Activated decay rate: finite barrier corrections

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The activated escape of a Brownian particle from a deep potential well is considered. The problem is characterized by two small parameters, one reflecting the weakness of dissipation, $\gamma \ll \omega$ (γ is the viscosity coefficient and ω a typical intrawell oscillation frequency), and the other corresponding to a large barrier height, $U_0 \gg T$ (U_0 is the barrier height and T the temperature). A previous approach to the decay rate calculation relies on the derivation of an integral equation and enables one to calculate the preexponential factor in the Arrhenius law by summing an infinite series in powers of the ratio $\gamma U_0/T\omega \sim 1$. The present paper shows that this result can be improved by corrections in the small parameter T/U_0. The leading correction is due to the particle motion slowing down at the top of the barrier and is of order $(T/U_0)\ln(U_0/T)$. To calculate it, a correction to the kernel of the aforementioned integral equation is required. In the next order of approximation, two factors giving corrections of order $T/U_0 \sim \gamma/\omega$ must be taken into account. First, thermal noise causes a partial return of the particles which cross the barrier with energies in the narrow interval $\sim \gamma T/\omega$; this can be accounted for straightforwardly in the general calculation scheme. Second, perturbations of the intrawell motion of the Brownian particle must be taken into account, which are caused by particle energy changes, on a scale of $T \ll U_0$, due to damping and thermal noise effects; this requires a more accurate calculation of the kernel of the integral equation. The suggested systematic approach to the small parameter expansion involved makes it possible to investigate quantitatively the transition from the low damping to the strong damping regime. Finite barrier corrections for the intermediate moderate-to-strong damping regime are also evaluated.

1. INTRODUCTION

The activated decay of metastable states is widespread in physical and chemical systems with phase space divided into one or more regions by high potential barriers. Transitions between individual potential minima are caused by thermal fluctuations and are exponentially rare if the barrier height U_0 is large compared to the temperature T. In this limit the rate of decay (or the inverse lifetime) of the metastable state is given by the Arrhenius law

$$\frac{1}{\tau}=\frac{\Omega}{2\pi}A\exp\left(-\frac{U_0}{T}\right), \tag{1}$$

where Ω is the frequency of small oscillations at the bottom of the potential well. All other details concerning the internal structure of the system and its interaction with the environment are incorporated into the preexponential factor A. The condition for metastability, $\tau\Omega \gg 1$, requires the inequality $U_0 \gg T$, but the exponential dependence of τ on the barrier height ensures a large lifetime even for moderate ratios $U_0/T > 5$. As the ratio is increased further, activated decay events become too rare to be observable. This implies that the ratio U_0/T should be considered large when it enters the argument of the exponential. At the same time, corrections in the inverse parameter T/U_0 may still be important in calculating the preexponential factor A. In what follows, this general statement is illustrated by using the example of a Brownian particle escaping from a one-dimensional potential well. This model was originally suggested by Kramers for describing the thermal dissociation of a molecule interacting with a light particle gas.[1] Many decades later, the resistively shunted Josephson junction[2] has proved to be virtually the only experimental system embodying all the details of the Kramers theoretical model. A comprehensive review of results relating to the Kramers problem is contained in Ref. 3. The problem of a weakly dissipated Brownian particle has called for a more sophisticated approach.[4] Since the physical aspects of the problem have already been discussed in the cited papers, the following analysis is mainly concerned with its mathematical details.

The starting point is the Fokker–Planck equation

$$\frac{p}{m}\frac{\partial f}{\partial x}-\frac{dU(x)}{dx}\frac{\partial f}{\partial p}=\gamma\frac{\partial}{\partial p}\left(mT\frac{\partial f}{\partial p}+pf\right) \tag{2}$$

for the distribution function $f(p,x)$ of a Brownian particle of mass m with position x and momentum p moving in the potential $U(x)$ with a damping coefficient γ. The function $f(p,x)$ is assumed to be normalized to one particle in the potential well,

$$\int f(p,x)\,dp\,dx=1. \tag{3}$$

Near the bottom of the well, the distribution function is close to its equilibrium form,

$$f(p,x)\approx\frac{\Omega}{2\pi T}\exp\left[-\frac{p^2}{2mT}-\frac{U(x)-U_0}{T}\right], \tag{4}$$

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where U_0 is the height of the barrier. For the sake of definiteness, the top of the barrier is located at $x=0$ and is described asymptotically by

$$U(x)\approx -\frac{1}{2}m\omega^2x^2. \tag{5}$$

Outside the well there are no particles except for those escaping the well, so the boundary condition is

$$f(p,x)\to 0, \quad x\to\infty. \tag{6}$$

The rate of decay is the flux of particles across the barrier top,

$$\frac{1}{\tau}=\int_{-\infty}^{\infty} f(p,0)\,\frac{p}{m}\,dp. \tag{7}$$

From this, substituting the Boltzmann distribution and integrating over the positive momenta we obtain (1) with the preexponential factor $A=1$. In the bulk of this paper, a weakly damped particle will be considered. In this case, for energies close to the barrier top the distribution function is depleted because of the particles escaping across the barrier, the thermal-noise particle excitation being too weak to restore the equilibrium distribution function. Under these conditions the preexponential factor A is less than unity and depends on the noise intensity, temperature, and other factors.

It should be noted that for weak damping, when

$$\gamma\ll\Omega,\omega, \tag{8}$$

the total energy of the particle

$$\varepsilon\equiv\frac{p^2}{2m}+U(x) \tag{9}$$

is almost conserved over one cycle of the particle's intrawell motion. This enables one to work out a perturbation theory such that its zeroth approximation is the particle trajectory for a given energy ε in the absence of damping and noise. Then, formally, the preexponential factor A can be treated as a function of two dimensionless parameters, γ/ω and T/U_0. To leading order, the expansion of A in terms of the small parameter γ/ω has the following form (Ref. 5):

$$A_0=\sum_{n=0}^{\infty} a_n(\gamma U_0/\omega T)^{1+n/2}. \tag{10}$$

This series has a finite radius of convergence. However, as an analytic function the series is meaningful for all values of the argument

$$\Delta\equiv\frac{\delta}{T}\sim\frac{\gamma U_0}{\omega T}, \tag{11}$$

where $\delta\sim\gamma U_0/\omega$ is the energy loss over one cycle of motion at the barrier top. For the function $A_0(\Delta)$ it has been found[5] that

$$A_0(\Delta)=\exp\left\{\frac{1}{\pi}\int_0^{\infty}\frac{\ln\{1-\exp[-\Delta(\lambda^2+1/4)]\}d\lambda}{\lambda^2+1/4}\right\}. \tag{12}$$

Asymptotically, this implies that

$$A_0(\Delta)\approx\Delta+\zeta(1/2)\Delta^{3/2}/\pi^{1/2}\approx\Delta-0.82\Delta^{3/2}, \quad \Delta\ll 1, \tag{13}$$

where $\zeta(x)$ is the Riemann zeta function, and

$$A_0(\Delta)\approx 1-2(\pi\Delta)^{-1/2}\exp(-\Delta/4), \quad \Delta\gg 1. \tag{14}$$

It might be assumed that corrections to this result should be expandable in powers of the small parameters T/U_0 and γ/ω. It will be shown, however, that because the particle is slowed down at the top of the barrier, the perturbation expansion for the function A is

$$A(\gamma/\omega,T/U_0)\approx A_0(\Delta)-\frac{T}{U_0}\frac{U_0}{\omega S}\left[A_1(\Delta)\ln\frac{U_0}{T}+B_1(\Delta)\right], \tag{15}$$

where S is the action per cycle of the motion of the escaping particle (an explicit expression for S is given below); the dependence on the ratio $\gamma U_0/\omega T$ is given by the positive-valued functions $A_1(\Delta)$ and $B_1(\Delta)$ of the argument

$$\Delta\equiv\frac{\gamma S}{T}=\frac{\gamma}{\omega}\frac{U_0}{T}\frac{\omega S}{U_0}, \tag{16}$$

and the ratio $\omega S/U_0$ is a numerical factor dependent on the potential shape $U_0(x)$.

The paper is arranged as follows. In Sec. 2 we solve the energy diffusion equation and discuss the general structure of the expansion of $A(\gamma/\omega,T/U_0)$ in the limit of small Δ, when $\gamma/\omega\ll T/U_0$. In Sec. 3, the opposite limit of $1\gg\gamma/\omega\gg T/U_0$ is considered. In this limit, the flow of escaping particles will be of the Boltzmann type, except for the narrow energy region $\varepsilon\sim\gamma T/\omega\ll T$ where damping and noise give rise to a fine structure in the distribution function. This leads to a relative suppression of the decay rate, by an amount related to the small parameter $\gamma/\omega\ll 1$. In Sec. 4 we present the general integral equation approach and derive an explicit expression, Eq. (12), for $A_0(\Delta)$. In Sec. 5, the kernel of the integral equation is evaluated to leading order in $(T/U_0)\ln(U_0/T)$, where the dominant contribution comes from the slowing down of the particle near the barrier top. In Sec. 6, the integral equation with the revised kernel is solved, and an expression for the function $A_1(\Delta)$ is derived. In Sec. 7, the contribution to the function $B_1(\Delta)$ from noise-induced reflection and recrossing processes is calculated. In Sec. 8 we calculate the kernel of the integral equation beyond the leading logarithmic approximation, and in Sec. 9 we find the full expression for $B_1(\Delta)$. Section 10 gives preliminary numerical estimates, and in Sec. 11 linear corrections in the parameter T/U_0 are discussed in the moderate-to-strong damping crossover regime.

