Onset of nondissipative shock waves and the "nonperturbative" quantum theory of gravitation

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We consider the special Gurevich–Pitaevskii solution of the Korteweg–de Vries equation that describes the onset of nondissipative shock waves when "wave fronts" steepen in low-dispersion media. We show that this solution is simultaneously the exact solution of a fourth-order ordinary differential equation. At $t=0$, the Gurevich–Pitaevskii solution is the same as a known nonlinear special function that arises in a certain problem in the quantum theory of gravitation.

1. In this paper, we consider the special Gurevich–Pitaevskii (GP) solution of the Korteweg–de Vries (KdV) equation

$$v_t+v_3+vv_1=0 \tag{1}$$

(subscripts denote the order of derivatives with respect to x), which describes[1,2] the onset of nondissipative shock waves when "wave fronts" steepen in low-dispersion media.

We continue the work begun in Ref. 3 by showing that besides Eq. (1), the special GP solution satisfies a fourth-order ordinary differential equation (ODE),

$$v_4+5vv_2/3+5(v_1)^2/6+5[v^3+x-tv]/18=0. \tag{2}$$

The relationship between the onset of nondissipative shocks and problems in the quantum theory of gravitation is surprising, in our opinion: at $t=0$, the GP solution is identical with the well known[4–8] special solution $V^s(x)$ of the equation

$$v_4+5vv_2/3+5(v_1)^2/6+5[v^3+x]/18=0, \tag{3}$$

which comes up in calculations of the "nonperturbative string effect" in two-dimensional quantum gravity (the ODE (3) applies to the number of so-called massive string equations).

In this paper, we also discuss various analytic properties of the GP solution.

2. The physically interesting solution $V^s(x)$ of Eq. (3) satisfies the boundary conditions[4]

$$v\to -x^{1/3} \quad \text{for } x\to\pm\infty. \tag{4}$$

Based on numerical calculations, it was concluded in Ref. 4 that the limiting behavior (4) guarantees that Eq. (3) will have a unique smooth, real solution. This uniqueness was later confirmed[5] by an explicit calculation of the so-called monodromy data[5–8] that parametrize the various solutions of (3). We show below that in fact the initial data

$$v(t,x)\,|_{t=0}=V^s(x) \tag{5}$$

prescribe a globally smooth solution of the KdV equation that satisfies both the ODE (2) and the boundary conditions (4) for all t. [We emphasize that in doing so, we build upon the properties of $V^s(x)$ described in Refs. 4–6. Specifically, we make use of the aforementioned smoothness[4] of $V^s(x)$ at all x.]

According to Ref. 1, the asymptotic form (4) is *exactly* the behavior to be expected of the GP solution.

3. In a previous paper[3] devoted to the special GP solution, I suggested that apart from satisfying from (1), it also satisfies a seventh-order ODE:

$$\begin{aligned} &v_7+7[vv_5+3v_1v_4+5v_2v_3]/3+35[v_3v^2+(v_1)^3\\ &\quad+4v_2v_1v]/18+35v_1v^3/54\\ &\quad+5[xv_1-3t(v_3+vv_1)+2v]/54=0. \end{aligned} \tag{6}$$

One can check directly that solutions of (2) also satisfy (6).

It was shown in Ref. 3 that the joint solution of (1) and (6) satisfies the monodromy theorem.[9] (This class of solutions was considered somewhat earlier[10,11] in implicit form.) Thus, in addition to the equations of the inverse-scattering method[12]

$$\Psi_x=\left\{-i\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\begin{pmatrix}0 & v/6\\ -1 & 0\end{pmatrix}\right\}\Psi, \tag{7}$$

$$\begin{aligned}\Psi_t=\Bigg\{&(-4i\lambda^3+iv\lambda/3-v_1/6)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &+4\lambda^2\begin{pmatrix}0 & v/6\\ -1 & 0\end{pmatrix}+i\lambda(v_1/3)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\\ &+\begin{pmatrix}0 & -v_2/6-v^2/18\\ v/3 & 0\end{pmatrix}\Bigg\}\Psi,\end{aligned} \tag{8}$$

we also have a linear system for the corresponding functions,

$$\Psi_\lambda=A(x,t,\lambda)\Psi. \tag{9}$$

When (2) holds, the matrix $A(x,t,\lambda)$ is simply a polynomial:

$$\begin{aligned}A=(1/5)\Big\{&[34561\lambda^6-2881\lambda^4v+144\lambda^3v_1\\ &+1\lambda^2(-60t/9+72(v_2+v^2/2))-36\lambda(v_3+vv_1)]\end{aligned}$$

