Self-focusing instability of plane solitons and chains of two-dimensional solitons in positive-dispersion media

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Analysis of self-focusing instability of plane solitons and chains of two-dimensional solitons in the framework of the Kadomtsev–Petviashvili equation for positive-dispersion media is presented. Exact solutions of this problem by the inverse scattering method are obtained. It is shown that in the simplest case, a chain of two-dimensional solitons and a new plane soliton are created as a result of the initial plane soliton decay. Decay processes are shown to be caused by specific resonant interaction of two-dimensional quasi-plane structures when the phase shift of their trajectories tends to infinity. A new type of wave synchronism condition is established for this case.

1. INTRODUCTION

At present it is well-known that in isotropic nonlinear media the development of multidimensional perturbations depends essentially on the form of the dispersion. This fact was first discovered by B. B. Kadomtsev and V. I. Petviashvili,[1] who obtained an equation (later called the Kadomtsev-Petviashvili (KP) equation) describing quasi-one-dimensional waves and generalizing the classical Korteweg–de Vries (KdV) equation, and revealed that in some cases plane solitons are unstable with respect to self-focusing processes. Moreover, soliton stability or instability depends directly on the sign of the dispersion: in negative-dispersion media (phase velocity of linear perturbations decreases with increasing wave number) solitons are stable, while in positive-dispersion media they are unstable. (To be precise, the instability of plane solitons is determined by the the way the spectrum decays for small perturbations. The spectrum is really decaying in isotropic homogeneous media with weak positive dispersion. In anisotropic media, however, the spectrum may be decaying even for negative dispersion and all plane solitons are unstable there.[2]) Note that in the one-dimensional case the form of the dispersion is not essential: the KdV equation can be reduced to standard form by renormalization, independently of the signs of its coefficients.[3]

The linear and nonlinear stages of the self-focusing instability in the framework of the KP1-equation describing waves in positive-dispersion media have been studied by many workers (see, e.g., Refs. 4–7). In particular, Zakharov[4] claimed that a plane soliton decays as a result of the excitation at its front of small oscillations which move faster than the soliton, overtake it and take away some energy that is scattered in space by dispersive effects. However, our study of this problem shows that the nonlinear development of small perturbations at the soliton front gives rise, in an intermediate stage of instability, to periodic chains of two-dimensional (2D) solitons which have larger amplitude and smaller velocity than the initial plane soliton. The latter decreases in amplitude and moves faster after losing some of its energy. (The first stage of this process is clearly seen in the numerical simulations of Ref. 6.)

It should be mentioned that chains of 2D solitons are also unstable against deformation of their fronts. This was discovered by Burtsev,[8] who constructed a growing mode of the discrete spectrum of a chain. As we shall show, if this mode is given as an initial perturbation, the original chain decays into two others with greater separations between the 2D solitons. We can therefore expect that the development of unstable quasiplane solitary structures under the action of real, disordered perturbations gives rise to a disordered ensemble of two-dimensional solitons, the role of which in the evolution of the wave field in media with positive dispersion has been unclear until now (see Ref. 9).

In addition, the analysis presented in this paper shows that the plane soliton instability is closely associated with the resonant interaction between plane solitons and chains of 2D solitons when the space phase shift of their trajectories grows without bound. The soliton resonance is well-known for negative-dispersion media (for the KP2-equation);[7,10] the simplest example of such resonance is the exact solution found in Ref. 10 which describes a triad of plane solitons at certain angles to each other. Until now it was thought that there is no soliton resonance in the KP1-equation.[7] However, as will be shown below, the soliton resonance is possible in this model too and plays an important role in the decay of quasiplane structures.