2. ENERGY DIFFUSION

In order to consider the $\gamma\to 0$ limit, Eq. (2) must be transformed to the energy variable

$$\frac{\partial f}{\partial x}=\pm\gamma\frac{\partial}{\partial\varepsilon}p(\varepsilon,x)\left(T\frac{\partial f}{\partial\varepsilon}+f\right), \tag{17}$$

where

$$p(\varepsilon,x)\equiv\{2m[\varepsilon-U(x)]\}^{1/2} \tag{18}$$

is the absolute value of the momentum for a particle of energy ε at point x, and the signs $\pm$ indicate the direction of particle motion. Averaging (17) over x at fixed energy leads to the energy diffusion equation[1]

$$\frac{\partial}{\partial\varepsilon}\delta(\varepsilon)\left(T\frac{\partial f}{\partial\varepsilon}+f\right)=0, \tag{19}$$

where the diffusion coefficient $\delta(\varepsilon)$ is given by

$$\delta(\varepsilon)=\gamma S(\varepsilon)=2\gamma\int_{x_1}^{x_2}\{2m[\varepsilon-U(x)]\}^{1/2}dx. \tag{20}$$

The points $x_{1,2}(\varepsilon)$ are the turning points:

$$U(x_{1,2})=\varepsilon, \tag{21}$$

and the function $S(\varepsilon)$ is the action per cycle of the motion. The solution to (19) with the boundary condition

$$f(0)=0 \tag{22}$$

is given by

$$f(\varepsilon)=\frac{\Omega}{2\pi T}\exp\left(-\frac{\varepsilon+U_0}{T}\right)\int_\varepsilon^0\frac{\exp(\varepsilon/T)d\varepsilon}{\delta(\varepsilon)}\times\left[\int_0^\infty\frac{\exp(-\varepsilon'/T)}{\delta(-\varepsilon')}d\varepsilon'\right]^{-1}. \tag{23}$$

The correction arising from the energy dependence of Ω at the bottom of the well is left out because our primary concern is with the leading correction to A due to the $\delta(\varepsilon)$ dependence. The relation

$$\frac{1}{\tau}=-\delta(\varepsilon)T\left.\frac{df(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}, \tag{24}$$

which can be derived from (2) and (22) (see also Ref. 1), gives

$$\frac{1}{\tau}=\frac{\Omega}{2\pi}\exp\left(-\frac{U_0}{T}\right)\left[\int_0^\infty\frac{\exp(-\varepsilon/T)d\varepsilon}{\delta(-\varepsilon)}\right]^{-1}. \tag{25}$$

The function $\delta(\varepsilon)$ can be expanded into an asymptotic series in ε. For our purposes, we must consider the first-order correction

$$\delta(\varepsilon)\approx\delta+\frac{\gamma}{\omega}\varepsilon\left(\ln\frac{U_0}{T}+C_U+1+\ln 2-\ln\frac{|\varepsilon|}{T}\right), \tag{26}$$

where

$$\delta\equiv\delta(0)\equiv\gamma S, \tag{27}$$

S is the action per cycle of the motion at energy $\varepsilon=0$,

$$S=2\int_{x_1}^0[-2mU(x)]^{1/2}dx, \tag{28}$$

and C_U is a number that depends on the shape of the potential $U(x)$,

$$C_U\equiv2\int_{x_1}^0dx\left\{\frac{m\omega}{[-2mU(x)]^{1/2}}+\frac{1}{x}\right\}+\ln\frac{m\omega^2x_1^2}{U_0}. \tag{29}$$

Physically, δ is the energy loss per cycle of the motion calculated to linear order in γ,

$$\delta\equiv\gamma m\oint\left(\frac{dx}{dt}\right)^2dt,$$

where $dx(t)/dt$ is the solution of Newton's equation in the potential $U(x)$ for total energy $\varepsilon=0$. Substitution of (26) into (25) and expansion in terms of the small parameter γ/ω then yield,

$$A(\gamma/\omega,T/U_0)\approx\Delta-\frac{T}{U_0}\frac{U_0}{\omega S}\left[\Delta\ln\frac{U_0}{T}+\Delta(C_U+2+\ln 2-C)\right], \tag{31}$$

where $C=0.5772$ is Euler's constant. Comparing this with the definition (15) yields the asymptotic forms

$$A_0(\Delta)\approx\Delta,\quad\Delta\ll1, \tag{32}$$

$$A_1(\Delta)\approx\Delta,\quad\Delta\ll1, \tag{33}$$

$$B_1(\Delta)\approx\Delta(C_U+2+\ln 2-C),\quad\Delta\ll1, \tag{34}$$

At this stage all the parameters of the problem have been introduced. Let us now compute them for two typical potentials. For the cubic potential

$$U(x)=-\frac{m\omega^2x^2}{2}\left(1-\frac{x}{x_1}\right) \tag{35}$$

we obtain

$$\frac{U_0}{\omega S}=\frac{5}{36}\approx0.1389, \tag{36}$$

$$C_U=3\ln 6\approx5.375. \tag{37}$$

For the fourth-order parameter

$$U(x)=-\frac{m\omega^2x^2}{2}\left(1-\frac{x^2}{x_1^2}\right) \tag{38}$$

we have

$$\frac{U_0}{\omega S}=\frac{3}{16}\approx0.1875, \tag{39}$$

$$C_U=5\ln 2\approx3.466. \tag{40}$$

The above results, obtained to linear order in γ/ω, reveal a rather complex structure of the asymptotic expansion of A in terms of the small parameter T/U_0. Taking account of the higher terms, this expansion takes the form

$$A(\gamma/\omega,T/U_0)\approx\Delta\sum_{n=0}^\infty\left(\frac{T}{U_0}\right)^nP_n\left(\frac{\ln U_0}{T}\right),\quad\gamma/\omega\to0, \tag{41}$$

where $P_n(x)$ is a polynomial in x. A number of workers suggest a somewhat different expression for A (Refs. 6,7). In fact, this expression uses Eq. (12) with Δ multiplied by a certain function of γ/ω (Refs. 8 and 9). Naturally, in the limit $\gamma/\omega\to 0$, only the first term of (41) is reproduced.

3. NOISE-INDUCED RECROSSING FLOW

If energy losses are large enough, $\delta\gg T$, there is enough time for the particles to get thermalized in their motion in the well, and the flow of escaping particles turns out to be of very nearly Boltzmann type in the vicinity of the barrier top. In this case the approximate equation

$$\frac{p}{m}\frac{\partial f}{\partial x}+m\omega^2 x\frac{\partial f}{\partial p}=\gamma\frac{\partial}{\partial p}\left(mT\frac{\partial f}{\partial p}+pf\right), \tag{42}$$

adequate at the top of the barrier, has a solution, in the limit $\gamma/\omega\ll 1$, of the form[1]

$$f(p,x)=\frac{\Omega}{2\pi T}\exp\left[-\frac{U_0}{T}-\frac{p^2-m^2\omega^2x^2}{2mT}\right]\left(\frac{1}{2\pi m\gamma T}\right)^{1/2}\times\int_{-\infty}^{p-m\omega x}\exp\left(-\frac{\omega u^2}{2m\gamma T}\right)du. \tag{43}$$

The flow of escaping particles determines the decay rate,

$$\frac{1}{\tau}=\int_{-\infty}^{\infty}\frac{p}{m}f(p,0)dp=\frac{\Omega}{2\pi}\left(1-\frac{\gamma}{2\omega}\right)\exp\left(-\frac{U_0}{T}\right). \tag{44}$$

Comparison with Eq. (1) gives

$$A\approx 1-\frac{\gamma}{2\omega}\equiv 1-\frac{1}{2}\frac{T}{U_0}\frac{U_0}{\omega S}\Delta,\quad 1\ll\Delta\ll U_0/T. \tag{45}$$

In turn, comparison with (15) yields the following asymptotics:

$$A_0(\Delta)\approx 1,\quad \Delta\gg 1, \tag{46}$$

$$A_1(\Delta)\approx 0,\quad \Delta\gg 1, \tag{47}$$

$$B_1(\Delta)\approx\Delta/2,\quad \Delta\gg 1. \tag{48}$$

In the next section we will find explicit expressions for the functions $A_0(\Delta)$, $A_1(\Delta)$, and $B_1(\Delta)$ that can be used with arbitrary Δ and that have as their asymptotic forms the expressions (32)–(34) for $\Delta\ll 1$ and (46)–(48) for $\Delta\gg 1$.

4. INTEGRAL EQUATION FORMALISM

In order to extend the above results to finite values of $\Delta\equiv\delta/T$, it is necessary to find the solution of the Fokker–Planck equation (17) from perturbation theory. This section gives a brief summary of the earlier approach to the calculation of $A_0(\Delta)$. Some of the quantities and concepts we introduce in this way will be used in our subsequent generalization of the method.

