# Possible deviations from the Kolmogorov spectrum of developed turbulence

N. V. Antonov

*St. Petersburg State University, 199034 St. Petersburg, Russia*
(Submitted 20 May 1993; resubmitted 25 November 1993)
Zh. Eksp. Teor. Fiz. **105**, 614–625 (March 1994)

We consider a semiphenomenological model of developed turbulence based on the Navier–Stokes equation with an eddy viscosity $\nu(k)k^2 \sim k^{2/3}$ and a random external force. We use an infrared perturbation theory (and also a direct single-loop approximation calculation) to show that all IR divergences (i.e., the singularities as $L\to\infty$, where $L$ is the external scale of the turbulence) vanish in the single-time correlation functions. This means, in particular, that in the model considered, the energy spectrum is described by the Kolmogorov "5/3 law." We calculate the velocity pair correlator and the response function in the simplest approximation of the IR theory and discuss their analytic properties.

## 1. INTRODUCTION

An important characteristic of (homogeneous, isotropic) developed turbulence of an incompressible viscous liquid is the turbulent energy spectrum, which is determined by the single-time velocity pair correlator

$$\langle \varphi_i(\mathbf{x},t)\varphi_j(\mathbf{x}+\mathbf{r},t)\rangle = \frac{1}{(2\pi)^d}\int d\mathbf{k}e^{i\mathbf{kr}}P_{ij}(\mathbf{k})D(k) \quad (1)$$

through the relation

$$E(k)=C_d k^{d-1}D(k)/2, \quad C_d\equiv(d-1)S_d/(2\pi)^d,$$
$$k\equiv|\mathbf{k}|. \quad (2)$$

Here $\varphi_i(\mathbf{x},t)$ is the transverse (by virtue of the incompressibility condition) velocity vector field, $d>2$ is the arbitrary (for generality) dimensionality of $\mathbf{x}$ space, $\mathbf{k}$ is the momentum (wave vector), $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ is the transverse projector, and $S_d=2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $d$-dimensional space.

According to the main assumption of the phenomenological Kolmogorov–Obukhov theory (Ref. 1, §21; Ref. 2, Ch. 3, §33), the turbulent energy spectrum in the inertial range $L\gg 1/k\gg l_D$ (where $L$ is the external scale of the turbulence, i.e., the characteristic size of the large-scale vortices which pump in the energy, and $l_D$ is the dissipative length at which viscosity becomes important) is determined by a single parameter $\bar{W}$, the average energy pumping power (equal, with a minus sign, to the average rate of dissipation). Dimensionality considerations then determine the spectrum, apart from a numerical factor, in the form

$$E_0(k)=A\bar{W}^{2/3}k^{-5/3}, \quad (3)$$

where $A=1.3$ to 2.3 is Kolmogorov's constant.

Both experimental and theoretical evidence is known in favor of some dependence of the turbulent spectrum in the inertial range on the external scale of the turbulence; see the discussion in §25 of Ref. 1, Ch. 3 of Ref. 2, and p. 323 of Ref. 3 and the literature cited there, and also the later Refs. 4, 5, and 6. The deviation from the spectrum (3) is usually written in the form

$$E(k)=E_0(k)f(kL), \quad f(kL)\simeq(kL)^{-\delta}, \quad \delta\leqslant 10^{-1}, \quad (4)$$

and it is explained by the strongly developed fluctuations in the parameter $W$. In the framework of various models, the index $\delta$ in (4) is connected with the statistical characteristics of the large-scale vortices which pump in energy (Ref. 1, §25) or with the dimensionality of fractal structures formed by small-scale vortices in the dissipation region.[6]

As a microscopic model of developed turbulence one considers usually the Navier–Stokes equation with an external random force which imitates the pumping of energy by large-scale pulsations (Ref. 1, §28.4):

$$\nabla_t\varphi_i+R\varphi_i+\partial_i p=F_i, \quad \nabla_t\equiv\partial_t+(\varphi\partial). \quad (5)$$

Here $\varphi$ is the velocity field, $p$ and $F$ are, respectively, the pressure and the transverse external random force per unit mass, $R=-\nu\partial^2$, or in the momentum representation, $R_k=\nu k^2$ with the kinematic viscosity coefficient $\nu$. We assume for $F$ a Gaussian distribution with zero mean and a given correlator:

$$\langle F_i(\mathbf{x},t)F_j(\mathbf{x}+\mathbf{r},t')\rangle=\delta(t-t')(2\pi)^{-d}\times\int d\mathbf{k}e^{i\mathbf{kr}}P_{ij}(\mathbf{k})d_F(k). \quad (6)$$

The pumping function $d_F(k)$ is assumed to be concentrated in a region of small $k\simeq 1/L$, the parameter $W$ takes on the meaning of the power conveyed by the random force, and its average value is connected with the function $d_F$ through the relation:

$$\bar{W}=(d-1)/2(2\pi)^d\int d\mathbf{k}d_F(k). \quad (7)$$

In a number of papers[7–9] a modification of the model (5) and (6) was considered in which one took

$$R_k=\gamma k^{2/3}, \quad d_F(k)=\alpha k^{-d}. \quad (8)$$

The choice (8) corresponds to the assumptions of the Kolmogorov–Obukhov theory in the sense that the parameters $\alpha$ and $\gamma^3$ have the dimensionality of $W$ while the