2. PLANE SOLITON DECAY

As has been shown by various methods,[11–13] the broad class of solutions to the KP1 equation, which we shall treat in the normalization

$$\partial_x(4\partial_t u+6u\partial_x u+\partial_x^3 u)=3\partial_y^2 u, \tag{2.1}$$

can be expressed in determinant form:

$$u(x,y,t)=2\partial_x^2 \ln \Phi,$$

$$\Phi(x,y,t)=\det\Big|\delta_{nm}+\int_{-\infty}^{x}\Psi_n^+(x',y,t)$$

$$\times\Psi_m^-(x',y,t)dx'\Bigg|, \quad (2.2)$$

where $n,m=1,2,...,N$, and $\Psi_n^\pm$ are solutions of the set of linear equations

$$\partial_t\Psi_n^\pm+\partial_x^3\Psi_n^\pm=0, \quad i\partial_y\Psi_n^\pm\pm\partial_x^2\Psi_n^\pm=0.$$

These solutions can be written in integral form:

$$\Psi_n^\pm=\int_0^\infty S_n^\pm(p)\exp(px\pm ip^2y-p^3t)dp, \quad (2.3)$$

where $S_n^\pm(p)$ is an arbitrary function of p. In particular, one can easily find soliton solutions for $S_n^\pm(p)\propto\delta(p-p_{n0}^\pm)$, which corresponds to the following choice of the functions $\Psi_n^\pm$:

$$\Psi_{n0}^\pm\propto\exp[p_{n0}^\pm x\pm i(p_{n0}^\pm)^2y-(p_{n0}^\pm)^3t+\gamma_{n0}], \quad (2.4)$$

where γ_{n0} is phase constant.

Analysis of the Zakharov–Shabat theory[11] presented by Newell and Redekopp[10] revealed that under certain conditions the solution determined by the functions (2.4) formally diverges. Actually, however, this divergence can be eliminated by proper renormalization of the phase constants γ_{n0}. Then the corresponding solution describes degenerate plane-soliton interaction processes in the case of the KP2 equation. Following the approach used by Zakharov,[4] we shall take into account such processes not by renormalizing the constants γ_{n0}, but by choosing the functions $\Psi_n^\pm$ in the form

$$\Psi_n^\pm=\sum_{m=0}^{M_n^\pm}C_m^\pm\exp[p_{nm}^\pm x\pm i(p_{nm}^\pm)^2y-(p_{nm}^\pm)^3t], \quad (2.5a)$$

which corresponds to the following kernel $S_n^\pm(p)$ in the integral (2.3):

$$S_n^\pm(p)=\sum_{m=0}^{M_n^\pm}C_m^\pm\delta(p-p_{nm}^\pm). \quad (2.5b)$$

The solutions (2.2), (2.5), as will become clear, describe not only the usual collision between plane solitons and chains of two-dimensional solitons, but, also their resonant (degenerate) interaction. The physical meaning of these solutions becomes obvious from analysis of the simplest case $N=1$, $\Psi_1^-=\Psi_1^+$. Simplifying the calculations, we shall assume that the solitons move only in the x direction with fronts parallel to the y axis; then $p_{1n}\equiv p_n$ are real parameters. In this case the solution (2.2), (2.5a) is as follows:

$$\Phi=1+\sum_{n=0}^{M}C_n^2\exp(2p_n\eta_n)$$
$$+\sum_{n=0}^{M}\sum_{m>n}\frac{4\sqrt{p_mp_n}C_mC_n}{p_m+p_n}\cos[(p_m^2-p_n^2)y]$$
$$\times\exp[(p_m+p_n)\mu_{mn}], \quad (2.6)$$

where $\eta_n=x-p_n^2t$ and $\mu_{mn}=x-(p_m^2+p_n^2-p_mp_n)t$.

We order the constants p_n according to

$$p_0>p_1>p_2>...>p_M>0. \quad (2.7)$$

Then it is not difficult to see that in the limit $t\to-\infty$ the solution (2.6) describes a unique plane soliton which is determined by the parameter p_0:

$$\Phi_p(\eta_0;p_0)=1+\exp(2p_0\eta_0);$$
$$u_0(\eta_0;p_0)=2p_0^2/\text{ch}^2(p_0\eta_0). \quad (2.8)$$