Typical energies of the escaping particles are small compared to the typical potential energy, i.e., $\varepsilon\sim T\ll U_0$. Neglecting corrections of order $T/U_0\ll 1$, the differential equation (17) is equivalent to the integral equation

$$f(\varepsilon,x)=\int_{-\infty}^{\infty}g(\varepsilon,\varepsilon';x,x')f(\varepsilon',x')d\varepsilon'. \tag{49}$$

The Green's function $g(\varepsilon,\varepsilon',x,x')$ obeys the equation

$$\frac{\partial g}{\partial x}=\pm\gamma\frac{\partial}{\partial\varepsilon}[-2mU(x)]^{1/2}\left(T\frac{\partial g}{\partial\varepsilon}+g\right). \tag{50}$$

The initial condition at $x=x'$ has the form

$$g(\varepsilon,\varepsilon';x,x)=\delta_D(\varepsilon-\varepsilon'), \tag{51}$$

where $\delta_D(z)$ is the Dirac delta function. Obviously, the solution of Eq. (50) is a certain Gaussian function of the argument $\varepsilon-\varepsilon'$. For our purposes, we need only the Green's function for the base trajectory which starts at the barrier top $x=0$, reaches the turning point $x=x_1$ and goes back,

$$g_0(\varepsilon-\varepsilon')=(4\pi\delta T)^{-1/2}\exp\left[-\frac{(\varepsilon-\varepsilon'+\delta)^2}{4\delta T}\right]. \tag{52}$$

The parameter δ equals the energy loss per cycle and is given by (27). The Green's function thus obtained gives the probability of a particle, having performed one oscillation in the well, to return to the barrier with energy ε under the condition that it was reflected from the barrier at energy ε'. The energy distribution function near the barrier top, $f(\varepsilon)$, is formed by the particles which failed in their previous attempt to overcome the barrier because of their energies ε' being negative. This permits to write down the integral equation[10]

$$f(\varepsilon)=\int_{-\infty}^{0}g_0(\varepsilon-\varepsilon')f(\varepsilon')d\varepsilon'. \tag{53}$$

Equation (53) is to be solved for the boundary condition that deep in the well, $f(\varepsilon)$ is a Boltzmann distribution function,

$$f(\varepsilon)\approx\frac{\Omega}{2\pi T}\exp\left(-\frac{\varepsilon+U_0}{T}\right),\quad |\varepsilon|\gg T,\quad \varepsilon<0. \tag{54}$$

The decay rate is then given by the expression

$$\frac{1}{\tau}=\int_0^{\infty}f(\varepsilon)d\varepsilon, \tag{55}$$

in which we have explicitly ignored the contribution from the recrossing particles. On order to solve the integral equation by the Wiener–Hopf method, let us introduce the one-sided Fourier transform,

$$f_{\pm}(\lambda)=\frac{2\pi}{\Omega}\int_{-\infty}^{\infty}\theta(\pm\varepsilon)\exp\left(\frac{i\lambda\varepsilon}{T}+\frac{\varepsilon}{2T}+\frac{U_0}{T}\right)f(\varepsilon)d\varepsilon. \tag{56}$$

The integral equation takes the form

$$f_+(\lambda)+f_-(\lambda)=g_0(\lambda)f_-(\lambda), \tag{57}$$

where

$$g_0(\lambda)=\exp[-\Delta(\lambda^2+1/4)] \tag{58}$$

and

$$\Delta\equiv\delta/T. \tag{59}$$

Let us rewrite the equation in the form

$$f_+(\lambda)=-G(\lambda)f_-(\lambda), \tag{60}$$

and represent the function

$$+G(\lambda)\equiv 1-g_0(\lambda) \quad (61)$$

as a product of two factors that are analytic in the upper and lower complex λ half-planes, respectively:

$$G(\lambda)=G_+(\lambda)G_-(\lambda), \quad (62)$$

where

$$\ln G_\pm(\lambda)=\pm\int\frac{d\lambda'}{2\pi i}\frac{\ln G(\lambda')}{\lambda'-\lambda\mp i0}. \quad (63)$$

In the very weak damping limit we have

$$G_\pm(\lambda)=\mp i\Delta^{1/2}(\lambda\pm i/2), \quad \Delta\ll 1. \quad (64)$$

Equation (60) can now be rewritten as

$$\frac{f_+(\lambda)}{G_+(\lambda)}=-f_-(\lambda)G_-(\lambda). \quad (65)$$

The left-hand and right-hand sides of this equation are analytic in, respectively, the upper and lower complex λ half-planes and have a common analyticity strip. This means that they are equal to a certain simple function of λ, which has to be found from the boundary condition. In the present case this condition is given by the expression (54), which under the Fourier transform (56) becomes

$$f_-(\lambda)\approx-\frac{1}{\lambda+i/2}, \quad |\lambda+i/2|\ll 1. \quad (66)$$

The solution of Eq. (65) for the boundary condition (66) has the form[5]

$$f_+(\lambda)=\frac{iG_+(\lambda)G_-(-i/2)}{\lambda+i/2}, \quad (67)$$

$$f_-(\lambda)=-\frac{iG_-(-i/2)}{G_-(\lambda)(\lambda+i/2)}. \quad (68)$$

The function $f_+(\lambda)$ is analytic over the entire complex λ plane because $f(\varepsilon)$, in accordance with (53), decreases faster than any exponential as $\varepsilon\to\infty$:

$$f(\varepsilon)\propto\exp(-\varepsilon^2/4T\delta), \quad \varepsilon\gg(T\delta)^{1/2}. \quad (69)$$

On the contrary, $f_-(\lambda)$ is analytic for $\mathrm{Im}\,\lambda<-1/2$. The preexponential factor is obtained from

$$A_0(\Delta)=f_+(i/2), \quad (70)$$

giving (12).

5. LEADING LOGARITHMIC APPROXIMATION

To find the function $A_1(\Delta)$, one needs to solve the equation

$$\frac{\partial f}{\partial x}=\pm\gamma\frac{\partial}{\partial\varepsilon}\{2m[\varepsilon-U(x)]\}^{1/2}\left(T\frac{\partial f}{\partial\varepsilon}+f\right) \quad (71)$$

to leading order in $T/U_0\ln(U_0/T)$. Now let us turn to the integral equation

$$f(\varepsilon)=\int_{-\infty}^{0}g(\varepsilon,\varepsilon')f(\varepsilon)d\varepsilon, \quad (72)$$

where the Green's function $g(\varepsilon,\varepsilon')$ is the solution of the revised equation

$$\partial g\partial x=\pm\gamma\frac{\partial}{\partial\varepsilon}\{[-2mU(x)]^{1/2}+\varepsilon[-m/2U(x)]^{1/2}\}\times\left(T\frac{\partial g}{\partial\varepsilon}+g\right), \quad (73)$$

along the base trajectory. Introduce a new function

$$g(\lambda,\varepsilon',x)\equiv\int g(\varepsilon,\varepsilon',x)\exp\left[\frac{i\lambda(\varepsilon-\varepsilon')}{T}+\frac{\varepsilon-\varepsilon'}{2T}\right]d\varepsilon, \quad (74)$$

which obeys the equation

$$\frac{\partial g(\lambda,\varepsilon',x)}{\partial x}=\pm\frac{\gamma}{T}[-2mU(x)]^{1/2}\left(\lambda^2+\frac{1}{4}\right)g(\lambda,\varepsilon',x)\pm\gamma[-m/2U(x)]^{1/2}\left(\lambda-\frac{i}{2}\right)\times\left(-i\frac{\partial}{\partial\lambda}+\frac{\varepsilon'}{T}\right)\left(\lambda+\frac{i}{2}\right)g(\lambda,\varepsilon',x) \quad (75)$$

with the initial condition

$$g(\lambda,\varepsilon',\tilde{x})=1. \quad (76)$$

Here we have introduced a coordinate $\tilde{x}$, to be used as a cutoff parameter below. The substitution

$$g=g_0[1+g_1], \quad g_1\ll 1, \quad (77)$$

yields a simple equation

$$\frac{\partial g_1(\lambda,\varepsilon',x)}{\partial x}=\pm\gamma[-m/2U(x)]^{1/2}\times\left[\left(\lambda^2+\frac{1}{4}\right)\frac{\varepsilon'+2i\delta(x)}{T}-i\lambda-\frac{1}{2}\right], \quad (78)$$

where $\delta(x)$ is the energy dissipated by the particle in its motion along the base trajectory from point $\tilde{x}$ to point x. To calculate $g_1(\lambda,\varepsilon)$, we integrate the right-hand side of (78) from $\tilde{x}$ to the left turning point x_1 and then back. This integral diverges logarithmically,

$$\int_{x_1}^{\tilde{x}}\left[-\frac{m}{2U(x)}\right]^{1/2}dx\approx\ln\frac{x_1}{\tilde{x}}, \quad |\tilde{x}|\ll|x_1|. \quad (79)$$