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$$\times\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+[-576\lambda^5 v-288i\lambda^4 v_1+144\lambda^3(v_2$$

$$+v^2/3)+72i\lambda^2(v_3+vv_1)+\lambda(10x+12vv_2$$

$$-6(v_1)^2+4v^3)+5i]\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+[3456\lambda^5$$

$$-288\lambda^3 v-\lambda(72(v_2+v^2/2)-60t)]\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\Bigg\}. \quad (10)$$

For any sector $S_j=\{\lambda:\pi(j-1)/7<\arg\lambda<\pi(j+1)/7$ $(j = 1.14)\}$, there exists (see Ref. 13, § 12) a fundamental solution, Ψ_j of (9) and (10) with the following asymptotic behavior in that sector as $\lambda\to\infty$:

$$\Psi_j\sim\exp\Bigg\{[-i(\lambda x+4t\lambda^3-3456\lambda^7/35)+2^{-1}\ln\lambda]$$

$$\times\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Bigg\}. \quad (11)$$

The so-called Stokes phenomenon (Ref. 13, § 15), whereby

$$M_j(x,t)=\Psi_j^{-1}\Psi_{j+1} \quad (12)$$

are not unit matrices, comes into play in the fundamental solutions of Eq. (9).

A direct check shows that when the ODE (2) holds, (7) will be compatible with (9) and (10). Clearly, the corresponding Stokes matrices (12) will then be independent of x. Conversely, the x-independence of these matrices implies[9] the validity of (7), and it also means that the potential v satisfies (2). Likewise, the subscripts on the v_j in (10) denote the order of derivatives of v with respect to x.

It can also easily be shown that compatibility of (8) and (9) gives rise to four ODE's in the variable t,

$$v_t=-(v_3+vv_1),$$

$$(v_1)_t=2vv_2/3-(v_1)^2/6+5[x-tv+v^3]/18,$$

$$(v_2)_t=2vv_3/3+v_1v_2/3+5[1-tv_1+3v_1v^2]/18, \quad (13)$$

$$(v_3)_t=v_3v_1+(v_2)^2/3-5v^2v_2/18+10v(v_1)^2/9-5v^4/27$$

$$-5xv/27-5tv_2/18+5tv^2/27.$$

These equations imply that the M_j are independent of t.

We thus find that the solution v of the Cauchy problem (1), (5) is in fact simultaneously a solution of the ODE (2) at all t (for which this Cauchy problem is solvable).

4. In the limit as $\lambda\to\infty$ [see (11)], the leading term $(\Psi_j)^2$ in the sectors S_j (as noted in Ref. 3) is the same as the exponential in the Fourier integral $J=\int_R\lambda\exp\{-2i(x\lambda+4t\lambda^3-3456\lambda^7/35)\}d\lambda$, which satisfies the linear part of the KdV equation and the linear part of the ODE (6). I therefore suggested in Ref. 3 that the GP solution might be considered an analog of J. Note that this is also a reasonable assumption from the vantage point of Ref. 1 (where Gurevich and Pitaevskii cast the solution of the linearized kinetic equations of a tenuous plasma as a linear combination of integrals like J).

This "consistency" is typical[3,8,14–16] of nonlinear isomonodromy analogs of the special functions of wave catastrophes (Ref. 17, Ch. VI, §4), which are Fourier integrals in canonical form, and which play a fundamental role in studies of rapidly oscillating solutions of linear problems.[17–19]

Note that it is just this association between the GP solution and the integral J that plays such an important role in deriving Eq. (6). Completely in keeping with the plan described in Ref. 16, the derivation was carried out by

a) finding the linear ODE $J_7+5[xJ_1\ -3tJ_3+2J]/54=0$ satisfied by the Fourier integral J, and

b) replacing all derivatives in that equation by their nonlinear generalizations—the stationary parts of the commutative symmetries of the KdV equation[20] of corresponding order.