 1063-7761/94/030327-07$10.00  

zeroth approximation of perturbation theory in the dimensionless parameter $g \equiv \alpha/\gamma^3$ reproduces the spectrum (3). In correspondence with the ideas of Ref. 2 one can consider[9] the model (8) as the analog of the fluctuation theories of critical phenomena which for the Kolmogorov–Obukhov theory corresponds to the self-consistent field approximation. We must note some inconsistencies in the model (5)–(8). Simultaneously with an effective turbulent viscosity and the correlator (8) of the random force, the contribution of the nonlinearity in the Navier–Stokes equation (5) is completely taken into account and it shapes them. [Usually in papers on the application of the renormalization group (RG) in the theory of turbulence, the correlator of the random force is chosen in the form (8) and its independence of the viscosity is proven; see Refs. 10 and 11.]

The external scale $L$ of the turbulence and the dissipation length $l_D$ are taken into account in the model (5) to (8) through an infrared (IR) cutoff of the momenta in all perturbation theory diagrams at the magnitude $m \simeq L^{-1}$ and an ultraviolet (UV) cutoff at $l_D^{-1}$. The Ward identity, which expresses the Galilean invariance of the theory, guarantees the cancellation of the UV singularities of the diagrams in each order of perturbation theory, so that the correlation functions have a finite limit as $l_D \to 0$ and are independent of $l_D$ in the inertial range. In the $m \to 0$ region, the separate contributions of perturbation theory contain strong (powers of $m$) IR singularities, but these singularities are not connected with dynamic interactions (which shape the spectrum) of the vortices but with the kinematic effect of the transfer of vortices in the inertial range as a whole by large-scale vortices of size $\simeq L$, so that they are removed by transforming to a frame of reference moving with the constant random velocity of the large-scale vortices and do not thereby affect the shape of the spectrum.[12] Technically, the splitting off of singular contributions connected with the transfer is realized by means of various schemes in infrared perturbation theory (IRPT; see Refs. 13, 14, and 15).

After eliminating the power singularities there remain weaker singularities (logarithmic in $m$ as $m \to 0$). Their contributions were summed in Refs. 7 and 9 using a RG technique (in the form of Wilson recursion relations in Ref. 7, and in a quantum field formulation in the two-loop approximation in Ref. 9), applied directly to the IR divergences of the perturbation theory diagrams (as for the legitimacy of such an approach, see below). As a result, one found in Ref. 7 for the spectrum (4): $f(kL) \simeq 1/\ln(kL)$, and in Ref. 9: $f(kL) \simeq (kL)^{-\delta}$, $\delta \simeq 1.7 \times 10^{-2}$ (the difference between the results of Refs. 7 and 9 is explained, in particular, by the fact that in Ref. 7 one assumed that $\gamma < 0$, which changes the nature of the stability of the fixed point in the RG.

We show in the present paper that all contributions which are singular as $m \to 0$ in perturbation theory for the model (5)–(8), both the power and the logarithmic ones, vanish completely when one changes to single-time correlation functions. This proves, in particular, that the energy spectrum in the model (5)–(8) has the Kolmogorov form (3). The vanishing of the singular contributions in single-time correlators is explained by the fact that they are connected with the kinematic effect of the transfer in the inertial range of turbulent vortices by large-scale vortices, the velocity of which depends on the time, and that they are completely removed by transforming to the appropriate frame of reference. The neglect of the time-dependence of the velocity of the large-scale vortices is the cause of the apparent "penetration" of the external scale $L$ in the inertial range in Refs. 7 and 9. A Galilean transformation with a time-dependent parameter was considered in Refs. 10 and 11.

A great many papers have been devoted to the problem of a foundation of the Kolmogorov scaling in the framework of a micromodel such as (5), (6) (see Refs. 13 and 14 and the literature cited there). In those papers one usually considered skeleton self-consistency equations with "dressed" lines (Wyld's diagram technique), and the problem posed is to prove the existence in those equations of Kolmogorov type solutions. Skeleton diagrams with Kolmogorov lines contain IR divergences of the kind described above, and the problem is reduced to proving that they vanish in single-time correlation functions. Such a proof was obtained in a paper by Belinicher and L'vov[13] in the framework of an "internal diagram technique," and in a preprint by Tur and Yanovskiĭ[14] using the so-called "ballistic mode" elimination procedure.

It is necessary to make clear that in the present paper, in contrast to Refs. 13 and 14, we consider not a microscopic, but a semiphenomenological model of the kind of Refs. 7–9, with an effective vortex viscosity and a random force correlator of the form (8). Instead of the self-consistency equations, we use in this case the usual perturbation theory in the coupling constant $g = \alpha/\gamma^3$, which turns out to be possible since the correlator (8) is also nonvanishing in the inertial range.

