\mu\nu}=h^{cl}_{\mu\nu}+\delta h_{\mu\nu}, \tag{101}$$

where $h^{cl}_{\mu\nu}$ is the perturbation of the classical solution and $\delta h_{\mu\nu}$ is the quantum correction. For $\delta h_{\mu\nu}$ we obtain the equation

$$(L\delta h)_{\mu\nu}=8\pi T^{\text{vac}}_{\mu\nu}, \tag{102}$$

where

$$(L\delta h)_{\mu\nu}=\tfrac{1}{2}(\tilde{\nabla}^{\sigma}\tilde{\nabla}_{\mu}\delta h_{\sigma\nu}+\tilde{\nabla}^{\sigma}\tilde{\nabla}_{\nu}\delta h_{\mu\sigma}-\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\tilde{g}^{\alpha\beta}\delta h_{\alpha\beta}-\tilde{\Box}\delta h_{\mu\nu}-\tilde{g}_{\mu\nu}\tilde{\nabla}^{\alpha}\tilde{\nabla}^{\beta}\delta h_{\alpha\beta}+\tilde{g}_{\mu\nu}\tilde{\Box}\tilde{g}^{\alpha\beta}\delta h_{\alpha\beta}), \tag{103}$$

and the operators with a tilde pertain to the flat metric $\tilde{g}_{\mu\nu}$. For simplicity we shall confine ourselves to the case of spherical symmetry.

We shall consider first the behavior of the solution at spatial infinity. We fix the coordinates of the metric $g_{\mu\nu}$ by the expression

$$g_{\mu\nu}dx^{\mu}dx^{\nu}=g_{tt}dt^2+g_{rr}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{104}$$

and the coordinates of the metric $\tilde{g}_{\mu\nu}$ by the values $g_{tt}=-1$ and $g_{rr}=1$ in (104). In these coordinates the conditions for asymptotic flatness at i^0 have the form ($\mathscr{O}\to 0$ at i^0)

$$h_{tt}=\frac{2M_0}{r}+\frac{\mathscr{O}}{r},\quad h_{rr}=\frac{2M_0}{r}+\frac{\mathscr{O}}{r},\quad h_{rt}\equiv 0, \tag{105}$$

where M_0 is the conserved ADM mass. Since the perturbation of the classical solution has the form (105), the quantum correction $\delta h_{\mu\nu}$ should also have the form (105), with a certain value of δM_0. Substituting $\delta h_{\mu\nu}$ into (103), we obtain

$$(L\delta h)_{AB}=\frac{\mathscr{O}}{r^3},\quad A,B=r,t \tag{106}$$

for all projections onto the Lorentzian subspace r, t. For compatibility of Eqs. (102) the projections $8\pi T^{\text{vac}}_{\mu\nu}$ should also behave in the same way, and it can be proved that this condition is necessary and sufficient for asymptotic flatness of the solution at i^0. From (99), with allowance for the local differential operators in (95), we find

$$8\pi T_{AB}^{\text{vac}}(x)=O(r^{-5+2\lambda}), \quad x\to i^0, \tag{107}$$

which, for $\lambda<1$, satisfies the required condition. More precisely,

$$\delta h_{AB}=O(r^{-3+2\lambda}). \tag{108}$$

The latter implies that not only is the solution asymptotically flat at i^0, but the quantum correction δM_0 to the ADM mass is equal to zero as well. This result also holds with the correct form factors (15), (16), and is an important property of the in-state.[4]

We now consider the behavior of the solution at light infinity. The coordinates of the metric $g_{\mu\nu}$ that cover $\mathscr{I}^+$ (in the asymptotic sense) are determined by the expression[10,11]

$$g_{\mu\nu}dx^\mu dx^\nu=g_{uu}du^2+2g_{ur}dudr+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{109}$$

while the coordinates of the metric $\tilde{g}_{\mu\nu}$ are fixed by the values $g_{uu}=-1$, $g_{ur}=-1$ in (109). The conditions for asymptotic flatness on $\mathscr{I}^+$ have the form[11]

$$h_{uu}=\frac{2M(u)}{r}+O\left(\frac{1}{r^2}\right), \quad h_{ur}=O\left(\frac{1}{r^3}\right), \quad h_{rr}\equiv 0, \tag{110}$$

where $M(u)$ is the Bondi mass. Since $T_{\text{source}}^{\mu\nu}$ has compact spatial support, the perturbation of the classical solution has the form (110) with $M(u)$ replaced by M_0, while the quantum correction $\delta h_{\mu\nu}$ should have the form (110) with a certain function $\delta M(u)$. It will be sufficient to consider the uu component of Eq. (102). Calculating the left-hand side, we find

$$(L\delta h)_{uu}=-\frac{1}{r}\partial_u\delta h_{uu}+O\left(\frac{1}{r^3}\right)$$
$$=-\frac{2}{r^2}\frac{d}{du}\delta M(u)+O\left(\frac{1}{r^3}\right). \tag{111}$$