Within the leading logarithmic approximation we are not interested in the numerical factor under the logarithm, and should therefore write the result in the form $(1/2)\ln(U_0/T)$. In the $\delta(x)$ term, only the final portion of the trajectory contributes to the logarithmic divergence, with $\delta(x)$ being replaced by $\delta\equiv T\Delta$. The result is

$$g_1^{LL}(\lambda,\varepsilon')=-\frac{\gamma}{\omega}\ln\frac{U_0}{T}\left[\left(\lambda^2+\frac{1}{4}\right)(\varepsilon T'+i\Delta\lambda)-i\lambda-\frac{1}{2}\right]. \quad (80)$$

The inverse Fourier transform then yields

$$g_1^{LL}(\varepsilon,\varepsilon')=-\frac{\gamma}{\omega}\ln\frac{U_0}{T}\left\{(\varepsilon+\varepsilon')\left[\frac{1}{8}+\frac{1}{4\Delta}-\frac{(\varepsilon-\varepsilon')^2}{8\Delta^2T^2}\right]-\frac{1}{2}\right\}. \tag{81}$$

6. SOLUTION OF THE REVISED INTEGRAL EQUATION

In its Fourier representation, the revised integral equation

$$f_+(\lambda)+G(\lambda)f_-(\lambda)$$
$$=g_0(\lambda)\int_{-\infty}^{0} g_1(\lambda,\varepsilon')\exp\left(\frac{i\lambda\varepsilon'}{T}+\frac{\varepsilon'}{2T}\right)f(\varepsilon')d\varepsilon', \tag{82}$$

is equivalent to

$$f_+(\lambda)+G(\lambda)f_-(\lambda)$$
$$=-\frac{\gamma}{\omega}\ln\frac{U_0}{T}g_0(\lambda)\left[\left(\lambda^2+\frac{1}{4}\right)\left(i\Delta\lambda-i\frac{\partial}{\partial\lambda}\right)-i\lambda-\frac{1}{2}\right]f_-(\lambda). \tag{83}$$

In the correction term we must replace $f_-(\lambda)$ by (68). Then using the factorization of the function $G(\lambda)$, the equation can be divided into terms analytic in the upper and lower λ half-planes, after which one should apply the relation

$$A(\Delta)=f_+(i/2). \tag{84}$$

For the function $A_1(\Delta)$ defined by (15) we obtain

$$A_1(\Delta)=A_0(\Delta)\Delta\left\{\frac{g_0(\lambda)}{G_+(\lambda)}\left[\left(\lambda^2+\frac{1}{4}\right)\left(i\Delta\lambda-i\frac{\partial}{\partial\lambda}\right)-i\lambda-\frac{1}{2}\right]\frac{i}{G_-(\lambda)(\lambda+i/2)}\right\}_+(i/2), \tag{85}$$

where the notation $\{...\}_+(i/2)$ indicates that the expression inside the braces should be integrated as in the Cauchy relation Eq. (63) for $\lambda=i/2$,

$$\{...\}_+(i/2)\equiv\int\frac{d\lambda}{2\pi i}\frac{\{...\}}{\lambda-i/2}. \tag{86}$$

The Δ term in brackets in Eq. (85) is odd in λ and hence vanishes. The product $G_+(\lambda)G_-(\lambda)$ should be replaced by $1-g_0(\lambda)$, and the derivative of $\ln G_-(\lambda)$ can be written in the form

$$\frac{\partial\ln G_-(\lambda)}{\partial\lambda}=-\int\frac{d\lambda'}{2\pi i}\frac{\partial\ln G(\lambda')}{\partial\lambda'}\frac{1}{\lambda'-\lambda+i0}$$
$$=-\int\frac{d\lambda'}{2\pi i}\frac{g_0(\lambda')}{1-g_0(\lambda')}\frac{2\Delta\lambda'}{\lambda'-\lambda+i0}. \tag{87}$$

The expression for $A_1(\Delta)$ then simplifies to

$$A_1(\Delta)=A_0(\Delta)\Delta\int\frac{d\lambda}{2\pi}\frac{g_0(\lambda)}{1-g_0(\lambda)}\times\int\frac{d\lambda'}{2\pi}\frac{g_0(\lambda')}{1-g_0(\lambda')}\frac{2\Delta\lambda'}{\lambda'-\lambda+i0}. \tag{88}$$

The symmetrization with respect to λ,λ' is equivalent to replacing the last factor by Δ, which leads to the final result

$$A_1(\Delta)=A_0(\Delta)\Phi^2(\Delta), \tag{89}$$

where the function $A_0(\Delta)$ is given by (12), and

$$\Phi(\Delta)=\Delta\int\frac{d\lambda}{2\pi}\frac{g_0(\lambda)}{1-g_0(\lambda)}$$
$$=\int\frac{d\lambda}{2\pi}\frac{\Delta}{\exp[\Delta(\lambda^2+1/4)]-1}. \tag{90}$$

Asymptotically we obtain

$$\Phi(\Delta)\approx1+\left(\frac{\Delta}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}n\left[\frac{1}{n^{1/2}}-\frac{1}{2n^{3/2}}-\frac{1}{(n+1)^{1/2}}\right]$$
$$\approx1-0.407\Delta^{1/2},\quad \Delta\ll1, \tag{91}$$

$$\Phi(\Delta)\approx\frac{1}{2}\left(\frac{\Delta}{\pi}\right)^{1/2}\exp\left(-\frac{\Delta}{4}\right),\quad \Delta\gg1. \tag{92}$$

Taking into account the asymptotic behavior of $A_0(\Delta)$, we find

$$A_1(\Delta)\approx\Delta-1.63\Delta^{3/2},\quad \Delta\ll1, \tag{93}$$

$$A_1(\Delta)\approx(\Delta/4\pi)\exp(-\Delta/2),\quad \Delta\gg1. \tag{94}$$

7. CONTRIBUTION FROM RECROSSING PARTICLES

As discussed in Sec. 3, in the narrow phase space region

$$|p-m\omega x|\sim(\gamma mT/\omega)^{1/2} \tag{95}$$

the distribution function acquires fine structure under the influence of the damping and noise effects. Outside this region, the solution of the Fokker–Planck equation is satisfactorily approximated by the solution of the integral equation obtained in Sec. 4. Matching these two solutions over the range $T\gg\varepsilon\gg\gamma T/\omega$ where they are both applicable, we obtain for the distribution function the following expression:

$$f(p,x)=f_B\left(\frac{1}{2\pi m\gamma T}\right)^{1/2}\int_{-\infty}^{p-m\omega x}\exp\left(-\frac{\omega u^2}{2m\gamma T}\right)du, \tag{96}$$

where f_B is the limiting value of $f(\varepsilon)$ at low energy,

$$f_B\equiv f(\varepsilon\to0). \tag{97}$$

The function (96) yields the correct boundary condition for the distribution functions of right- and left-going particles at the turning points ($p=0$) for $\varepsilon<0$. On the other hand, for $\varepsilon>0$ (or more accurately for $\varepsilon>\gamma T/\omega$) and $p<0$, this function goes to zero, showing that there are no particles entering the potential well from the outside.

According to the weak noise assumption, the recrossing particles occupy only a small region in phase space. To take them into account, the small parameter $\gamma/\omega\ll1$ should be used. To leading order, the function $f(p,0)$ is given by $f_B\theta(p)$. The correction to the decay rate is then given by the integral [see also Eq. (44)]

$$\left(\frac{1}{\tau}\right)^{(1)}=\int_{-\infty}^{\infty}[f(p,0)-f_B\theta(p)]\frac{p}{m}dp. \tag{98}$$

The difference between the right- and left-going particles becomes patently clear for $x<0$, $|x|\gg(\gamma T/m\omega^2)^{1/2}$, where the distribution of the reflected particles can be approximated by the function

$$f^{refl}(\varepsilon)=f(\varepsilon)\theta(-\varepsilon)+\frac{\gamma}{2\omega}f_BT\delta_D(\varepsilon),\quad |\varepsilon|\ll T,\delta. \tag{99}$$

As in to Sec. 4, the function $f(\varepsilon)$ is obtained from the one-sided convolution of the above function with the kernel $g_0(\varepsilon-\varepsilon)$, giving a closed integral equation for $f(\varepsilon)$. Expanding the distribution function in terms of the parameter γ/ω, we obtain for the first-order correction $f^{(1)}(\varepsilon)$ the inhomogeneous integral equation

$$f^{(1)}(\varepsilon)=\int_{-\infty}^{0}g_0(\varepsilon-\varepsilon')f^{(1)}(\varepsilon')d\varepsilon'+\frac{\gamma}{2\omega}f_BTg_0(\varepsilon). \tag{100}$$

The correction to the decay rate is then

$$\left(\frac{1}{\tau}\right)^{(1)}=-\frac{\gamma}{2\omega}f_BT+\int_0^{\infty}f^{(1)}(\varepsilon)d\varepsilon, \tag{101}$$

where the first term corresponds to noise-induced recrossing events, and the second term to the outgoing flux of originally reflected particles.