The plan of our paper is the following. We perform in §2 an explicit calculation of the contributions which are singular as $m \to 0$ in the single-loop approximation for the velocity pair correlator, and we verify that they vanish in the single-time correlator. We briefly discuss the applicability of the RG technique and the renormalization theory to remove IR divergences. We give in, Sec. 3 a general proof of the vanishing of singular contributions for single-time correlators in all orders of perturbation theory. It is based on a generalization of IRPT in the form of Ref. 15, which makes it possible to take into account the time dependence of the large-scale field. We obtain explicit expressions for the pair correlator and the response function in the simplest IRPT approximation, the first of which reproduces the single-loop result of §2 when we expand the perturbation theory series in $g$. We note that the IR divergences lead to a vanishing of the singularities of the response function in the $\omega$ plane. In §4 we discuss the results obtained.

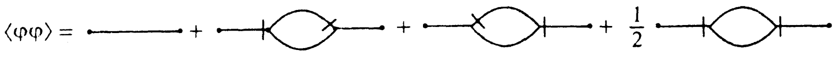


FIG. 1. Pair correlator $\langle\varphi\varphi\rangle$ in the one-loop approximation.

## 2. SINGLE-LOOP APPROXIMATION

The stochastic problem (5)–(8) is equivalent to the quantum theory of two transverse vector fields $\Phi\equiv\varphi,\varphi'$ with the action[6, 10,11]

$$S(\Phi)=\varphi' D_F\varphi'/2+\varphi'[-\partial_t\varphi-R\varphi-(\varphi\partial)\varphi]. \qquad (9)$$

Here $D_F$ is the random force correlator (6) with the function $d_F$ from (8), the operation $R$ has the form (8) in the $\mathbf{k}$ representation, and we assume the necessary summations over the vector indices of the fields and integration over their arguments.

The action (9) corresponds to the standard quantum field diagram perturbation theory with bar propagators, which in the $\omega$, $\mathbf{k}$ representation have the form

$$\langle\varphi\varphi'\rangle_0=\langle\varphi'\varphi\rangle_0^*=(-i\omega+R_k)^{-1},\quad \langle\varphi'\varphi'\rangle_0=0, \qquad (10)$$
$$\langle\varphi\varphi\rangle_0=\langle\varphi\varphi'\rangle_0\, d_F(k)\langle\varphi'\varphi\rangle_0.$$

(The vector indices omitted from (10) are all common to the transverse projector), and the triple vertex $\varphi'(\varphi\partial)\varphi=\varphi'_i V_{ijs}\varphi_j\varphi_s/2$ with vertex factor

$$V_{ijs}(\mathbf{k})=ik_sP_{ij}(\mathbf{k})+ik_jP_{is}(\mathbf{k}), \qquad (11)$$

where $\mathbf{k}$ is the momentum entering into the vertex through the field $\varphi'$. All momenta are bounded from below $|\mathbf{k}|\leqslant m$, by the quantity $m\simeq L^{-1}$, which is of the order of the reciprocal of the external turbulence scale. The role of the coupling constant is played here by the dimensionless parameter $g=\alpha/\gamma^3$, with $\alpha$ and $\gamma$ from (8).

We give in Fig. 1 a diagram representation of the velocity pair correlator $\langle\varphi\varphi\rangle$ in the single-loop approximation, i.e., up to $g^2$. The lines without cancellations correspond to the bare propagator $\langle\varphi\varphi\rangle_0$, the lines with cancellations to $\langle\varphi\varphi'\rangle_0$ with the end $\varphi'$ canceled. We first evaluate the contributions of the 1-irreducible self-energy loops which enter through factors in $\langle\varphi\varphi\rangle$, and which are shown in Fig. 2. According to (10) and (11) the loop in Fig. 2(a) corresponds to the analytic expression (compare Ref. 9)

$$D_{ij}^{(a)}(\Omega,\mathbf{p})=\frac{\alpha}{(2\pi)^{d+1}}\int d\omega\int d\mathbf{k}\,\times\frac{A_{ij}(\mathbf{p},\mathbf{k})}{k^d(\omega^2+R_k^2)(-i(\Omega-\omega)+R_q)}, \qquad (12)$$

in which $\Omega$ and $\mathbf{p}$ are the external frequency and momentum, $\omega$ and $\mathbf{k}$ are integration variables in the loop, we have written $\mathbf{q}\equiv\mathbf{p}-\mathbf{k}$, and

$$A_{ij}(\mathbf{p},\mathbf{k})=V_{imn}(\mathbf{p})P_{mm'}(\mathbf{k})P_{nn'}(\mathbf{q})V_{n'm'j}(\mathbf{q})$$

is the contribution of the vector index factors in (10) and (11). The integration region over $\mathbf{k}$ in (12) is restricted by the conditions $k,q\leqslant m$. Integrating over $\omega$, we are led to the expression

$$D_{ij}^{(a)}=\frac{\alpha}{2(2\pi)^d}\int d\mathbf{k}\,\frac{A_{ij}}{k^dR_k(-i\Omega+R_q+R_k)}, \qquad (13)$$