In the right-hand side of the uu component of the expression (95), the leading term is that with two derivatives with respect to u:

$$8\pi T_{uu}^{\text{vac}}=\frac{a}{\pi}\frac{d^2}{du^2}[(-\Box_{\text{ret}})^{-\lambda}R](1+\mathscr{O}), \tag{112}$$

whence, according to (100),

$$8\pi T_{uu}^{\text{vac}}(x)=r^{-2+\lambda}\frac{a}{\pi}\frac{d^2}{du^2}\bar{c}(u)(1+\mathscr{O}), \quad x\to\mathscr{I}^+. \tag{113}$$

As a result, for $\lambda>0$ Eqs. (102) are incompatible, i.e., the solution is not asymptotically flat on $\mathscr{I}^+$, in full agreement with the principal result of this paper. If instead of (95) we take the general expression (6) for $T_{\text{vac}}^{\mu\nu}$, the uu component of Eq. (102) will have the form

$$-\frac{d}{du}\delta M(u)+O\left(\frac{1}{r}\right)=\frac{r^2}{4\pi}\frac{d^2}{du^2}\{[\gamma_1(-\Box_{\text{ret}})$$
$$+2\gamma_2(-\Box_{\text{ret}})]R\}(1+\mathscr{O}). \tag{114}$$

But this is the equation obtained above for the radiative energy in the spherically symmetric case. For this to be compatible it is necessary that the form factor have the asymptotic form (60).

The situation when the solution is asymptotically flat at i^0 and is not asymptotically flat on $\mathscr{I}$ is exotic in classical theory, in which the perturbation of the metric behaves in a power-law manner both at i^0 and on $\mathscr{I}$, but in quantum theory it is entirely natural. The point is that at i^0 the quantum correction in the self-consistent solution should fall off exponentially,[12,13] as is the case in a wide class of form factors, but on $\mathscr{I}$ the decrease should be a power-law decrease because of the presence of radiation. Here, the selection of the necessary power leads to a stringent restriction on the asymptotic form of the form factors.

As already noted, the behavior of the form factors (15) and (16) coincides with the one-loop behavior in field theory, and, therefore, in this theory our result for the energy of the radiation should be obtainable by direct calculation. Below, we shall perform this calculation, using traditional methods of quantum field theory, and answer a number of questions that arise in connection with the result obtained. In particular, an explanation is needed for the fact that vacuum radiation of first order in the curvature is present at all, since it is clear that the probability of creation of particles by the external field, and the number of particles created, cannot be lower than second-order in the perturbation of the external field (see, e.g., Ref. 14). The answer, as we shall show, is that we are calculating the energy radiated up to a given moment of retarded time u, and it does indeed contain a contribution linear in the curvature, whereas the total energy radiated over all history is proportional to the number of particles created and is of second (and higher) order. This situation is possible because we are concerned not with classical observables but with averages over a quantum state.

Since in the one-loop approximation $T_{\text{vac}}^{\mu\nu}$ is the sum of the contributions of the individual kinds of particles, we shall consider one such contribution—the contribution of spin-zero particles described by a real (Hermitian) scalar field with action

$$S=-\frac{1}{2}\int d^4x\sqrt{-g}(\nabla_\mu\Phi\nabla^\mu\Phi+\xi R\Phi^2), \tag{115}$$

where Φ is a scalar massless field and ξ is a numerical parameter. The energy-momentum tensor of this field has the form

$$T^{\mu\nu}=(1-2\xi)\nabla^\mu\Phi\nabla^\nu\Phi-2\xi\Phi\nabla^\mu\nabla^\nu\Phi+\xi R^{\mu\nu}\Phi^2$$
$$+g^{\mu\nu}\left[\left(-\frac{1}{2}+2\xi\right)(\nabla\Phi)^2+2\xi\Phi\Box\Phi-\frac{1}{2}\xi R\Phi^2\right] \tag{116}$$

and depends on ξ even in flat space. Here, $\xi=0$ corresponds to the canonical energy-momentum tensor, and $\xi=1/6$ to the so-called metric energy-momentum tensor.[15] The advantages of the latter have been noted repeatedly in the literature,[15–21] and are related to the fact that for

$\xi=1/6$ the action (115) possesses conformal invariance, and the trace of $T^{\mu\nu}$ is equal to zero in the equations of motion

$$(\Box-\xi R)\Phi=0 \tag{117}$$

of the field Φ. Conformal invariance is an important property of massless fields, and, therefore, the massless spin-zero field is described by the action (115) with $\xi=1/6$ (Ref. 17), and the values given in (17) for the spectral weights pertain to this case. Below, all the calculations are performed for arbitrary ξ.