Let $C_n=\varepsilon\tilde{C}_n$, where $\varepsilon\ll1$, $n=1,2,...,M$. Then for $t\sim0$ the solution (2.6) can be regarded as a perturbed plane soliton (2.8). Expanding $u(x,y,t)$ in powers of ε, in the first-order approximation we obtain the perturbation to the plane soliton expressed by Eq. (2.8) in the form

$$u_1(\eta_0,y,t)=\sum_{n=0}^{M}\frac{2\sqrt{p_mp_n}\tilde{C}_n}{p_0+p_n}\cos(\varkappa_ny)\exp(\lambda_nt)$$
$$\times W(\eta_0;p_0,p_n), \quad (2.9)$$

where $\varkappa_n=p_0^2-p_n^2$, $\lambda_n=p_n\varkappa_n>0$ and

$$W(x;p_0,p_n)=\frac{\partial^2}{\partial x^2}\left(\frac{\exp(p_nx)}{\text{ch}(p_0x)}\right).$$

The perturbation (2.9) corresponds exactly to linear superposition of M discrete modes of the problem obtained by linearizing about the solution (2.8). Here the parameters λ_n and $\varkappa_n$ determine the growth rate of the perturbations and the period of the transversal modulation.[4] Thus, for $t>0$ the complete solution (2.6) describes the nonlinear stage of instability development for a plane soliton perturbed by M modes of the discrete spectrum.

Note that the mode $W(x;p_0,p_n)$ is localized under the condition (2.7), but the degree of localization at $x\to+\infty$ ($W\propto\exp[-(p_0-p_n)x]$) is less than at $x\to-\infty$ ($W\propto\exp[(p_0+p_n)x]$). This means that for $t>0$ "radiation" propagates from plane soliton towards $x>0$. However, as follows from (2.1), small-amplitude wave perturbations move in the opposite direction ($x<0$) in the reference frame used. Hence, as a result of the plane soliton instability, nonlinear solitary structures rather than linear dispersive waves are formed.

2.1. The elementary decay of a plane soliton

Here we shall study the elementary process of plane soliton decay for a one-mode perturbation ($M=1$). It is easy to see that as $t\to+\infty$ the solution (2.6) is nontrivial only along two curves:

1) $\eta_1=x_1-p_1^2t$, for which

$$\Phi\propto1+\exp(2p_1\eta_1)=\Phi_p(\eta_1;p_1); \quad (2.10a)$$

and

2) $\theta_{01}=x-(p_0^2+p_1^2+p_0p_1)t$, for which

$$\Phi\propto\Phi_r(\theta_{01};p_0,p_1), \quad (2.10b)$$

where

$$\Phi_r(\theta_{01};p_0,p_1)\equiv\text{ch}[(p_0-p_1)\theta_{01}]$$
$$+\frac{2\sqrt{p_0p_1}}{p_0+p_1}\cos[(p_0^2-p_1^2)y].$$

A plane soliton having smaller amplitude and velocity than the original one moves along the first curve. Along the second curve a nonlinear wave propagates which is periodic in y and localized in x. It was discovered in the Refs. 14 and 15. In Ref. 14 it was shown that this solution can be regarded as a chain of two-dimensional solitons which are localized in all directions and vanish at infinity according to a power law $\propto (x^2+y^2)^{-1}$ (Ref. 16). Thus, for a periodic single-mode perturbation plane soliton self-focusing results in the formation of a 2D soliton chain located parallel to the front of the original wave at a distance equal to the period of initial modulation. This process is shown in Fig. 1.

The properties of the solution (2.10b) indicate that there is a threshold level of transversal modulation period located within the instability region ($\lambda_1>0$). At one boundary of this region where $p_1\sim p_0$, $\varkappa_1\sim 0$, the solution (2.10b) describes 2D solitons spaced far apart ($\sim 1/\varkappa_1$). The parameters (amplitude, velocity, half-width) differ little between the final and initial plane solitons.

At the other boundary where $p_1\sim 0$, $\varkappa_1\sim p_0^2$, the distance between the 2D solitons in the solution (2.10b) is so small that they merge to reconstruct the initial almost plane soliton (2.8) with a small residual amplitude modulation proportional to $\sqrt{p_1/p_0}$. The amplitude of the other plane soliton (2.10a) is infinitesimal, $\sim p_1^2$. Hence the complete annihilation of the initial plane soliton, which was claimed in Ref. 4, never occurs.