and from its analysis it is clear that the contributions which are singular as $m\to 0$ are generated by the region of small $k\simeq m$, i.e., by the "soft" line $\langle\varphi\varphi\rangle_0$; when we evaluate them we can completely neglect the $\mathbf{k}$ dependence of the "hard" momentum $\mathbf{q}$ in $A_{ij}$ and $R_q$, i.e., we can set $\mathbf{q}=\mathbf{p}$. By virtue of isotropy we can make the substitution $P_{mm'}(\mathbf{k})\to\delta_{mm'}(d-1)/d$, after which we can change to an integral over $k=|\mathbf{k}|$,

$$D_{ij}^{(a)}=-P_{ij}(\mathbf{p})C_d\frac{\alpha p^2}{2d}I(-i\Omega+R_p) \qquad (14)$$

with the coefficient $C_d$ from (2). In (14) we have written

$$I(z)=\int_m^\infty\frac{dk}{kR_k(R_k+z)}. \qquad (15)$$

One can easily evaluate the integral (15), which gives (compare Ref. 9):

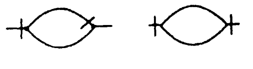
$$I(z)=\frac{3}{2z}\{R_m^{-1}+z^{-1}\ln(R_m/R_p)$$
$$-z^{-1}\ln((R_m+z)/R_p)\}. \qquad (16)$$

The first two terms are singular as $m\to 0$ and the last one is finite; we neglect it in what follows. (Note that there is apparently an error in the coefficient of $\ln R_m$ in the expression in Ref. 9 corresponding to (16). Expressions (14) to (16) are thus the singular part of the loop in Fig. 2(a). We now turn to the diagram in Fig. 2(b), for which

$$D_{ij}^{((b)}(\Omega,\mathbf{p})=\frac{\alpha^2}{(2\pi)^{d+1}}\int d\omega\int d\mathbf{k}\,\times\frac{B_{ij}(\mathbf{p},\mathbf{k})}{k^dq^d(\omega^2+R_k^2)[(\Omega-\omega)^2+R_q^2]}$$
$$=\frac{\alpha^2}{4(2\pi)^d}\int d\mathbf{k}\,\frac{B_{ij}}{k^dq^dR_kR_q}\left\{\frac{1}{i\Omega+R_k+R_q}+\frac{1}{-i\Omega+R_k+R_q}\right\}, \qquad (17)$$

where

$$B_{ij}=V_{imn}(\mathbf{p})P_{mm'}(\mathbf{k})P_{nn'}(\mathbf{q})V_{jm'n'}(-\mathbf{p}),$$



a b

FIG. 2. Single-loop contributions to the 1-irreducible functions $\langle\varphi'\varphi\rangle$ and $\langle\varphi'\varphi'\rangle$.

and the rest of the notation is as in (13). The singularities as $m \to 0$ are again produced by the soft lines $\langle \varphi\varphi \rangle_0$, i.e., the region of small $k$ and $q$. Their contributions are the same because of the symmetry of the diagram in Fig. 2(b), so we consider only the contribution from small $k \simeq m$ and take the second contribution into account by a factor 2. As before, we make the substitution $\mathbf{q} \to \mathbf{p}$, $P_{mm'}(\mathbf{k}) \to \delta_{mm'}(d-1)/d$ and change to an integral over $k$:

$$D_{ij}^{(b)} = P_{ij}(\mathbf{p}) \frac{\alpha^2 p^2 C_d}{2dp^d R_p} \{ I(i\Omega + R_p) + I(-i\Omega + R_p) \}, \tag{18}$$

where $I$ is the integral (15). Expression (18) is the singular contribution of the diagram of Fig. 2(b).

Comparing Figs. 1 and 2 for the singular part of the correlator $\langle \varphi\varphi \rangle$ in the single-loop approximation, we find

$$\langle \varphi\varphi \rangle = \frac{d_F(p)}{a} + \frac{d_F(p)}{a} \left\{ \frac{1}{z} D^{(a)} + \text{c.c.} \right\} + \frac{1}{2a} D^{(b)}, \tag{19}$$

where we have written $z \equiv -i\Omega + R_p$, $a \equiv zz^* = \Omega^2 + R_p^2$. All terms are proportional to the projector $P_{ij}(\mathbf{p})$; the first one is the contribution from the bare propagator and the second and third are complex conjugates by virtue of (10). Substituting Eqs. (14), (16), and (18) into (19), we find

$$\langle \varphi\varphi \rangle = \langle \varphi\varphi \rangle_0 - \frac{3C_d \alpha^2 p^2}{8dp^d R_p} \{ A_3 R_m^{-1} + A_4 \ln(R_m/R_p) \}. \tag{20}$$

The $\Omega$-dependence is contained in the factors $A_n = A_n(\Omega, p)$, which can be written in the form

$$A_n(\Omega,p) = (-i\Omega + R_p)^{-n} + (i\Omega + R_p)^{-n}, \quad n = 3,4. \tag{21}$$

The single-time correlator is obtained by integrating (20) over $\Omega$. It is clear from the representation (21) that both the power and the logarithmic singular parts of (20) will then vanish. One can also easily obtain an explicit expression for the singular contribution (20) in the time representation:

$$\langle \varphi\varphi \rangle = \langle \varphi\varphi \rangle_0 \left\{ 1 - \frac{3\alpha C_d p^2}{4d} \left( \frac{1}{2} \left| t \right|^2 R_m^{-1} + \frac{1}{6} \left| t \right|^3 \ln \frac{R_m}{R_p} \right) \right\},$$

$$\langle \varphi\varphi \rangle_0 = \frac{d_F(p)}{2R_p} e^{-R_p|t|}. \tag{22}$$

Here $t$ is the difference in the times of the fields occurring in the correlator; the singular contribution vanishes for $t = 0$.