Regarding the field Φ as the source of the gravitational field, we shall calculate its energy-flux density across $\mathscr{I}^+$. Near $\mathscr{I}^+$ we use the coordinates $x=(r,u,\theta,\varphi)$ (109). Since in asymptotically flat space,

$$\Phi|_{\mathscr{I}^+}=O\left(\frac{1}{r}\right), \tag{118}$$

we find, in the notation of Eq. (19),

$$\frac{1}{4}T^{\mu\nu}\nabla_\mu v\nabla_\nu v|_{\mathscr{I}^+}=(\partial_u\Phi)^2-\xi\partial^2_{uu}(\Phi^2)+O\left(\frac{1}{r^3}\right). \tag{119}$$

After averaging of the equations of gravitation over the in-vacuum state, the energy balance (19) takes the form

$$-\frac{dM(u)}{du}=\int d^2\mathscr{S}\lim_{r\to\infty}r^2[\langle\text{in vac}|(\partial_u\Phi)^2|\text{in vac}\rangle -\xi\partial^2_{uu}\langle\text{in vac}|\Phi^2|\text{in vac}\rangle], \tag{120}$$

where we have taken into account only the vacuum of the field Φ, and the left-hand side is the expectation value of the corresponding observable. We note that even for a classical field the expression (120) contains, for $\xi\neq0$, a term of nonfixed sign in the form of a total derivative. Here, as a consequence of the condition that the energy be finite, the quantity $r\Phi$, taken on $\mathscr{I}^+$, decreases sufficiently rapidly as $u\to\pm\infty$ for this term not to contribute to the total radiated energy

$$-[M(+\infty)-M(-\infty)] =\int_{-\infty}^{\infty}du\int d^2\mathscr{S}\lim_{r\to\infty}r^2\langle\text{in vac}|(\partial_u\Phi)^2|\text{in vac}\rangle. \tag{121}$$

In the case of the expectation values, however, on account of the first average in (120), as we shall show, additional terms similar to $\partial^2_{uu}\langle\Phi^2\rangle$ appear, and these are now nonzero for any ξ, although they also do not contribute to the total radiated energy. The reason is that the energy density of the classical field is positive, at least for $\xi=0$, whereas the energy density of the vacuum is of nonfixed sign (even on $\mathscr{I}^+$) for any ξ.

For the calculation of the averages in (120) we shall expand the field operator in the basis of the solutions of the wave equation (117). We can then choose a basis of solutions that have positive frequency on $\mathscr{I}^-$:

$$\Phi(x)=\sum_A(f^A_{\text{in}}(x)a^A_{\text{in}}+f^{*A}_{\text{in}}(x)a^{+A}_{\text{in}}) \tag{122}$$

or a basis of solutions that have positive frequency on $\mathscr{I}^+$:

$$\Phi(x)=\sum_A(f^A_{\text{out}}(x)a^A_{\text{out}}+f^{*A}_{\text{out}}(x)a^{+A}_{\text{out}}). \tag{123}$$

Here, A is a set of quantum numbers, f^A_{in} and f^A_{out} are the corresponding basis functions, and a^A_{in} and a^A_{out} are the annihilation operators that define the vacuum of the particles that can be detected on $\mathscr{I}^-$ and $\mathscr{I}^+$, respectively:[21]

$$a^A_{\text{in}}|\text{in vac}\rangle=0,\quad a^A_{\text{out}}|\text{out vac}\rangle=0. \tag{124}$$

The basis functions f_{in} and f_{out} are related by the Bogolyubov transformation[15,21,22]

$$f^A_{\text{out}}(x)=\sum_B[\alpha(A,B)f^B_{\text{in}}(x)+\beta(A,B)f^{*B}_{\text{in}}(x)], \tag{125}$$

in which the matrices α and β are combinations of the particle-creation and particle-scattering amplitudes. In the perturbation-theory calculation, the basis $\tilde{f}^A(x)$ of solutions of the free wave equation (with a flat metric) is also involved, and is positive-frequency both on $\mathscr{I}^-$ and on $\mathscr{I}^+$. We have

$$f^A_{\text{in}}(x)|_{\mathscr{I}^-}=\tilde{f}^A(x)|_{\mathscr{I}^-},\quad f^A_{\text{out}}(x)|_{\mathscr{I}^+}=\tilde{f}^A(x)|_{\mathscr{I}^+}. \tag{126}$$

Henceforth, all quantities with a tilde pertain to the flat metric $\tilde{g}^{\mu\nu}$ in (92).