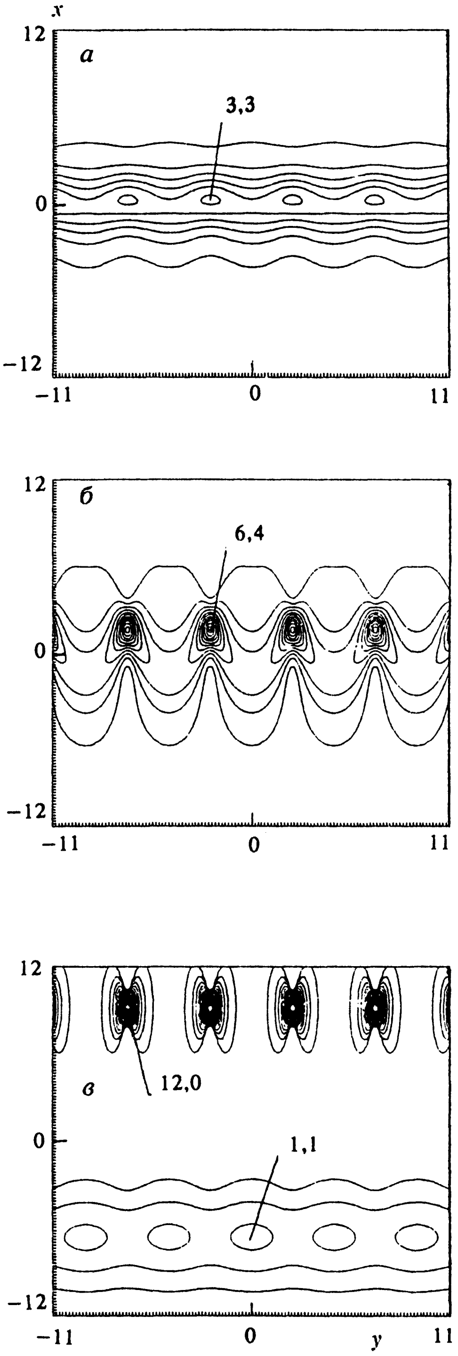

FIG. 1. The birth of 2D soliton chain as a result of the plane soliton instability described by Eq. (2.6) for $M=1$: a—$t=0$; b—$t=10$; c—$t=20$.

2.2. Nonlinear interaction of discrete modes

The period of a single-mode initial perturbation determines the distance between solitons in the chain formed by the self-focusing process. Now we shall consider a more complicated process of plane soliton evolution under the action of a quasiperiodic perturbation. The corresponding solution follows from Eq. (2.6) at $M=2$.

If a superposition of modes (2.9) with the wave numbers $\varkappa_1$, $\varkappa_2$ is specified initially, their nonlinear interaction gives rise to a perturbation with the wave number $k_2=\varkappa_2-\varkappa_1$. As a result, besides the 2D soliton chain $\Phi_r(\theta_0;p_0,p_1)$ and the plane soliton with the amplitude $2p_2^2$, which is smaller than for the plane soliton (2.10a), another 2D soliton chain $\Phi_r(\theta_{12};p_1,p_2)$ forms with a period which does not coincide with any period of the initial perturbation. Although the result of plane soliton decay in the case of the two-mode perturbation, as in the other cases, is determined by the relation (2.7), the intermediate picture depends on the ratio of the growth rates of the two modes.

If $\lambda_1>\lambda_2$ holds, 2D soliton chains are formed due to the subsequent single-mode decay of the plane soliton $\Phi_p(\eta_0;p_0)$ first to the chain $\Phi_r(\theta_{01};p_0,p_1)$ and the plane soliton $\Phi_p(\eta_1;p_1)$, and then through secondary decay of the latter to the chain $\Phi_r(\theta_{12};p_1,p_2)$ and the plane soliton $\Phi_p(\eta_2;p_2)$ (Fig. 2a–f). Alternatively, if $\lambda_1<\lambda_2$, holds, a perturbed chain with the wave number $\varkappa_2$ first is separated in the course of the primary decay from the plane soliton with the amplitude $2p_2^2$, and then decays into two chains $\Phi_r(\theta_{01};p_0,p_1)$ and $\Phi_r(\theta_{12};p_1,p_2)$ (Fig. 2a–f). Thus, the wavenumber displacement of the second soliton chain $\Phi_r(\theta_{12};p_1,p_2)$ is accounted for by the two stages of decay and formation of metastable soliton structure in the intermediate stage. It should be added that for $\lambda_1\approx\lambda_2$ both stages of the plane soliton decay take place at the same time, and no metastable structure can be distinguished in the intermediate stage.