It was assumed in Refs. 7 and 9 that only the power singularities vanish in the single-time correlator; the logarithmic ones were summed using the RG technique applied directly to the IR divergences of the model (5)–(8): the RG equation in Ref. 9 expressed the arbitrariness in the renormalization procedure removing the IR (rather than, as usual, the UV divergences of the diagrams which in the language used in Ref. 7 for the Wilson recursion-relation scheme corresponds to eliminating parts of the momenta near the IR rather than the UV cutoff. A similar application of the RG method is also found in Ref. 17. It is clear that it is then impossible to rely on general theorems of quantum field renormalization theory, which were developed exclusively in connection with the problem removing UV divergences (see Ref. 18), and one needs to prove the IR renormalizability of the model considered here, or at least verify it in the perturbation theory order considered (we note, however, that the use of the RG method even in the lowest single-loop approximation assumes renormalizability of the theory to all orders). It is clear in the present case from Eqs. (14), (15), and (19) that the contributions that are logarithmically singular as $m \to 0$ in the 1-irreducible single-loop diagrams can certainly not be removed by counterterms which arise by a multiplicative renormalization of the action (9) of the form $\text{const} \cdot i\Omega + \text{const} \cdot p^{2/3}$ for $D^{(a)}$, and $\text{const} \cdot p^{-d}$ for $D^{(b)}$, for any choice of constants, and hence the model considered here is not IR renormalizable in the aforementioned sense even in the single-loop approximation.

## 3. INFRARED PERTURBATION THEORY

We now proceed to a general proof of the vanishing of contributions that are singular as $m \to 0$ in all perturbation theory orders for the single-time correlators of the model (9). It is based upon a generalization of IRPT in the form of Ref. 15.

We write the field $\Phi \equiv \varphi, \varphi'$ as a sum $\Phi = \Phi_< + \Phi_>$ of a soft, $\Phi_<$, and a hard, $\Phi_>$, component, referring to $\Phi_<$ all Fourier components with $k = |\mathbf{k}| \leqslant \lambda$, i.e.,

$$\Phi_<(\mathbf{x},t) = \frac{1}{(2\pi)^d} \int_{k \leqslant \lambda} d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \Phi(\mathbf{k},t),$$

and to $\Phi_>$ the remaining contributions with $k > \lambda$. Here $\lambda$ is a fixed arbitrary separating scale which satisfies the relations $p_i \gg \lambda \gg m$, where $m \simeq L^{-1}$ is the reciprocal of the external turbulence scale and the $p_i$ are the external momenta of the correlation function considered. The proof given below is valid for any single-time correlators of the two fields $\Phi = \varphi, \varphi'$, and we shall for definiteness consider the pair correlator of the hard field $\varphi_>$ in which we are now interested, which we write in the form of a functional integral

$$D_> = \langle \varphi_> \varphi_> \rangle = \int \mathscr{D} \Phi \varphi_< \varphi_< \exp S(\Phi), \tag{23}$$

where $S(\Phi)$ is the action (9), and all normalizing factors are considered to be included in the differential $\mathscr{D}\Phi = \mathscr{D}\varphi \mathscr{D}\varphi'$. Using the equation

$$\int \mathscr{D}\Phi ... = \int \mathscr{D}\Phi_> \int \mathscr{D}\Phi_< ...$$

we can write the correlator (23) in the form[15]

$$D_> = \int \mathscr{D}\Phi_< D_>(\Phi_<) \int \mathscr{D}\Phi_> \exp S(\Phi), \tag{24}$$

where we have written

$$D_>(\Phi_<) \equiv \frac{\int \mathscr{D}\Phi_> \varphi_> \exp S(\Phi)}{\int \mathscr{D}\Phi_> \exp S(\Phi)}. \tag{25}$$

Thus (25) is the propagator of the hard field in a fixed external soft field, and (24) is the average over a distribution of the soft field taken exactly, i.e., determined by the total action $S(\Phi)$.