Before we solve Eq. (100), it is useful to find an explicit form for the quantity f_B defined by (97). An inverse Fourier transform gives

$$f(\varepsilon)=\frac{\Omega}{2\pi T}\exp\left(-\frac{U_0}{T}\right)\int[f_+(\lambda)+f_-(\lambda)]\times\exp\left(\frac{i\lambda\varepsilon}{T}-\frac{\varepsilon}{2T}\right)\frac{d\lambda}{2\pi}. \tag{102}$$

This should be considered a symbolic expression, since the functions $f_+(\lambda)$ and $f_-(\lambda)$ actually result from two different Laplace transforms. For this reason the λ integration of these two functions must be performed along two different contours passing respectively above and below their singular points. The function $f_+(\lambda)$ is analytic in the upper complex λ half-plane. Consequently, the real λ axis may be used as the contour of integration in (102). On the other hand, the function $f_-(\lambda)$ has a pole at $\lambda=-i/2$, and the contour of integration must go above this point. In order to push the above contour onto the real λ axis, the residue at the singular point $\lambda=-i/2$ must be taken into account. Finally,

$$f(\varepsilon)=\frac{\Omega}{2\pi T}\exp\left(-\frac{\varepsilon+U_0}{T}\right)\left\{1+\int_{-\infty}^{\infty}[f_+(\lambda)+f_-(\lambda)]\times\exp\left(-\frac{i\lambda\varepsilon}{T}+\frac{\varepsilon}{2T}\right)\frac{d\lambda}{2\pi}\right\}. \tag{103}$$

Using (67) and (68) for $f_+(\lambda)$ and $f_-(\lambda)$, the sum of these can be expressed in terms of $G_-(\lambda)$. To calculate f_B it suffices to set $\varepsilon=0$ in (103), giving

$$f_B=\frac{\Omega}{2\pi T}[1-I(\Delta)]\exp\left(-\frac{U_0}{T}\right), \tag{104}$$

where

$$I(\Delta)\equiv A_0^{1/2}(\Delta)\int\frac{d\lambda}{2\pi}\frac{ig_0(\lambda)}{(\lambda+i/2)G_-(\lambda)}. \tag{105}$$

In the very weak damping limit we obtain

$$I(\Delta)\approx 1-1.0\Delta^{1/2},\quad \Delta\ll 1. \tag{106}$$

For large Δ we have the asymptotic form

$$I(\Delta)\approx(\pi\Delta)^{1/2}\exp(-\Delta/4). \tag{107}$$

To solve Eq. (100) we again employ the one-sided Fourier transform (56), giving

$$f_+^{(1)}(\lambda)+G(\lambda)f_-^{(1)}(\lambda)=\frac{\gamma}{2\omega}[1-I(\Delta)]g_0(\lambda). \tag{108}$$

This equation can be solved by the Wiener–Hopf method:

$$f_+^{(1)}(\lambda)=\frac{\gamma}{2\omega}[1-I(\Delta)]G_+(\lambda)\times\int\frac{d\lambda'}{2\pi i}\frac{g_0(\lambda')/G_+(\lambda')}{\lambda'-\lambda-i0}. \tag{109}$$

The integral in Eq. (101) is proportional to $f_+^{(1)}(i/2)$. Using the relations $G_+(\lambda)G_-(\lambda)=1-g_0(\lambda)$ and $G_+(i/2)=A_0^{1/2}(\Delta)$ and the representation (105), the result is expressible in terms of the function $I(\Delta)$,

$$f_+^{(1)}(i/2)=\frac{\gamma}{2\omega}I(\Delta)[1-I(\Delta)]. \tag{110}$$

The correction to the preexponential factor $A(\gamma/\omega,T/U_0)$ has the form

$$A^{(1)}=-\frac{\gamma}{2\omega}[1-I(\Delta)]^2\equiv-\frac{T}{U_0}\frac{U_0}{\omega S}\frac{\Delta}{2}[1-I(\Delta)]^2. \tag{111}$$

Comparison with (15) shows that this correction contributes to the function $B_1(\Delta)$. We denote this contribution by $B_1^{(r)}(\Delta)$ to emphasize that it is due to particle reflection and recrossing processes. Thus,

$$B_1^{(r)}(\Delta)=\frac{\Delta}{2}[1-I(\Delta)]^2. \tag{112}$$

8. CORRECTIONS TO THE GREEN'S FUNCTION BEYOND THE LEADING LOGARITHMIC APPROXIMATION

In Sec. 5, we found the leading logarithmic correction $g_1^{LL}(\varepsilon,\varepsilon')$ to the kernel of the integral equation, and evaluated the function $A_1(\Delta)$. To calculate $B_1(\Delta)$, it is necessary to obtain the kernel $g_1(\varepsilon,\varepsilon')$ more accurately. Introducing the coordinate $\tilde{x}$ and using the inequalities

$$(T/m\omega^2)^{1/2}\ll|\tilde{x}|\ll|x_1|, \tag{113}$$

it proves possible to decompose the problem into the following two subproblems. In the main part of the potential well, for $x_1 < x < \tilde{x}$, the function $g_1(\varepsilon,\varepsilon')$ can be calculated by expansion in terms of the small ratio $|\varepsilon/U(x)|$. We denote this result by $g_1^{BL}(\varepsilon,\varepsilon',\tilde{x})$, having in mind that this correction is beyond the logarithmic approximation which gives $g_1^{LL}(\varepsilon,\varepsilon')$. Near the top of the barrier, i.e., between $\tilde{x}$ and the turning points $x(\varepsilon)$ and $x(\varepsilon')$, a parabolic approximation for $U(x)$ can be used and, noting that these regions are relatively narrow, the Fokker–Planck equation can be solved iteratively. As a reminder of the origin of this contribution, we denote it by $g_1^{VB}(\varepsilon,\varepsilon',\tilde{x})$. The final result for that part of $g_1(\varepsilon,\varepsilon')$ that contributes to the function $B_1(\Delta)$ is of the form

$$g_1^{B}(\varepsilon,\varepsilon')=g_1^{BL}(\varepsilon,\varepsilon',\tilde{x})+g_1^{VB}(\varepsilon,\varepsilon',\tilde{x}). \tag{114}$$

Once the two contributions have been added, the dependence on $\tilde{x}$ disappears, thus justifying the introduction of $\tilde{x}$ as an auxiliary parameter. The functions g_1^{BL} and g_1^{VB} are evaluated separately in the subsections that follow.

8.1. The main part of the potential well

In order to calculate the function $g_1^{BL}(\varepsilon,\varepsilon',\tilde{x})$, it is necessary to calculate more accurately the integral of the right-hand side of (78). There are two different integrals to carry out. The first of these, by subtracting the term which has already been accounted for in the function $g_1^{LL}(\varepsilon,\varepsilon')$, can be written in the form

$$2\gamma\int_{x_1}^{\tilde{x}}\left[-\frac{m}{2U(x)}\right]^{1/2}dx-\frac{\gamma}{\omega}\ln\frac{U_0}{T}=\frac{\gamma}{\omega}\left(C_U+\ln\frac{T}{m\omega^2\tilde{x}^2}\right), \tag{115}$$

where the number C_U is given by (29). The second integral can be expressed in terms of the same number,

$$\gamma\oint_{\tilde{x}}^{\tilde{x}}\left[-\frac{m}{2U(x)}\right]^{1/2}\frac{\delta(x)}{T}dx-\frac{\gamma}{\omega}\Delta\ln\frac{U_0}{T}$$

$$\equiv\gamma^2\int_{x_1}^{\tilde{x}}\left[-\frac{m}{2U(x)}\right]^{1/2}\left(\int_x^{\tilde{x}}+\int_{x_1}^{\tilde{x}}+\int_{x_1}^{x}\right)$$

$$\times[-2mU(x')]^{1/2}dx'-\frac{\gamma}{\omega}\Delta\ln\frac{U_0}{T}$$

$$=2\gamma^2\int_{x_1}^{\tilde{x}}\left[-\frac{m}{2U(x)}\right]^{1/2}\int_{x_1}^{\tilde{x}}[-2mU(x')]^{1/2}dx'$$

$$-\frac{\gamma}{\omega}\Delta\ln\frac{U_0}{T}\approx\frac{\gamma}{2\omega}\Delta\left(C_U+\ln\frac{T}{m\omega^2\tilde{x}^2}\right), \tag{116}$$

where we have neglected a small term of order $(\tilde{x}/x_1)^2 \ll 1$. The final result for the contribution from the main part of the potential well can be written in the form

$$g_1^{BL}(\varepsilon,\varepsilon',\tilde{x})=-\frac{\gamma}{\omega}\left(C_U+\ln\frac{T}{m\omega^2\tilde{x}^2}\right)$$

$$\times\left\{\frac{\varepsilon+\varepsilon'}{T}\left[\frac{1}{8}+\frac{1}{4\Delta}-\frac{(\varepsilon-\varepsilon')^2}{8\Delta^2T^2}\right]-\frac{1}{2}\right\}. \tag{117}$$