2.3. Plane soliton decay for an *M*-mode perturbation

The results presented above can be generalized for an initial perturbation with M discrete modes. In this case, the

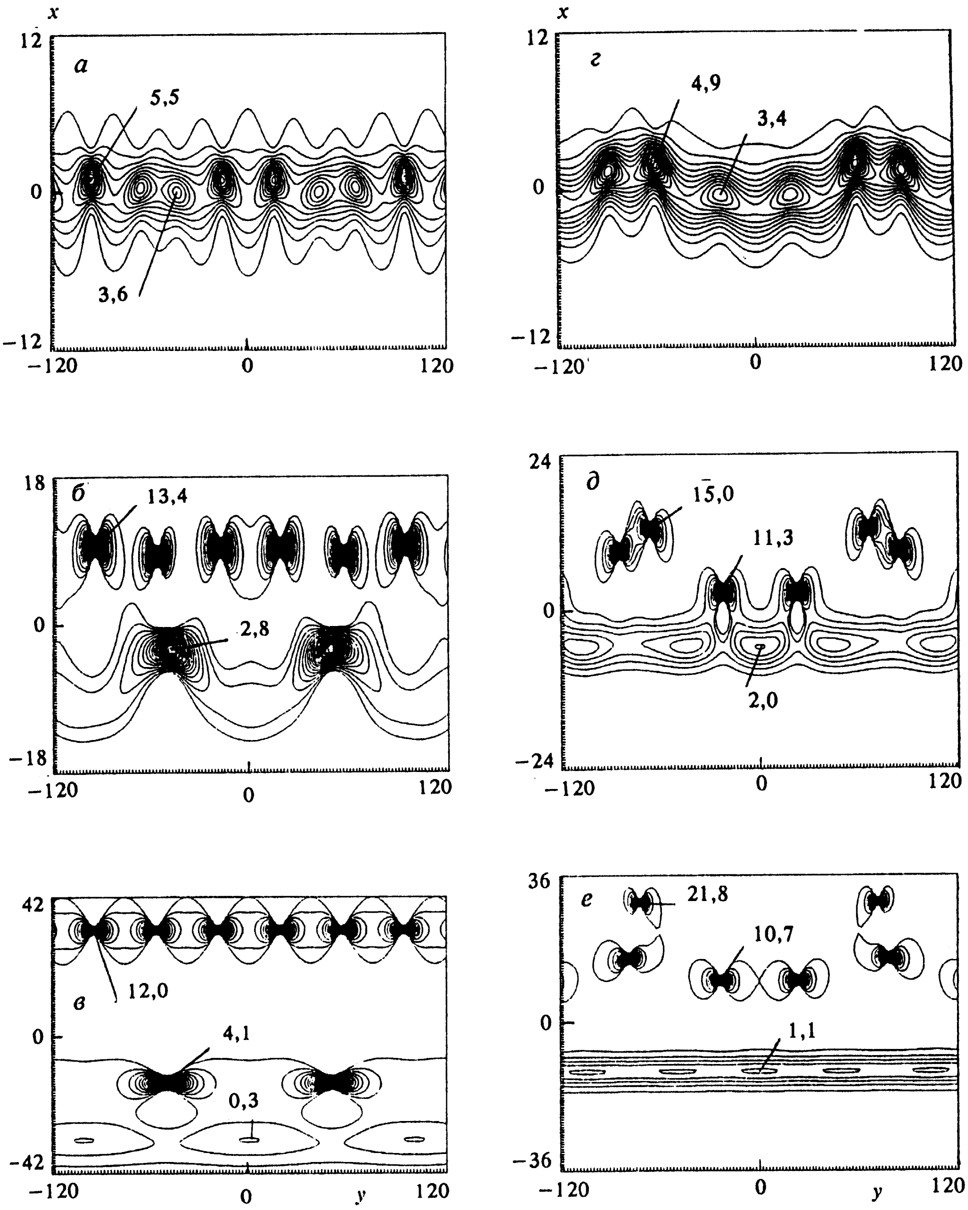

FIG. 2. The plane soliton decay for a two-period initial perturbation for $\lambda_1>\lambda_2$: a—$t=0$; b—$t=15$; c—$t=40$, but for $\lambda_2>\lambda_1$; d—$t=0$; e—$t=10$; f—$t=25$.

development of the self-focusing instability gives rise to M parallel chains of two-dimensional solitons $\Phi_r(\theta_{n-1,n};p_{n-1},p_n)$ propagating one after another with transverse wave numbers $k_n=\varkappa_n-\varkappa_{n-1}=p_{n-1}^2-p_n^2$ and velocities $V_n=p_n^2+p_{n-1}^2+p_np_{n-1}$, $n=1,2,\ldots,M$, and to a single plane soliton $\Phi_p(\eta_M;p_M)$ with the smallest possible amplitude and velocity which are determined by the parameter p_M (see 2.7).