We have seen from the example of the single-loop diagrams in Sec. 2 that when calculating the singular contributions we can completely neglect the dependence of the hard lines on the soft momenta. This is equivalent in the coordinate representation to neglecting the dependence of the soft field on the spatial coordinates, which means replacing the averages $\langle \Phi_<(\mathbf{x}_1,t_1)...\Phi_<(\mathbf{x}_n,t_n)\rangle$ that occur in (24) (and they are just the ones that contain all IR singularities) by the averages $\langle \Phi_<(\mathbf{x},t_1)...\Phi_<(\mathbf{x},t_n)\rangle$, which are local in space and independent of the argument $\mathbf{x}$ common to all fields. It can easily be shown that the contributions of the soft field $\varphi'_<$ then vanish in (24) and (25), and the denominator in (25) becomes a constant. The soft field $\varphi_<(\mathbf{x},t)$ can now be replaced by the random quantity $v \equiv v_i(t)$, which depends solely on the time, and whose distribution is given by

$$\begin{aligned}\langle\langle v(t_1)...v(t_n)\rangle\rangle &\equiv \langle \varphi_<(\mathbf{x},t_1)...\varphi_<(\mathbf{x},t_n)\rangle \\ &= \int \mathscr{D}\Phi_< \varphi_<(\mathbf{x},t_1)...\varphi_<(\mathbf{x},t_n) \\ &\quad \times \int \mathscr{D}\Phi_> \exp S(\Phi).\end{aligned} \tag{26}$$

By virtue of what we said above, Eq. (24) takes the form

$$\begin{aligned}\langle \varphi_>(\mathbf{x},t)\varphi_>(\mathbf{x}',t')\rangle = \Big\langle \Big\langle \int \mathscr{D}\Phi_> \varphi_>(\mathbf{x},t)\varphi_>(\mathbf{x}',t') \\ \times \exp[S(\Phi_>) - \varphi'_>(v\partial)\varphi_>] \Big\rangle \Big\rangle,\end{aligned} \tag{27}$$

and the dependence of the hard propagator on the soft field remains only in the last term in the exponent; $\langle\langle ... \rangle\rangle$ indicates the average (26).

We change variables in the functional integral over the hard fields in (27):

$$\Phi_>(\mathbf{x},t) \to \Phi_>(\mathbf{x}-\mathbf{u}(t),t), \quad \mathbf{u}(t) \equiv \int_{-\infty}^{t} dt' \mathbf{v}(t'), \tag{28}$$

where $\mathbf{v}(t)$ is the random quantity (26). Differentiation with respect to the time then produces in $S(\Phi_>)$ a contribution which cancels the $v$-dependent term in the exponent in Eq. (27), and the latter takes the form

$$\begin{aligned}\langle \varphi_>(\mathbf{x},t)\varphi_>(\mathbf{x}',t')\rangle = \Big\langle \Big\langle \int \mathscr{D}\Phi_> \varphi_>(\mathbf{x}-\mathbf{u}(t),t) \\ \times \varphi_>(\mathbf{x}'-\mathbf{u}(t'),t') \exp S(\Phi_>) \Big\rangle \Big\rangle.\end{aligned} \tag{29}$$

By virtue of the translational invariance of the theory, the hard propagator subject to the averaging $\langle\langle ... \rangle\rangle$ in (29) is a function of two arguments, $t-t'$ and $\mathbf{x}-\mathbf{x}'-\mathbf{u}(t)+\mathbf{u}(t')$. The whole of the dependence on the soft field is contained in the second argument, and hence vanishes for a single-time correlator, i.e., for $t=t'$ in (29). All IR singularities contained in the averages of (26) then also vanish, as we set out to prove.

Note that the separating scale $\lambda$, which had to be introduced in the framework of the IRPT as a result of the approximations made there, occurs in Eq. (29) and is canceled only when we take into account the nonsingular contributions that we dropped. The use of the quantum field technique of an operator expansion enables us to renormalize the IRPT without introducing any intermediate separating scale.[19] However, we have proposed a more natural technique in this case that allows a simple interpretation of the results: for instance, one can, in accordance with Ref. 12, consider the representation (29) as describing the transfer of turbulent vortices by the large-scale field $\mathbf{v}(t)$.

We give explicit expressions for the simplest approximation of the IRPT in which the self-action of the soft and the hard fields is neglected. Neglecting in (27) the self-action of the hard field we have:

$$\begin{aligned}\langle \varphi\varphi \rangle = \Big\langle \Big\langle \int \mathscr{D}\Phi_> \varphi_> \varphi_> \exp[S_0(\Phi_>) \\ - \varphi'_>(v\partial)\varphi_>] \Big\rangle \Big\rangle,\end{aligned} \tag{30}$$

where $S_0$ is the part of the action (9) quadratic in the fields. One can easily evaluate the Gaussian integral over $\Phi_>$ in (30), which represents the bare hard propagator in the given external field $v(t)$; in the $\mathbf{p},t$ representation, it gives

$$\langle \varphi\varphi \rangle = \langle \varphi\varphi \rangle_0 \Big\langle \Big\langle \exp i\mathbf{p} \int_t^{t'} \mathbf{v}(\tau) d\tau \Big\rangle \Big\rangle, \tag{31}$$

with $\langle \varphi\varphi \rangle_0$ from (22).