8.2. Top of the barrier

To find the contribution to $g_1(\varepsilon,\varepsilon')$ from the region $[\tilde{x},x(\varepsilon)]$, one can use the equation

$$\frac{\partial g(\varepsilon,\varepsilon';x,\tilde{x})}{\partial x}=\gamma\frac{\partial}{\partial\varepsilon}(2m\varepsilon+m^2\omega^2x^2)^{1/2}$$

$$\times\left(T\frac{\partial}{\partial\varepsilon}+1\right)g_0(\varepsilon-\varepsilon'), \tag{118}$$

where on the right we have taken the function $g_0(\varepsilon-\varepsilon')$ as a zeroth approximation. The integration over x should be performed from $\tilde{x}$ to $x(\varepsilon)$, where

$$x(\varepsilon)=0,\quad \varepsilon>0, \tag{119}$$

$$x(\varepsilon)=-(-2\varepsilon/m)^{1/2}/\omega,\quad \varepsilon<0. \tag{120}$$

The result is

$$g(\varepsilon,\varepsilon';x(\varepsilon),\tilde{x})=g(\varepsilon,\varepsilon';\tilde{x},\tilde{x})+\frac{\gamma}{2\omega}\frac{\partial}{\partial\varepsilon}\left(m\omega^2\tilde{x}^2+\varepsilon\right.$$

$$\left.+\varepsilon\ln\frac{2m\omega^2\tilde{x}^2}{|\varepsilon|}\right)\left(T\frac{\partial}{\partial\varepsilon}+1\right)g_0(\varepsilon-\varepsilon').$$

The term $\propto\tilde{x}^2$ should be dropped as it has already been taken into account in deriving $g_0(\varepsilon-\varepsilon')$. Substituting for this function from (52), differentiating and separating out the common factor $g_0(\varepsilon-\varepsilon')$, we obtain the contribution to $g_1^{VB}(\varepsilon,\varepsilon';\tilde{x})$. An analogous contribution comes from the initial portion of the trajectory, $\tilde{x}<x<x(\varepsilon')$, where we must use the equation

$$\frac{\partial g(\varepsilon,\varepsilon';\tilde{x},x')}{\partial x'}=\gamma\left(T\frac{\partial}{\partial\varepsilon'}-1\right)(2m\varepsilon'+m^2\omega^2x'^2)^{1/2}$$

$$\times\frac{\partial}{\partial\varepsilon'}g_0(\varepsilon-\varepsilon'). \tag{121}$$

Combining the two contributions yields

$$g_1^{VB}(\varepsilon,\varepsilon';\tilde{x})=-\frac{\gamma}{\omega}\left\{\left[\frac{\varepsilon+\varepsilon'}{T}\left(\ln\frac{2m\omega^2\tilde{x}^2}{T}+1\right)\right.\right.$$

$$\left.-\frac{\varepsilon}{T}\ln\frac{|\varepsilon|}{T}-\frac{\varepsilon'}{T}\ln\frac{|\varepsilon'|}{T}\right]\left[\frac{1}{8}+\frac{1}{4\Delta}\right.$$

$$\left.-\frac{(\varepsilon-\varepsilon')^2}{8\Delta^2T^2}\right]+\frac{1}{4}\left[\operatorname{sgn}\varepsilon+\operatorname{sgn}\varepsilon'-2\right.$$

$$-2\ln 2-2\ln\frac{m\omega^2\tilde{x}^2}{T}+\ln\frac{|\varepsilon|}{T}+\ln\frac{|\varepsilon'|}{T}$$

$$\left.\left.-\frac{\varepsilon-\varepsilon'}{T}\left(\ln\left|\frac{\varepsilon}{\varepsilon'}\right|+\operatorname{sgn}\varepsilon-\operatorname{sgn}\varepsilon'\right)\right]\right\}, \tag{122}$$

where sgn x is the sign function. According to the definition (114), this expression must be added to (117). Then the dependence on the auxiliary parameter $\tilde{x}$ disappears, and the final result can be written in the form

$$g_1^B(\varepsilon,\varepsilon') = -\frac{T}{U_0}\frac{U_0}{\omega S}(C_U+1+\ln 2)\left\{\frac{\varepsilon+\varepsilon'}{T}\left[\frac{1}{8}+\frac{1}{4\Delta}-\frac{(\varepsilon-\varepsilon')^2}{8\Delta^2T^2}\right]-\frac{1}{2}\right\}-\frac{T}{U_0}\frac{U_0}{\omega S}\mathscr{K}(\varepsilon,\varepsilon'), \quad (123)$$

where

$$\mathscr{K}(\varepsilon,\varepsilon') \equiv \Delta\left(\frac{\varepsilon}{T}\ln\frac{|\varepsilon|}{T}+\frac{\varepsilon'}{T}\ln\frac{|\varepsilon'|}{T}\right)\left[\frac{1}{8}+\frac{1}{4\Delta}-\frac{(\varepsilon-\varepsilon')^2}{8\Delta^2T^2}\right] -\frac{1}{4}\left[\ln\frac{|\varepsilon|}{T}+\ln\frac{|\varepsilon'|}{T} +\operatorname{sgn}\varepsilon+\operatorname{sgn}\varepsilon' -\frac{\varepsilon-\varepsilon'}{T}\left(\ln\left|\frac{\varepsilon}{\varepsilon'}\right| +\operatorname{sgn}\varepsilon-\operatorname{sgn}\varepsilon'\right)\right]. \quad (124)$$

From this expression, it follows that the function $f(\varepsilon)$ is singular at small ε; in particular, at $\varepsilon=0$ it has a finite jump due to the term $\propto \operatorname{sgn}\varepsilon$ and diverges logarithmically. These singularities reflect a qualitative difference between the escaping ($\varepsilon>0$) and reflected ($\varepsilon<0$) particles. For energies $\sim\gamma T/\omega$ these singularities will be smoothed out analogously to (96).

The correction to the function $B_1(\Delta)$ coming from the first term in (123) can be expressed in terms of the function $A_1(\Delta)$, because this term differs from (81) only by a constant factor. Accordingly, $B_1(\Delta)$ can be expressed in the form

$$B_1(\Delta) = (\Delta/2)[1-I(\Delta)]^2 + A_1(\Delta)(C_U+1+\ln 2) + \int_0^\infty f^{(1)}(\varepsilon)d\varepsilon, \quad (125)$$

where $f^{(1)}(\varepsilon)$ corresponds to the correction to the distribution function $f(\varepsilon)$ arising from the last term in (123).

9. SOLUTION OF THE REVISED INTEGRAL EQUATION

The function $f^{(1)}(\varepsilon)$ obeys the integral equation

$$f^{(1)}(\varepsilon) - \int_{-\infty}^0 g_0(\varepsilon-\varepsilon')f^{(1)}(\varepsilon')d\varepsilon' = \int\frac{d\lambda'}{2\pi}f_-(\lambda')\int_{-\infty}^\infty g_0(\varepsilon-\varepsilon')\mathscr{K}(\varepsilon,\varepsilon') \times\exp\left[-\frac{i\lambda'\varepsilon'}{T}-\frac{\varepsilon'}{2T}\right]d\varepsilon'. \quad (126)$$

The integration over ε' on the right extends to infinity, since the contribution from positive ε' is identically zero in accordance with the analytic properties of the function $f_-(\lambda')$ [see (68)]. Introducing the Fourier transform (56) in the usual fashion, we obtain

$$\int_0^\infty f^{(1)}(\varepsilon)d\varepsilon = f_+^{(1)}(i/2). \quad (127)$$

For conciseness, we introduce the Green's function of the integral equation,

$$\mathscr{G}(\varepsilon,\varepsilon') - \int_{-\infty}^0 g_0(\varepsilon,\varepsilon'')\mathscr{G}(\varepsilon'',\varepsilon') = \delta_D(\varepsilon-\varepsilon')\exp\left(-\frac{\varepsilon}{2T}\right). \quad (128)$$

In the Fourier representation we obtain

$$\mathscr{G}_+(\lambda,\varepsilon') - [1-g_0(\lambda)]\mathscr{G}_-(\lambda,\varepsilon') = \exp\left(\frac{i\lambda\varepsilon'}{T}\right). \quad (129)$$

The expression for $\mathscr{G}_+(i/2,\varepsilon')$ needed to calculate $f_+^{(1)}(i/2)$ is given by the function

$$\mathscr{G}(\varepsilon') = A^{1/2}(\Delta)\int\frac{d\lambda}{2\pi i}\frac{\exp(i\lambda\varepsilon'/T)}{(\lambda-i/2)G_+(\lambda)}. \quad (130)$$

Using these results we obtain

$$f_+^{(1)}(i/2) = A_0(\Delta)\int\frac{d\lambda}{2\pi}\int\frac{d\lambda'}{2\pi} \times\frac{\mathscr{K}(\lambda,\lambda')}{(\lambda-i/2)G_+(\lambda)(\lambda'+i/2)G_-(\lambda')}, \quad (131)$$

where

$$\mathscr{K}(\lambda,\lambda') \equiv -\iint d\varepsilon d\varepsilon' g_0(\varepsilon-\varepsilon')\mathscr{K}(\varepsilon,\varepsilon') \times\exp\left(\frac{i\lambda\varepsilon-i\lambda'\varepsilon'}{T}+\frac{\varepsilon-\varepsilon'}{2T}\right). \quad (132)$$

To proceed further, let us introduce the notation

$$g_0(\varepsilon-\varepsilon')\mathscr{K}(\varepsilon,\varepsilon') = \frac{\Delta}{2}\left[\frac{\partial}{\partial\varepsilon}\epsilon\ln\frac{|\varepsilon|}{T}\left(T\frac{\partial}{\partial\varepsilon}+1\right) +\left(T\frac{\partial}{\partial\varepsilon'}-1\right)\varepsilon' \times\ln\frac{|\varepsilon'|}{T}\frac{\partial}{\partial\varepsilon'}\right]g_0(\varepsilon-\varepsilon'). \quad (133)$$

Substituting this into (132) gives

$$\mathscr{K}(\lambda,\lambda') = -(\Delta/2)J(\lambda-\lambda')(\lambda-i/2)(\lambda'+i/2) \times[g_0(\lambda)+g_0(\lambda')], \quad (134)$$

where the singular function $J(\lambda)$ is defined by the integral

$$J(\lambda) \equiv \int_{-\infty}^\infty x\ln|x|\exp(i\lambda x)dx. \quad (135)$$

This function has a singularity at $\lambda=0$, and we must now consider the result of integrating this function with a function $\varphi(\lambda)$ that is analytic near the real λ axis.