Our study of the ODE (6) reduced to an investigation of the special solution (2) on the basis of an extremely lengthy analysis (using the isomonodromy method described in Ref. 3) of the behavior of the solutions Φ as $\lambda\to 0$. Here we choose to forgo more detailed discussion.

5. Outside the zone of rapid oscillations, the full asymptotic expansion of the GP solution as $|t|\to\infty$ takes the form[3] $(s=x/|t|^{3/2})$

$$v=|t|^{1/2}\left[f(s)+\sum_{j=1}^{\infty}\left|t\right|^{-7j/2}g_j(s)\right] \quad (|t|\gg 1). \quad (14)$$

The leading term $u(t,x)=|t|^{1/2}f(s)$ is a solution of the equation

$$x-tu+u^3=0. \quad (15)$$

The discussion in Ref. 3 actually also makes it possible to write out the full asymptotic GP solution for $|x|\to\infty$. Outside the oscillation zone (at $t=0$, in particular), it is given by the series $(r=t/x^{2/3})$

$$v(t,x)=x^{1/3}\left[g(r)+\sum_{j=1}^{\infty}x^{-7j/3}p_j(r)\right], \quad (16)$$

where $g(r)$ is the one real root of the equation

$$1-rg+g^3=0, \quad (17)$$

and the remaining series coefficients $p_j(r)$ are specific smooth functions. At $t<0$, this series is identical to the formal series obtained from (14) by substituting $t=x^{2/3}r$ and $s=r^{-3/2}$.

6. Equation (16) for the asymptotic GP solution shows that it is a member of the class of infinitely rising solutions of the KdV equation previously considered by Bondareva and Shubin.[21–23] [This can easily be confirmed by expanding the coefficients in (16) in a Taylor series as $r\to 0$, replacing r by $t/x^{2/3}$, and grouping equal powers of x.] Bondareva and Shubin have thus shown that it is only possible to guarantee that the solution of the Cauchy problem for the KdV equation (belonging to the class of rising functions considered in Refs. 21–23) is globally smooth and unique if the initial data are given on the line $t=0$.

We can easily verify the existence of formal solutions of the ODE (3) that take the form

$$v_{\pm}(x)=-x^{1/3}\left[1+\sum_{j=1}^{\infty} x^{-7j/3}p_j^{\pm}\right] \tag{18}$$

(the $p_j^{\pm}$ are constants). The principal result obtained by Kuznetsov[24] was that solutions like (3) with the asymptotic expansion (18) will exist at sufficiently large $|x|$ (although they may not be unique). On the other hand, Kapaev[6] has shown that all solutions $V_{\pm}$ of the ODE (3) with asymptotic behavior (4) (that are defined at large enough $|x|$) will depend on two parameters that affect only the exponentially small corrections to the power-law background. Thus, the series (18) yields the full asymptotic expansion of the smooth solution $V^s(x)$ of Eq. (3) at $|x|\to\infty$.

In its turn, this circumstances allow, according to Refs. 21–23, guarantee the global (and unique) solution of the Cauchy problem (1), (5). For any known t and the large enough $|x|$, this solution will have a necessary asymptotic expansion (16).

In the two concluding sections, we discuss problems that bear on the behavior of the GP solution in the fast oscillatory zone and on the lines $x=0$ and $t=0$.