If for the calculation of (120), where $\Phi(x)$ is taken on $\mathscr{I}^+$, we use the expansion (122), the problem reduces to the determination of the basis functions $f^A_{\text{in}}(x)$ on $\mathscr{I}^+$. But if we use the expansion (123), the basis functions are trivial and the problem reduces to the determination of the coefficients α and β. For our purposes the second route is more convenient. We obtain

$$\begin{aligned}\langle\text{in vac}|[\partial_u\Phi(y)]^2|\text{in vac}\rangle|_{y\to\mathscr{I}^+} &=\sum_A\partial_u\tilde{f}^A(y)\partial_u\tilde{f}^{*A}(y)+2\,\text{Re}\sum_{A,B}\partial_u\tilde{f}^A(y)\partial_u\tilde{f}^B(y)\\ &\quad\times\langle\text{in vac}|a^A_{\text{out}}a^B_{\text{out}}|\text{in vac}\rangle\\ &\quad+2\sum_{A,B}\partial_u\tilde{f}^{*A}(y)\partial_u\tilde{f}^B(y)\\ &\quad\times\langle\text{in vac}|a^{+A}_{\text{out}}a^B_{\text{out}}|\text{in vac}\rangle\end{aligned} \tag{127}$$

and an analogous expression for $\langle\Phi^2\rangle$. Then

$$\langle\text{in vac}|a^A_{\text{out}}a^B_{\text{out}}|\text{in vac}\rangle=-\sum_C\alpha^*(A,C)\beta^*(B,C), \tag{128}$$

$$\langle\text{in vac}|a^{+A}_{\text{out}}a^B_{\text{out}}|\text{in vac}\rangle=\sum_C\beta(A,C)\beta^*(B,C). \tag{129}$$

In the expansion in the perturbation of the external field in the expression (127), there is already a term of zeroth power in the curvature. This is the energy of vacuum fluctuations of the field in flat space—the first term in (127). It gives rise to a nonzero $T^{\mu\nu}_{\rm vac}$ even on $\mathscr{I}^-$, and so should be subtracted. Otherwise it would turn out that we would be considering a problem in which there is an incident flux of energy across $\mathscr{I}^-$, so that the presence of the emergent flux across $\mathscr{I}^+$ would not be due just to the creation of particles from the vacuum. It is well known that in the theory of massless fields there is an internal mechanism of cancellation of terms of zeroth power in the curvature in $T^{\mu\nu}_{\rm vac}$.[7)] In each formalism (in the method of functional integration,[23,24] in the Green-function formalism,[2] and in the canonical formalism[24,25]), it can be shown that by dealing with the divergences in a certain way, these terms cancel. In dimensional regularization[26,27] this cancellation occurs trivially. In the present context, to cancel the first term in (127) it is sufficient to separate the angular arguments of the operators Φ on $\mathscr{I}^+$ and to regard the operation of bringing them together as the last operation. Henceforth, this term will be discarded. We note that after it has been discarded we lose the formal positive definiteness of the expression (127).

The remaining two terms in (127) depend on the external field. From (125) it can be seen that upon expansion in the curvature we have

$$\beta=O[R_{..}],\quad \alpha=\mathbb{1}+O[R_{..}]. \tag{130}$$

Therefore, the third term in (127), which is quadratic in β, is $O[R^2_{..}]$, but the second term contains a contribution linear in the curvature, since

$$\langle \text{in vac}|a^A_{\rm out}a^B_{\rm out}|\text{in vac}\rangle=-\beta^*(B,A)+O[R^2_{..}]. \tag{131}$$

It is this which gives rise to the presence of vacuum radiation of first order in the curvature. The local terms in $T^{\mu\nu}_{\rm vac}$ that are linear in the curvature either fall off on $\mathscr{I}^+$ or renormalize the gravitational constant, but the matrix β is nonlocal in the external field. Therefore, the phenomenon of first-order radiation remains after the renormalization as well.[8)]