We start with the simple integral

$$\int_{-\infty}^{\infty}\varphi(\lambda)\frac{d\lambda}{2\pi}\int_{-\infty}^{\infty}\ln|x|\exp(i\lambda x)dx$$
$$=\int_{C_1}\varphi(\lambda)\frac{d\lambda}{2\pi}\int_{0}^{\infty}\ln|x|\exp(i\lambda x)dx$$
$$+\int_{C_2}\varphi(\lambda)\frac{d\lambda}{2\pi}\int_{-\infty}^{0}\ln|x|\exp(i\lambda x)dx, \qquad (136)$$

where the contour of integration C_1 (C_2) passes above (below) the real λ axis. Integrating over x, we obtain

$$\int_{C_1}\varphi(\lambda)\frac{C+\ln(-i\lambda)}{i\lambda}\frac{d\lambda}{2\pi}+\int_{C_2}\varphi(\lambda)\frac{C+\ln(i\lambda)}{-i\lambda}\frac{d\lambda}{2\pi}$$
$$=-(C+\ln\rho)\varphi(0)-\left(\int_{-\infty}^{-\rho}+\int_{\rho}^{\infty}\right)\frac{\varphi(\lambda)}{|\lambda|}\frac{d\lambda}{2}, \qquad (137)$$

where

$$C\equiv-\int_0^\infty e^{-x}\ln x dx\approx 0.5772\ldots. \qquad (138)$$

Integrating by parts on the right-hand side of (137), we find

$$\int_{-\infty}^{\infty}\varphi(\lambda)\frac{d\lambda}{2\pi}\int_{-\infty}^{\infty}\ln|x|\exp(i\lambda x)dx$$
$$=-C\varphi(0)+\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\varphi(\lambda)}{d\lambda}\ln|\lambda|\operatorname{sgn}\lambda d\lambda. \qquad (139)$$

For the function $J(\lambda)$, a similar calculation gives

$$\int_{-\infty}^{\infty}\varphi(\lambda)J(\lambda)\frac{d\lambda}{2\pi}$$
$$=-iC\varphi'(0)+i\pi\int_{-\infty}^{\infty}\frac{d^2\varphi(\lambda)}{d\lambda^2}\ln|\lambda|\operatorname{sgn}\lambda\frac{d\lambda}{2\pi}. \qquad (140)$$

Substitution of (134) and (140) into (131) leads to the result

$$f_+^{(1)}(i/2)=-CA_1(\Delta)+D(\Delta), \qquad (141)$$

where

$$D(\Delta)\equiv\frac{i\pi}{2}A_0(\Delta)\Delta\iint\frac{d\lambda}{2\pi}\frac{d\lambda'}{2\pi}\ln|\lambda-\lambda'|\operatorname{sgn}(\lambda-\lambda')$$
$$\times\frac{d^2}{d\lambda d\lambda'}\frac{g_0(\lambda)+g_0(\lambda')}{G_+(\lambda)G_-(\lambda')}. \qquad (142)$$

In the very weak damping limit $\Delta\ll 1$, it is possible to make the replacements $A_0(\Delta)\approx\Delta$, $A_1(\Delta)\approx\Delta$, and $g_0(\lambda)\approx 1$, and to use (64) for $G_\pm(\lambda)$. Then for the function $D(\Delta)$ we obtain

$$D(\Delta)\approx i\pi\Delta\iint\frac{d\lambda}{2\pi}\frac{d\lambda'}{2\pi}\frac{\ln|\lambda-\lambda'|\operatorname{sgn}(\lambda-\lambda')}{(\lambda+i/2)^2(\lambda'-i/2)^2}=\Delta. \qquad (143)$$

This yields the asymptotic form

$$\int_0^\infty f^{(1)}(\varepsilon)d\varepsilon\approx-C\Delta+\Delta,\quad \Delta\ll 1, \qquad (144)$$

which when substituted into (125) reproduces (34). Finally, for $B_1(\Delta)$ we have

$$B_1(\Delta)=(\Delta/2)[1-I(\Delta)]^2+(C_U+1+\ln 2-C)$$
$$\times A_1(\Delta)+D(\Delta). \qquad (145)$$

10. NUMERICAL ESTIMATES

Equations (12), (89), and (145) are sufficient for the calculation of the revised preexponential factor using (15). Note that the above results are exact as far as the calculation of A to order T/U_0 is concerned. As in a previous paper,[5] it is convenient to write down the following interpolation expression for A, which is exact as long as terms of order $(\gamma/\omega)^2$ can be neglected, and which reproduces the Kramers results to order γ/ω:

$$A(\gamma/\omega,T/U_0)=A_0(\Delta)\left(1+\frac{\gamma^2}{4\omega^2}\right)^{1/2}$$
$$-\frac{T}{U_0}\frac{U_0}{\omega S}\left[A_1(\Delta)\ln\frac{U_0}{T}+B_1(\Delta)\right]. \qquad (146)$$

It is worth noting that in the region $\Delta\gg 1$, the $B_1(\Delta)$ term reproduces correctly the linear decrease of A with $\gamma/2\omega$, as seen from Eqs. (45) and (48). When combined with the first term of (146), where $A_0\approx 1$, this yields the Kramers result

$$A(\gamma/\omega)=(1+\gamma^2/4\omega^2)^{1/2}-\gamma/2\omega. \qquad (147)$$

Thus the expression (146) for the preexponential factor A describes the transition from the weak-damping to the moderately strong-damping regime, with allowance for corrections of order T/U_0.

In the next section it is shown that in the moderately strong-damping regime, the corrections in the same small parameter, while remaining relatively small, are not small in comparison with the corrections found earlier. Therefore, Eq. (146) gives only a qualitative understanding of the behavior of the preexponential factor as it passes through a maximum. The final conclusion is that when damping is not too weak, the simplified interpolation given by the product of (12) and (147) [see (5)] is applicable even for relatively low barriers. For example, the amplitude and position of the maximum of A as a function of γ/ω are changed little by the introduction of the correction. For $T/U_0=1.0$, the position of the maximum of A shifts from $\gamma/\omega\approx 0.76$ to $\gamma/\omega\approx 0.58$, whereas the amplitude of A is reduced by 10%. For $T/U_0=0.5$, the fractional shift in the maximum position is less than 3%. In the weak damping regime, the small numerical factor in Eq. (31),

$$\frac{U_0}{\omega S}=\frac{5}{36}\approx\frac{1}{7}$$

is compensated by the second numerical factor,

$$C_U+2+\ln 2-C\approx 7.49$$

so that for a cubic potential, from (31), (36), and (37),

$$A\approx\Delta\left(1-1.0\frac{T}{U_0}-\frac{5}{36}\frac{T}{U_0}\ln\frac{U_0}{T}\right),\quad \Delta\ll 1. \qquad (148)$$

In accordance with the general properties of the asymptotic expansions, one would expect the expression in Ref. 5 to be a good approximation for A for $T/U_0<1/5$. Thus in the case of high barriers ($U_0>5T$), one can employ it as a zeroth approximation and estimate finite barrier corrections by use of Eq. (146). For $U_0<5T$, the preexponential factor A starts to depend on both the height and shape of the barrier. Fortunately, for low barriers the rate of activated processes can be effectively found by numerical methods.[9,15,16]

11. RELATIVELY STRONG DAMPING REGIME

In the strong damping regime, $\Delta\gg 1$, the parameter γ/ω reaches values of order unity, which makes it essential to account for corrections in this parameter. As shown by Kramers, for $\gamma\sim\omega$ the leading result can be obtained using a parabolic approximation to the top of the potential barrier. To correct this result, it is necessary to expand the potential up to the cubic and/or fourth-order term,

$$U(x)\approx-\frac{m\omega^2x^2}{2}+U^{(3)}\frac{x^3}{6}+U^{(4)}\frac{x^4}{24}. \qquad (149)$$

It will be shown below that the linear contribution in $U^{(3)}$ is zero, whereas the second-order contribution in this parameter is of the same order of magnitude as the linear contribution in $U^{(4)}$. The expression for the preexponential factor may be assumed to have the form

$$A=\left(1+\frac{\gamma^2}{4\omega^2}\right)^{1/2}-\frac{\gamma}{2\omega}-\frac{T}{U_0}\left[C_3a_3\left(\frac{\gamma}{\omega}\right)+C_4a_4\left(\frac{\gamma}{\omega}\right)\right], \qquad (150)$$

where the dependence on the shape of the potential enters through the numerical factors

$$C_3\equiv\frac{[U^{(3)}]^2U_0}{(m\omega^2)^3}, \qquad (151)$$

$$C_4\equiv\frac{U^{(4)}U_0}{(m\omega^2)^2}, \qquad (152)$$

and the $a_n(\gamma/\omega)$ are certain universal functions. For the cubic potential discussed above $C_3=2/3$, whereas for the fourth-order potential $C_4=3/2$, and for the cosine potential $C_4=2$.