7. In conventional hydrodynamics, the steepening of a wave is immediately followed by the formation of a shock whose intensity increases with time. Sagdeev showed in Ref. 25 that collisionless shocks tend to be oscillatory. It was therefore immediately clear that the GP solution of Eq. (1) describing the onset of such a wave should exhibit rapid oscillations after some sufficiently long time. To investigate the character of these oscillations quantitatively, Gurevich and Pitaevskii[2] suggested using the solutions of Whitham's equations, which were obtained by averaging over a period of the cnoidal wave:

$$V_0=2a\,\mathrm{dn}^2\{(a/6)^{1/2}\xi,\kappa)\}+\gamma, \tag{19}$$

where $\xi=x-\varphi t$, $\kappa^2=[(r_2-r_1)/(r_3-r_1)]$ is the squared modulus of the Jacobi elliptical function dn, $a=r_3-r_1$, $\gamma=r_1+r_2-r_3$, $\varphi=(r_1+r_2+r_3)/3$, and $r_1\leqslant r_2\leqslant r_3$.

Whitham's resulting equations[26]

$$(r_j)_t+P_j(r_j)_x=0 \quad (j=1,2,3) \tag{20}$$

are governed by the "group velocities"

$$P_1=[r_1+r_2+r_3+2(r_1-r_2)/(1-\mu)]/3,$$

$$P_2=[r_1+r_2+r_3-2(r_1-r_2)(1-\kappa^2)/(1-\mu-\kappa^2)]/3,$$

$$P_3=[r_1+r_2+r_3+2(r_3-r_2)/\mu]/3,$$

where $\mu=E/K$ is the ratio of the complete elliptic integrals $E=E(\kappa^2)$ and $K=K(\kappa^2)$ of the first and second kind.

The self-similar substitution suggested in Ref. 2,

$$r_j=t^{1/2}l_j(s), \quad (s=x/t^{3/2}), \tag{21}$$

transforms (20) into a system of ordinary differential equations. Using the generalized hodograph method devised by Tsarev[27] and the algebraic geometry procedure proposed by Krichiver,[28] Potemin[29] found the required self-similar solutions (21). These solutions can be obtained from the implicit equations

$$x-tP_j-\omega_j=0, \quad (j=1,2,3), \tag{22}$$

where $\omega_j=[U+(3P_j-A)(U)'_{r_j}]/35$, $U=5A^3-12AB+8C$, $A=r_1+r_2+r_3$, $B=r_1r_2+r_3r_1+r_2r_3$, and $C=r_1r_2r_3$.

In our previous brief note,[3] we pointed out that the results obtained by Kudashev and Sharapov[30] enable one to verify that the solution of Whitham's equation obtained by Potemin is an averaged corollary of the ODE (6). We now expand on this point.

In addition to satisfying (1), the GP solution satisfies the ODE (6), whose left-hand side $K_7(v)+5\tau(x,t,v)/54$ combines the seventh-order commutative symmetry of $K_7(v)$ with the dilation symmetry $\tau=2v+xv_x+3tv_t$ of the KdV equation.[20] The results found in Ref. 30 immediately yield the averaged corollary of (6) for $j=1,2,3$:

$$[(r_j)_t+R_j\cdot(r_j)_x]+5[2r_j+x(r_j)_x+3t(r_j)_t]/54=0, \tag{23}$$

where

$$R_1=[U+2(r_1-r_2)(U)'_{r_1}(1-\mu)^{-1}],$$

$$R_2=[U-2(r_1-r_2)(1-\kappa^2)(U)'_{r_2}(1-\mu-\kappa^2)^{-1}],$$

$$R_3=[U+2(r_3-r_2)(U)'_{r_3}\mu^{-1}].$$

Making the self-similar substitution (21) in (20) and (23) and eliminating the derivatives $(l_j)'_s$, we immediately obtain the explicit solution of (22).

The results obtained by Kudashev and Sharapov[30] thus lead us to believe, in particular, that the solutions (22) of Whitham's equations (20) confirm Dubrovin's suggestion[31] that "...the strong integrability and self-similarity of systems obtained from averaging in soliton theory should result from 'averaging' over the symmetry groups (Galilean and scale transformations) of the original equations."