To prove this statement we multiply the expression (2.6) by $\exp[-(p_k+p_l)\mu_{kl}]$ and choose a reference frame moving with the velocity of a certain fixed chain $\Phi_r(\theta_{kl};p_k,p_l)$, $k<l$. Then the solution (2.6) assumes the form

$$\Phi(\theta_{kl},y,t)=\exp[-(p_k+p_l)\theta_{kl}-2p_kp_l(p_k+p_l)t] + \sum_{n=0}^{M} C_n^2\exp[(2p_n-p_k-p_l)\theta_{kl} -2\sigma_nt] + \sum_{n=0}^{M}\sum_{m>n}\frac{4\sqrt{p_mp_n}C_mC_n}{p_m+p_n} \times\cos[(p_m^2-p_n^2)y]\exp(\varphi_{klmn}), \qquad (2.11)$$

where $\varphi_{klmn}=(p_m+p_n-p_k-p_l)\theta_{kl}-(\sigma_k+\sigma_l)t$ and $\sigma_n=(p_l-p_n)(p_k-p_n)(p_l+p_k+p_n)$.

One can easily see that $\sigma_n\geqslant 0$ holds for arbitrary n only if $l=k+1$. Therefore for $l\neq k+1$ we have

$$\lim_{t\to\infty}\Phi(\theta_{kl},y,t)=\infty.$$

From formula (2.2) it follows that the field $u(x,y,t)$ tends to zero asymptotically as $t\to+\infty$ on the corresponding characteristic θ_{kl}. Hence, a nontrivial solution is possible