We now show that the entire second term in (127), and hence the part of it linear in the curvature, gives no contribution to the total radiated energy. For this we shall write an explicit expression for the basis function $\tilde{f}^A(x)$ on $\mathscr{I}^+$, taking it, as usual,[22] in the form of a positive-frequency outgoing spherical wave:

$$\tilde{f}^A(x)\Big|_{\mathscr{I}^+}=\frac{1}{r}\frac{1}{\sqrt{4\pi\varepsilon}}e^{-i\varepsilon u}Y_{lm}(\theta,\varphi)+O\left(\frac{1}{r^2}\right), \tag{132}$$

where

$$x=(r,u,\theta,\varphi),\quad A=(\varepsilon,l,m) \tag{133}$$

Y is a spherical harmonic, and $r\to\infty$ for fixed u, θ, and φ. Adapting in an obvious manner the notation in (127), we find

$$\int d^2\mathscr{S}\lim_{r\to\infty}r^2\langle \text{in vac}|(\partial_u\Phi)^2|\text{in vac}\rangle=$$

$$=-\frac{1}{2\pi}\operatorname{Re}\int_0^\infty d\varepsilon\int_0^\infty d\varepsilon'\sqrt{\varepsilon\varepsilon'}$$
$$\times\exp[-i(\varepsilon+\varepsilon')u]$$
$$\times\sum_{l,m}(-1)^{|m|}\langle \text{in vac}|a^{lm}_{\rm out}(\varepsilon)a^{l(-m)}_{\rm out}$$
$$\times(\varepsilon')|\text{in vac}\rangle$$
$$+\frac{1}{2\pi}\int_0^\infty d\varepsilon\int_0^\infty d\varepsilon'\sqrt{\varepsilon\varepsilon'}\exp^{[i(\varepsilon-\varepsilon')u]}$$
$$\times\sum_{l,m}\langle \text{in vac}|a^{+lm}_{\rm out}(\varepsilon)a^{lm}_{\rm out}(\varepsilon')|\text{in vac}\rangle. \tag{134}$$

Here, the term with $\langle a^+a\rangle$ is manifestly positive, since it can be written in the form of the norm

$$\frac{1}{2\pi}\sum_{l,m}\left\|\int_0^\infty d\varepsilon\sqrt{\varepsilon}e^{-i\varepsilon u}a^{lm}_{\rm out}(\varepsilon)|\text{in vac}\right\|, \tag{135}$$

and this positivity is no longer formal but real, since the matrix β in (129) and contractions with this matrix do not contain ultraviolet divergences.[21] The term with $\langle aa\rangle$ does not have a fixed sign, and upon integration over u from $-\infty$ to ∞ the delta function $\delta(\varepsilon+\varepsilon')$ appears in it and makes it vanish. As a result, from (121) and (134) we obtain

$$-[M(+\infty)-M(-\infty)]$$
$$=\int_0^\infty d\varepsilon\varepsilon\sum_{l,m}\langle \text{in vac}|a^{+lm}_{\rm out}(\varepsilon)a^{lm}_{\rm out}(\varepsilon)|\text{in vac}\rangle. \tag{136}$$

The average that appears here is the number of out-particles with the given quantum numbers in the in-vacuum, and the inequality (136) states that the total radiated energy is equal to the total energy of the particles created. According to (129) and (130), this quantity is $O[R^2_{..}]$.

Returning to the energy radiated up to a finite time, we shall calculate its component linear in the curvature by perturbation theory. For this we introduce the S-matrix

$$S=T\exp^{(iS_{int})}, \tag{137}$$

$$S_{int}=S-S|_{g=\tilde{g}}$$
$$=\frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\Big[h_{\alpha\beta}\Big(\tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}-\frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{g}^{\mu\nu}\Big)\nabla_\mu\Phi\nabla_\nu\Phi$$
$$-\xi R\Phi^2\Big]+O[R^2_{..}], \tag{138}$$

where T is the time-ordering operator and, to the given accuracy, we can replace $\tilde{g}^{\mu\nu}$ by the complete metric $g^{\mu\nu}$. Using the relation

$$a^A_{\rm out}=\mathbf{S}^{-1}a^A_{\rm in}\mathbf{S}, \tag{139}$$

we find, by direct calculation,

$$\langle \text{in vac}|a^{+A}_{\rm out}a^B_{\rm out}|\text{in vac}\rangle=O[R^2_{..}], \tag{140}$$

$$\left\langle \text{in vac} \left| a^A_{\text{out}} a^B_{\text{out}} \right| \text{in vac} \right\rangle = i \int d^4x \sqrt{-g} \left[h_{\alpha\beta} \left(\tilde{g}^{\alpha\mu} \tilde{g}^{\beta\nu} - \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} \right) \nabla_\mu \tilde{f}^{*A} \nabla_\nu \tilde{f}^{*B} - \xi R \tilde{f}^{*A} \tilde{f}^{*B} \right] + O[R^2_{..}] . \tag{141}$$