Assuming a Gaussian distribution for $v(t)$, which corresponds to the leading order of the usual perturbation theory in $g=\alpha/\gamma^3$ for the soft averages (26), we get from (31)

$$\langle \varphi\varphi \rangle = \langle \varphi\varphi \rangle_0 \exp H(T,p), \quad T \equiv |t-t'|, \tag{32}$$

where we have written

$$H(T,p) \equiv -p_m p_n \int_t^{t'} d\tau \int_t^{t'} d\tau' \langle\langle v_m(\tau) v_n(\tau') \rangle\rangle. \tag{33}$$

The correlator in (33) depends only on the modulus of the difference $u \equiv |\tau-\tau'|$, which enables us to write (33) as a single integral

$$H(T,p) = -2p_m p_n \int_0^T du (T-u) \langle\langle v_m(\tau) v_n(\tau') \rangle\rangle. \tag{34}$$

Substituting the correlator $\langle \varphi_< \varphi_< \rangle$ in the form (10) into (26), we find

$$\langle\langle v_m(\tau)v_n(\tau')\rangle\rangle=\frac{1}{(2\pi)^d}\int d\mathbf{k}P_{mn}(\mathbf{k})\frac{d_F(k)}{2R_k}e^{-uR_k}$$
$$=\delta_{mn}\frac{\alpha C_d}{2d}\int_m^\lambda\frac{dk}{kR_k}e^{-uR_k},$$
$$u=|\tau-\tau'|. \tag{35}$$

Expression (35) has a finite limit as $\lambda\to\infty$, and its singular part is independent of $\lambda$ and has the form

$$\langle\langle v_m(\tau)v_n(\tau')\rangle\rangle=\delta_{mn}\frac{3\alpha C_d}{4d}\{R_m^{-1}+u\ \ln(uR_m)+...\}.$$

Substituting this expression into (34) and integrating over $u$, we have

$$H(T,p)=-\frac{3\alpha C_d p^2}{4d}\left\{\frac{1}{2}T^2R_m^{-1}+\frac{1}{6}T^3\ln\frac{R_m}{R_p}+...\right\}. \tag{36}$$

Expressions (32) and (36) are the part of the velocity pair correlator that is singular as $m\to 0$ in the IRPT approximation considered. One sees easily that its expansion in $\alpha$ (or equivalently in $g=\alpha/\gamma^3$) reproduces the singular part in the single-loop approximation (22) as given in §2.

For the response function $\langle\varphi\varphi'\rangle$ we have in the same approximation

$$\langle\varphi\varphi'\rangle=\langle\varphi\varphi'\rangle_0\exp H,\quad \langle\varphi\varphi'\rangle_0=\theta(t-t')e^{-R_p(t-t')} \tag{37}$$

with the function $H$ from (36). Expression (37) corresponds to a sum of diagrams denoted in Ref. 7 by $G_s$. The expression for $G_s$ was given in Ref. 7 in the $\omega$,$\mathbf{p}$ representation, but its proof, as far as we know, was not published, so we shall not compare it with (37).

We note that according to (32) and (37), the correlator and the response function decrease as $\exp(-\text{const}|t-t'|^2)$ as $|t-t'|\to\infty$, and the expressions corresponding to them in the $\omega$ representation do not have any singularities in the whole of the complex $\omega$ plane. Such vanishing of the singularities of the Green function in the $\omega$ representation by virtue of the IR singularities is also known in problems about the propagation of waves in media with strongly developed fluctuations, and can be compared with the well known confinement phenomenon.[20] This means, in particular, that the determination of turbulence viscosity via the position of the singularities of the response function, which is sometimes used, makes no sense outside the framework of the usual perturbation theory.

## 4. CONCLUSION

We have thus shown that all contributions to perturbation theory that are singular for small $m$ for the model (5)–(8) vanish when we go over to the single-time correlation functions, so that the latter have a finite limit as $m\to 0$. The energy spectrum thereby has the shape (3) in the model considered. The results obtained here are in agreement with the Kolmogorov–Obukhov theory, whereas experiments[4,5] favor the spectrum (4). We have noted the phenomenological nature and some inconsistency in the model considered, but studies based upon microscopic theories probably also lead to a justification of the Kolmogorov–Obukhov theory in its original formulation (see Refs. 13 and 14). Following Ref. 11, one may assume that Kolmogorov scaling is the exact IR asymptotic behavior of a microtheory. However, if formally small IR corrections decrease as $m\to 0$ sufficiently slowly (such corrections have, for instance, been observed in Ref. 21 in the framework of a micromodel based on the maximum entropy principle), an $m$ dependence will be observed in the inertial range of real turbulence far from its IR boundary.

The author expresses his gratitude to L. Ts. Adzhemyan for a number of useful discussions.

This work was carried out with financial support from the Russian Fund for Fundamental Research (project 93-02-14515).