8. We pointed out in Ref. 3 that the line $x=0$ has special significance in the GP solution considered here (the smoothness requirement on that line uniquely determines the coefficients in (14)). Its distinctiveness is actually a rather deep property. That straight line creates the so-called *Maxwell-set* structural catastrophe,[32] which corresponds to the defining equation (15). The distinctiveness of the Maxwell set is in general typical of physical processes that can be described by the various catastrophes.[33] In particular, the special role of the line $x=0$ is especially clear for the special solution of Bürgers' equation given by Il'in (Ref. 34, Ch. VI, §4),

$$\Gamma_t+\Gamma\Gamma_x-\Gamma_{xx}=0, \tag{24}$$

in the "general position" situation that describes the advent of an ordinary shock wave in low-dissipation media (which takes place when a "simple wave front" steepens). Il'in's solution is the logarithmic derivative

$$\Gamma(t,x)=-2[\ln\Lambda(t,x)]'_x \tag{25}$$

of the integral

$$\Lambda(t,x)=\int_R \exp\{-(4\lambda x-2t\lambda^2+\lambda^4)\}d\lambda, \qquad (26)$$

which is a modified version of Pearcey's integral.[35]

The special status of the line $x=0$ in the given solution of Bürgers' equation has two aspects. Firstly—as in the GP solution of the KdV equation as well—it is precisely the smoothness requirement at $x=0$ that uniquely determines the solution of the recursive series of ordinary differential equations that arises when the asymptotic series for $\Gamma(t,x)$ at $t\to-\infty$ is substituted into (24). Secondly, a "discontinuity" appears at that line as $t\to\infty$, signifying the onset of a dissipative shock. (The variables t and x, which are used to describe a small neighborhood of the point that serves as the origin for steepening of the wave, are dilated. In the language of matched asymptotic expansions,[34] they are used to determine the so-called *interior asymptotic expansion.*)

The Maxwell set also enjoys a special status with regard to Pearcey's integral itself, which describes the "steepening" of the rapidly oscillating phase among the solutions of a set of linear dispersion equations.[18] (Specifically, that integral was used in the classic work of Pearcey[35] to investigate electromagnetic field structure in the vicinity of a caustic cusp.) The special role played by the Maxwell set in this case has nothing to do with shock onset, but rather with the fact that the "three-phase" oscillation regime remaining after reversal degenerates at $x=0$ into a "one-phase" regime.

The indisputable distinctiveness of the line $t=0$ for the GP solution remains an obscure phenomenon for the present author. It seems likely not to be accidental, however, and it ought to show up in other nonlinear special functions for wave catastrophes. Following Kitaev,[8] it would appear that the connection with problems in the quantum theory of gravitation is probably also meaningful.

9. To summarize, in Ref. 3 and once again in the present paper we have confirmed the general hypothesis[15,16] of inevitable isomonodromy of the solutions of equations integrated via the inverse scattering method, which are nonlinear generalizations of the special functions of wave catastrophes.[17–19]

The results obtained in Ref. 3 and in the present work are to a large extent able to describe the onset of the shock waves that result from "wavefront" steepening, under both dissipative and nondissipative conditions. Furthermore, the fact that at the same time the GP solution satisfies (1), it also satisfies the ODE (2) in x and (along with the derivatives v_1,v_2,v_3) the system of ODE's (13) in t largely compensates for the lack of the explicit equations (25) and (26) in the nondissipative case.

The established isomonodromy of the GP solution leads us to believe that its asymptotic behavior at large arguments can be studied as thoroughly as that of the classical special functions (such as the Airy, Bessel, and Pearcey functions). Nevertheless, it should be pointed out that we have yet to conclusively determine even the form of the leading term in the asymptotic behavior of the GP solution in the rapid oscillation region. Where this problem has been studied—including Ref. 2—only the crude approximation obtained by averaging over the Whitham cnoidal wave (19) has been considered. Isomonodromy now makes it possible[36,38] to elucidate this question.

It may well be, however, that the most important consequence of the simultaneous validity of (2) and (13) is the feasibility, in principle, of a detailed numerical tabulation of the GP solution at *all* (and not just large) x and t. Although this is still a formidable problem, it is clear that solving an ODE numerically is much simpler than solving the KdV equation directly.

The relationship that we have established here between the onset of nondissipative shock waves and "nonperturbative" problems in the quantum theory of gravitation reflects that universal nature of the GP special solution of the KdV equation, one of the most interesting special functions of contemporary nonlinear mathematical physics.

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