To calculate $a_n(\gamma/\omega)$, we substitute into Eq. (2) the function

$$f(p,x)=\frac{\Omega}{2\pi T}\varphi(p,x)\exp\left(-\frac{p^2}{2mT}-\frac{U(x)}{T}\right) \qquad (153)$$

and rewrite the equation in the form

$$\gamma mT\frac{\partial^2\varphi}{\partial p^2}-\frac{p}{m}\frac{\partial\varphi}{\partial x}-(m\omega^2x+\gamma p)\frac{\partial\varphi}{\partial p}=-\left(\frac{\partial U}{\partial x}+m\omega^2x\right)\frac{\partial\varphi}{\partial p}. \qquad (154)$$

Following Kramers, we introduce instead of p a new variable

$$u=(p-\alpha m\omega x)(\omega/\alpha\gamma mT)^{1/2}, \qquad (155)$$

where

$$\alpha=\left(1+\frac{\gamma^2}{4\omega^2}\right)^{1/2}+\frac{\gamma}{2\omega}. \qquad (156)$$

If we neglect the right-hand side of (154), the solution is independent of x and is given by the error integral

$$\varphi_0(u)=(2\pi)^{-1/2}\int_{-\infty}^{u}\exp\left(-\frac{v^2}{2}\right)dv. \qquad (157)$$

To proceed further, we write $\varphi(x,u)$ in the form

$$\varphi(x,u)=\varphi_0(u)+(2\pi)^{-1/2}\psi(x,u)\exp(-u^2/2). \qquad (158)$$

Now we substitute

$$\frac{\partial U}{\partial x}+m\omega^2x=U^{(n)}\frac{x^{n-1}}{(n-1)!} \qquad (159)$$

on the right-hand side of (154) and consider the cases $n=3$ and $n=4$ separately below. The particle flux is calculated at $x=0$. It is therefore convenient to scale the coordinate like

$$x\to x(\alpha^3\gamma T/m\omega^3)^{1/2} \qquad (160)$$

and to calculate the expression for $\psi(0,u)$, which does not depend on this scale. The final equation for $\psi(x,u)$ is fairly compact,

$$\frac{\partial^2\psi}{\partial u^2}-u\frac{\partial\psi}{\partial u}-\psi-(u+\alpha^2x)\frac{\partial\psi}{\partial x}=-\mu_nx^{n-1}\left(1-u\psi+\frac{\partial\psi}{\partial u}\right), \qquad (161)$$

where

$$\mu_n\equiv C_n^{n/2-1}\left(\frac{\gamma T}{\omega U_0}\right)^{n/2-1}\frac{\alpha^{3n/2-1}}{(n-1)!},\quad n=3,4. \qquad (162)$$

The case $n=4$ is the simplest because the linear term already yields a nonvanishing contribution. In order to calculate it, the function $\psi(x,u)$ should be expanded in powers of x,

$$\psi(x,u)\equiv-\mu_4\sum_{l=0}^{3}P_l(u)x^l, \qquad (163)$$

where the $P_l(u)$ are polynomials in u. The system of equations thus obtained,

$$P_l''-uP_l'-(1+\alpha^2l)P_l-u(l+1)P_{l+1}=\delta_{l,3}, \qquad (164)$$

where $\delta_{n,l}$ is the Kronecker symbol, can be solved by descending iterations starting with $P_4=0$. We obtain

$$P_0(u)=\frac{3u[(u^2+3)(\alpha^2+1)+4]}{4(\alpha^2+1)^2(\alpha^2+3)(3\alpha^2+1)}. \quad (165)$$

The function a_4 is given by the integral

$$a_4=\frac{\alpha^6}{6}\left(\frac{\gamma}{\omega}\right)^2\int_{-\infty}^{\infty}\exp\left(-\frac{\alpha^2u^2}{2}\right)P_0(u)\frac{udu}{(2\pi)^{1/2}}$$

$$=\frac{1}{8}\left(\frac{\gamma}{\omega}\right)^2\frac{\alpha}{(\alpha^2+1)^2}. \quad (166)$$

In the weak damping limit $\gamma \ll \omega$, for the potential (38) we have

$$A\approx 1-\frac{\gamma}{2\omega}+\frac{1}{8}\left(\frac{\gamma}{\omega}\right)^2\left(1-\frac{3}{8}\frac{T}{U_0}\right). \quad (167)$$

In the opposite limit

$$A\approx\frac{\omega}{\gamma}\left(1-\frac{3}{16}\frac{T}{U_0}\right),\quad \gamma/\omega\gg 1. \quad (168)$$

From these expressions it follows that corrections in the parameter T/U_0 enter with numerical factors less than unity, which makes them relatively small even when $T/U_0\sim 1$. For the cubic potential, the contribution to $\psi(u)$ linear in μ_3 is even in u, so that the integral corresponding to (166) vanishes. To find the second-order correction, we introduce the expansion

$$\psi(x,u)=-\mu_3\sum_{l=0}^{2}P_l(u)x^l-\mu_3^2\sum_{l=0}^{4}Q_l(u)x^l. \quad (169)$$

For $P_l(u)$ and $Q_l(u)$ we obtain the system of equations

$$P_l''-uP_l'-(1+\alpha^2l)P_l-u(l+1)P_{l+1}=\delta_{l,2}, \quad (170)$$

$$Q_l''-uQ_l'-(1+\alpha^2l)Q_l-u(l+1)Q_{l+1}=uP_{l-2}-P_{l-2}'. \quad (171)$$

The function $a_3(\gamma/\omega)$ is then given by the integral (see also Ref. 17)

$$a_3(\gamma/\omega)=\frac{\alpha^8}{4}\left(\frac{\gamma}{\omega}\right)^2\int\exp\left(-\frac{\alpha^2u^2}{2}\right)Q_0(u)\frac{udu}{(2\pi)^{1/2}}. \quad (172)$$

A nearly ideal model system for the observation of activated decay events is the Josephson junction.[11,12] In the last few years, superconducting quantum interference devices have become more popular for the study of the activated decay, the reason being that they enable the activated processes to be investigated in a controlled manner in both one- and two-dimensional potentials.[13,14] Typically, they are operated under moderately strong damping conditions. Both the experimental data and numerical simulation results agree well with theoretical calculations. The results of our calculations enable one, in principle, to move further into the region of lower potential barriers, where finite barrier corrections are important. Particular applications of the expressions obtained requires, in our view, a derivation on their basis of some interpolation formula applicable to arbitrary damping. This work must probably be correlated with the interpretation of numerical simulation results.

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[1] H. A. Kramers, Physica **7**, 284 (1940).
[2] E. Turlot, J. M. Martinis, C. Urbina *et al.*, Phys. Rev. Lett. **62**, 1288 (1989).
[3] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[4] V. I. Mel'nikov, Phys. Rep. **209**, 1 (1991).
[5] V. I. Mel'nikov and S. V. Meshkov, J. Chem. Phys. **85**, 1018 (1986).
[6] H. Grabert, Phys. Rev. Lett. **61**, 1683 (1988).
[7] E. Pollack, H. Grabert, and P. Hänggi, J. Chem. Phys. **91**, 4073 (1989).
[8] I. Rips and E. Pollack, J. Chem. Phys. **xx**, 5366 (1990).
[9] S. Linkwitz, H. Grabert, E. Turlot *et al.*, Phys. Rev. **A45**, R3369 (1992).
[10] G. Iche and P. Nozières, J. Phys. (Paris) **37**, 1313 (1976); **40**, 225 (1979).
[11] J. M. Martinis, M. H. Devores, and J. Clarke, Phys. Rev. **B35**, 4682 (1985).
[12] E. Turlot, D. Esteve, C. Urbina *et al.*, Phys. Rev. Lett. **62**, 1788 (1989); H. Grabert and S. Linkwitz, Phys. Rev. **A37**, 963 (1988).
[13] V. Lefevre-Seguin, E. Turlot, C. Urbina *et al.*, Phys. Rev. **B46**, 5507 (1992).
[14] S. Han, J. Lapointe, and J. E. Lukens, Phys. Rev. **B46**, 6338 (1992).
[15] H. Risken, The Fokker–Planck Equations, Springer, Berlin (1989), and references therein.
[16] R. Fernando, R. Spadacini, and G. E. Tommei, Phys. Rev. **A46**, R699 (1992).
[17] H. Dekker and A. Maasen van den Brink, Preprint, Amsterdam (1993).

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