The matrix in (131) has thereby been calculated to order $O[R^2_{..}]$. For the average (127) we obtain the following result:

$$\langle \text{in vac} | [\partial_u \Phi(y)]^2 | \text{in vac} \rangle |_{y \to \mathscr{I}^+} = 2 \,\text{Im} \int d^4x \sqrt{-g} \left\{ h_{\alpha\beta}(x) \left(\tilde{g}^{\alpha\mu} \tilde{g}^{\beta\nu} - \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} \right) \times [\partial^y_u \nabla^x_\mu \tilde{G}^+(x,y)] [\partial^y_u \nabla^x_\nu \tilde{G}^+(x,y)] - \xi R(x) \times [\partial^y_u \tilde{G}^+(x,y)] [\partial^y_u \tilde{G}^+(x,y)] \right\} + O[R^2_{..}], \tag{142}$$

where

$$\tilde{G}^+(x,y) = i \sum_A \tilde{f}^{*A}(x) \tilde{f}^A(y) = i \langle \text{vac} | \Phi(y) \Phi(x) | \text{vac} \rangle |_{g = \tilde{g}} \tag{143}$$

is a positive-frequency function of the free wave equation.[29] We obtain an analogous expression for $\langle \Phi^2 \rangle$. The imaginary part of the product of two functions $\tilde{G}^+$ contains the real part of the function $\tilde{G}^+$. Therefore, in the expression (142) the point x lies on the light cone of the point y. Since the point y is on $\mathscr{I}^+$, in the expression (142) x always lies in the past of y, and the function $\tilde{G}^+$ is then equal to the Feynman Green function $\tilde{G}^F$. Therefore, in the expression (142) we can replace $\tilde{G}^+$ by $\tilde{G}^F$.

The function $\tilde{G}^F$ depends only on the world function:

$$\tilde{G}^F(x,y) = \tilde{G}^F[\tilde{\sigma}(x,y)], \tag{144}$$

and, as a generalized function, satisfies the equation

$$\tilde{\sigma} \frac{\partial}{\partial \tilde{\sigma}} \tilde{G}^F = -\tilde{G}^F \tag{145}$$

[see (162) below]. When y tends to $\mathscr{I}^+$, while x remains in the compact region, we have

$$\tilde{G}^F(x,y) |_{y \to \mathscr{I}^+} = O(r^{-1}(y)), \tag{146}$$

$$\tilde{\sigma}(x,y) |_{y \to \mathscr{I}^+} = O(r(y)), \tag{147}$$

$$\partial^y_u \tilde{\sigma}(x,y) |_{y \to \mathscr{I}^+} = -r(y) + O(1), \tag{148}$$

whence

$$\partial^y_u \nabla^x_\mu \tilde{\sigma}(x,y) |_{y \to \mathscr{I}^+} = O(1), \tag{149}$$

$$\partial^y_u \partial^y_u \tilde{\sigma}(x,y) |_{y \to \mathscr{I}^+} = O(1). \tag{150}$$

We also take into account that

$$\tilde{\nabla}^x_\mu \tilde{\nabla}^x_\nu \tilde{\sigma}(x,y) \equiv \tilde{g}_{\mu\nu}(x). \tag{151}$$

Using (144)–(151), it is not difficult to convince oneself that the following relations are valid:

$$\nabla^x_\mu \tilde{G}^F(x,y) \nabla^x_\nu \tilde{G}^F(x,y) = \frac{1}{6} \tilde{\nabla}^x_\mu \tilde{\nabla}^x_\nu [\tilde{G}^F(x,y) \tilde{G}^F(x,y)] + O(r^{-3}(y)), \tag{152}$$

$$\partial^y_u \tilde{G}^F(x,y) \partial^y_u \tilde{G}^F(x,y) = \frac{1}{6} \partial^y_u \partial^y_u (\tilde{G}^F(x,y) \tilde{G}^F(x,y)) + O(r^{-3}(y)), \tag{153}$$