[1] A. S. Monin and A. M. Yaglom, *Statistical Hydrodynamics*, Vol. 2, Nauka, Moscow (1967) [English translation published by MIT Press].
[2] L. D. Landau and E. M. Lifshitz, *Hyrodynamics*, Nauka, Moscow (1986) [English translation published by Pergamon Press, Oxford].
[3] P. L. Sulem, J.-D. Fournier, and A. Pouquet, Lecture Notes in Physics, Vol. 104, Springer, Berlin (1979).
[4] R. A. Antonia, E. Hopfinger, Y. Gagne, and F. Anselment, Pys. Rev. **A30**, 2704 (1984); R. A Antonia, B. R. Satyaprakash, and A. K. M. F. Hussain, J. Fluid Mech. **119**, 55 (1982); F. Anselmet, Y. Gagne, E. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[5] V. R. Kuznetsov, A. A. Praskovskiĭ, and V. A. Sabel'nikov, Izv. Akad. Nauk SSSR, Ser. Mekh. Zhidk. Gaza No. 6, 51 (1988).
[6] U. Frisch, P. L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978); B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974); H. G. E. Hentschel and I. Procaccia, Phys. Rev. **A27**, 1266 (1983); **A28**, 417 (1983); M. S. Borgas, Phys. Fluids **A4**, 2055 (1992); C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
[7] V. Yakhot, Phys. Rev. **A23**, 1486 (1981); **A24**, 3268 (1981).
[8] V. Yakhot and S. A. Orszag, J. Sci. Comp. **1**, 3 (1986); W. P. Dannevik, V. Yakhot, and S. A. Orszag, Phys. Fluids **30**, 2021 (1987).
[9] É. V. Teodorovich, Zh. Eksp. Teor. Fiz. **102**, 863 (1992) [Sov. Phys. JETP **75**, 472 (1992)].
[10] C. De Dominicis and P. C. Martin, Phys. Rev. **A19**, 419 (1979); Progr. Theor. Phys., Suppl. No. 64, 108 (1978).
[11] L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Teor. Mat. Fiz. **57**, 268 (1983); L. Ts. Adzhemyan, A. N. Vasil'ev, and M. Gnatich, Teor. Mat. Fiz. **74**, 180 (1988).
[12] B. B. Kadomtsev, in *Problems of Plasma Theory*, M. A. Leontovich (ed.), Vol. 4 (1964) [English translation: *Plasma Turbulence*, Academic Press, London (1965)]; R. Kraichnan, Phys. Fluids **7**, 1723 (1964); **8**, 575 (1965); **9**, 1728, 1884 (1966).
[13] V. S. L'vov, Preprint No. 53, Inst. Autom. Electrom., Sib. Branch, Acad. Sci. USSR, Novosibirsk (1977); V. I. Belinicher and V. S. L'vov, Preprint No. 333, Inst. Autom. Electrom., Sib. Branch, Acad. Sci. USSR, Novosibirsk (1986); Zh. Eksp. Teor. Fiz. **93**, 553 (1987) [Sov. Phys. JETP **66**, 314 (1987)]; V. S. Lvov, Phys. Rep. **207**, 1 (1991); É. V. Teodorovich, Preprint No. 388, Inst. Appl. Math., Acad. Sci. USSR, Moscow (1989).
[14] A. V. Tur and V. V. Yanovskiĭ, Preprint No. Pr-1203, Space Res. Inst. USSR, Acad. Sci., Moscow (1986); S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, and V. V. Yanovskiĭ, Dokl. Akad. Nauk SSSR **271**, 856 (1983) [Sov. Phys. Dokl. **28**, 643 (1983)]; S. S. Moiseev, A. V. Tur, and V. V. Yanovskiĭ, Dokl. Akad. Nauk SSSR **279**, 96 (1984) [Sov. Phys. Dokl. **29**, 926 (1984)]; R. Z. Sagdeev, S. S. Moiseev, A. V. Tur, and V. V. Yanovskiĭ, in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics* (R. Z. Sagdeev (ed.), Mir, Moscow (1986).
[15] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Zh. Eksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)]; N. V. Antonov, Zap. Nauchn. Seminar. LOMI **169**, 18 (1988).
[16] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. **A8**, 423 (1973);

H. K. Janssen, Z. Phys. **B23**, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, Z. Phys. **B24**, 113 (1976); C. De Dominicis, J. Phys. (Paris) **37**, Suppl. C1, 247 (1976); R. Phythian, J. Phys. **A10**, 777 (1977); C. De Dominicis and L. Peliti, Phys Rev. **B18**, 353 (1978).
[17] S. Grossmann and E. Schnedler, Z. Phys. **B26**, 307 (1977); E. Levich, Phys. Lett. **A79**, 171 (1980); P. Olla, Phys. Rev. Lett. **67**, 2465 (1991).
[18] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Nauka, Moscow (1976) [English translation published by Interscience, New York]; O. I. Zav'yalov, *Renormalized Feynman Diagrams*, Nauka, Moscow (1979); J. Collins, Renormalization [Russian translation], Mir, Moscow (1988).
[19] N. V. Antonov, Zap. Nauchn. Seminar. LOMI **189**, 15 (1991).
[20] L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Teor. Mat. Fiz. **68**, 323 (1986).
[21] L. Ts. Adzhemyan and M. Yu. Nalimov, Teor. Mat. Fiz. **91**, 294 (1992).

Translated by D. ter Haar