$$\partial^y_u \nabla^x_\mu \tilde{G}^F(x,y) \partial^y_u \nabla^x_\nu \left(\tilde{G}^F(x,y) \right) = \frac{1}{30} \partial^y_u \partial^y_u \tilde{\nabla}^x_\mu \tilde{\nabla}^x_\nu [\tilde{G}^F(x,y) \tilde{G}^F(x,y)] + O(r^{-3}(y)), \tag{154}$$

$$\tilde{g}^{\mu\nu} \nabla^x_\mu \tilde{G}^F(x,y) \nabla^x_\nu \tilde{G}^F(x,y) = O[r^{-3}(y)]. \tag{155}$$

The relation (154) makes it possible to "flip" (via integration by parts) the derivatives $\nabla^x_\mu \nabla^x_\nu$ in the expression (142) to the perturbation h. The resulting contraction

$$\left(\tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} \right) \tilde{\nabla}_\mu \tilde{\nabla}_\nu h_{\alpha\beta} \tag{156}$$

must be compared with the following expansion of the Ricci scalar:

$$R = (\tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} - \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta}) \tilde{\nabla}_\mu \tilde{\nabla}_\nu h_{\alpha\beta} + O[h^2_{..}]. \tag{157}$$

The coefficients of the trace terms are not equal, but these terms can be omitted altogether, since, according to (155), they do not contribute to the leading term of the asymptotic form on $\mathscr{I}^+$.

As a result, the averages of interest take the form

$$\langle \text{in vac} | [\partial_u \Phi(y)]^2 | \text{in vac} \rangle |_{y \to \mathscr{I}^+} = \left(\frac{1}{15} - \frac{1}{3} \xi \right) \partial^y_u \partial^y_u \int d^4x \sqrt{-g} R(x) \times \text{Im} [\tilde{G}^F(x,y) \tilde{G}^F(x,y)] + O(r^{-3}(y)) + O[R^2_{..}], \tag{158}$$

$$\langle \text{in vac} | \Phi^2(y) | \text{in vac} \rangle |_{y \to \mathscr{I}^+} = \left(\frac{1}{3} - 2\xi \right) \times \int d^4x \sqrt{-g} R(x) \text{Im} [\tilde{G}^F(x,y) \tilde{G}^F(x,y)] + O(r^{-3}(y)) + O[R^2_{..}]. \tag{159}$$

It remains to calculate the loop in (158) and (159). For this we make use of the following integral representation for the Feynman propagator:

$$\tilde{G}^F(x,y) = \frac{1}{8\pi^2} \int_0^\infty dt \exp[i(\tilde{\sigma} + i0)t], \tag{160}$$

$$\tilde{G}^F(x,y) \tilde{G}^F(x,y) = \frac{1}{(8\pi^2)^2} \int_0^\infty dt \int_0^\infty ds \exp[i(\tilde{\sigma} + i0) \times (t+s)]$$

$$=\frac{1}{(8\pi^2)^2}\int_0^\infty d\lambda\lambda\,\exp[i(\tilde{\sigma}+i0)\lambda]$$

$$=\frac{1}{8\pi^2 i}\frac{\partial}{\partial\tilde{\sigma}}\tilde{G}^F(x,y). \quad (161)$$

Since

$$\tilde{G}^F(x,y)=\frac{1}{8\pi}\delta(\tilde{\sigma})+\frac{i}{8\pi^2}\frac{1}{\tilde{\sigma}}, \quad (162)$$

we find

$$\mathrm{Im}[\tilde{G}^F(x,y)\tilde{G}^F(x,y)]=-\frac{1}{64\pi^3}\delta'(\tilde{\sigma}). \quad (163)$$

We finally obtain

$$\langle\text{in vac}|[\partial_u\Phi(y)]^2|\text{in vac}\rangle|_{y\to\mathscr{I}^+}=\frac{1}{32\pi^3}\left(\frac{1}{6}\xi-\frac{1}{30}\right)$$

$$\times\partial_u^y\partial_u^y\int d^4x\sqrt{-g}R(x)\delta'(\tilde{\sigma}(x,y))$$

$$+O[r^{-3}(y)]+O[R_{..}^2], \quad (164)$$

$$\langle\text{in vac}|\Phi^2(y)|\text{in vac}\rangle|_{y\to\mathscr{I}^+}=\frac{1}{32\pi^3}\left(\xi-\frac{1}{6}\right)$$

$$\times\int d^4x\sqrt{-g}R(x)\delta'[\tilde{\sigma}(x,y)]+O[r^{-3}(y)]$$

$$+O[R_{..}^2]. \quad (165)$$

Substitution of these expressions into (120) gives the desired result:

$$-\frac{dM(u)}{du}=-\frac{1}{32\pi^3}\left(\frac{1}{30}-\frac{1}{3}\xi+\xi^2\right)\partial_u^y\partial_u^y$$

$$\times\int d^2\mathscr{S}(y)\lim_{r(y)\to\infty}r^2(y)$$

$$\times\int d^4x\sqrt{-g}R(x)\delta'(\sigma(x,y))$$

$$+O[R_{..}^2], \quad (166)$$

which it is necessary to compare with the result (61) obtained above. The spectral weights $w_1(0)$ and $w_2(0)$ for the contribution of the scalar field to the effective action (2) are known for all values of ξ (Refs. 2, 3, 21, 28):

$$w_1(0)=\frac{1}{60},\quad w_2(0)=-\frac{1}{180}+\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2. \quad (167)$$

As a result,

$$w_1(0)+2w_2(0)=\frac{1}{30}-\frac{1}{3}\xi+\xi^2, \quad (168)$$

and, since in the derivation of (166) the support of the perturbation was assumed for simplicity to be compact, we obtain complete agreement of the results. We note that the quadratic polynomial (168) is positive-definite, so that the coefficient in (166) is nonzero for any ξ.

If the Hilbert spaces constructed on the in- and out-vacua are unitarily equivalent, i.e., $M(u)$ does not go to infinity, the entire first-order contribution to (166) is in the form of oscillations of alternating sign, which sum to zero over the entire history. This result can serve as an illustration of the arbitrary character of the separation of virtual vacuum effects from real ones (see, e.g., Ref. 15). It is clear that departure of the matrix β from zero leads both to the creation of real particles and to nonlocal polarization of the vacuum. The only fact that may be unexpected is that the energy density of the vacuum turns out to be of nonfixed sign not only in the inner regions of the space but also in the asymptotically flat region in which real particles are observed. In this region observations of two types are possible. First, it is possible to set up a particle detector that will certainly detect only real particles with positive energy. Second, it is possible to measure a classical observable (in the present case, the free-fall acceleration). With sufficiently high precision, its increase can actually be observed. However, there is no contradiction between these two types of measurement, since the dispersion of the classical observable should show that radiation of negative energy is statistically uncertain. In the example in Sec. 7 the quantity dM/du is suppressed by the factor (m_p/M_0), where m_P is the Planck mass. The dispersion of this quantity has the same order of magnitude, but exceeds it numerically by a factor of $\sqrt{2}$.

The author is grateful to G. A. Vilkovisky for scientific guidance.

The work was performed with the financial support of the Russian Fund for Fundamental Research (Grant No. 93-02-15594).

[1)]In the paper we adopt a Riemann tensor $R^\mu_{\alpha\nu\beta}=\partial_\nu\Gamma^\mu_{\alpha\beta}-\ldots$, $R_{\alpha\beta}=R^\mu_{\alpha\mu\beta}$, $R=g^{\alpha\beta}R_{\alpha\beta}$, and the metric signature $(-,+,+,+)$.

[2)]For the expectation values there does not exist an action functional in the usual sense,[6,7] but the equations for the expectation values in the in-vacuum state can be obtained by means of a definite procedure from the Euclidean action.[8] We write the Lorentzian action by changing the common sign in front of the Euclidean action and formally applying this procedure.

[3)]A classical source $T^{\mu\nu}_{\text{source}}$ may not contribute to the energy flux across $\mathscr{I}^+$ and may not generate gravitational waves, and the information functions generated by $T^{\mu\nu}_{\text{vac}}$ are equal to zero in the $O[R_{..}^2]$ approximation.

[4)]Because of the presence of vacuum radiation, the solution of the equations for the mean field does not have a light-like horizon (event horizon), but if the mass of the collapsing source is much greater than the Planck mass the visibility horizon has a part that is almost light-like.[12,13]

[5)]Usually, by perturbation theory one means the simultaneous application of these two expansions.

[6)]The one-loop effective action in third order in the curvature has been calculated only recently by means of covariant perturbation theory.[5]

[7)]In the case of massive fields this mechanism is not sufficient, and a cosmological term appears in the equations for the expectation values.[9,23]

[8)]The calculations performed below are in agreement with dimensional regularization, in which there is no renormalization of the gravitational constant.

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Translated by P. J. Shepherd