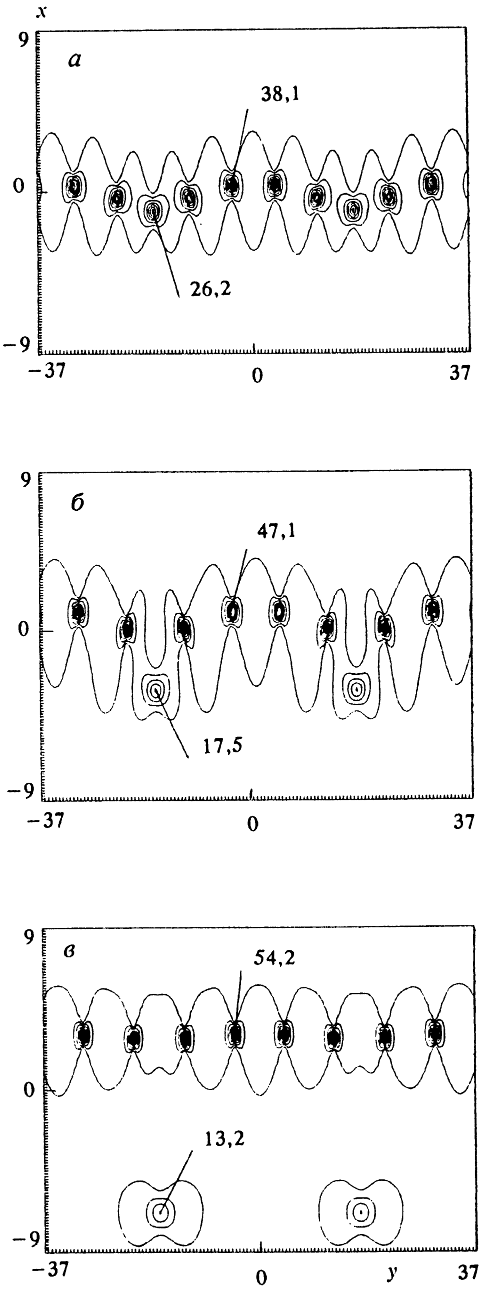

FIG. 3. The decay of 2D soliton chain described by Eq. (3.1): a) $t=0$; b) $t=10$; c) $t=20$.

only on the characteristic curve $\theta_{k,k+1}$ for $k=1,2,...,M$, along which the soliton chains $\Phi_r(\theta_{k,k+1};p_k,p_{k+1})$ propagate.

In order to find the form of the plane soliton that emerges as $t\to+\infty$, we choose the reference frame with the coordinate η_k in the solution (2.6):

$$\Phi(\eta_k,y,t)=1+\sum_{n=0}^{M} C_n^2 \exp\{2p_n[\eta_k+(p_k^2-p_n^2)t]\}$$
$$+\sum_{n=0}^{M}\sum_{m>n}\frac{4\sqrt{p_m p_n}C_m C_n}{p_m+p_n}$$
$$\times\cos[(p_m^2-p_n^2)y]\exp(\phi_{kmn}), \qquad (2.12)$$

where $\phi_{kmn}=(p_m+p_n)[\eta_k+(p_k^2-p_m^2+p_n p_m-p_n^2)t]$.

Hence, the solution $\Phi(\eta_k,y,t)$ is nontrivial in the limit $t\to+\infty$ only on the characteristic η_M, where the plane soliton $\Phi_p(\eta_M;p_M)$ with the smallest possible amplitude and velocity propagates.

Thus, the process of plane soliton decay for an M-mode perturbation is an M-fold decay of one of the two structures (either the perturbed plane soliton or the perturbed 2D soliton chain) formed at each intermediate stage. The specifics of this process depend on the ratio of the growth rates of the unstable modes. Consequently, M new characteristics of arising soliton chains split off from the characteristic of the original plane soliton.

To conclude this section we would like to point out that we have presented analysis of the solutions describing plane soliton self-focusing for the simplest perturbation type which is determined by formula (2.5). However, the results can be generalized when the kernel of the integral (2.3) is expressed by derivations of order k of the δ-function: $S_1(p)\propto\delta^{(k)}(p-p_m)$, $p_m<p_0$. A similar analysis shows that as $t\to+\infty$ the corresponding initial perturbations give rise not only to the structures described above (plane soliton and 2D soliton chains), but also to single two-dimensional solitons following the soliton chains equal in number to the order k of the derivative. Obviously this self-focusing mechanism for an arbitrary perturbation specified by the function $S_1(p)$ in general form will give rise to a disordered ensemble of individual two-dimensional solitons distinct from the original plane soliton.

3. SOLITON CHAIN DECAY

Here we shall consider the decay of a nonlinear wave periodic in y and localized in the direction of propagation. The corresponding solution can be constructed in explicit form, if in (2.2), (2.5) we take $N=2$, $\Phi_1^-=\bar{\Phi}_2^+$, $\Phi_2^-=\bar{\Phi}_1^+$, $M_1=0$, $M_2=1$; $p_{10}\equiv p$, $p_{20}\equiv q$, $p_{21}\equiv s$. As a result, we have the following solution:

$$\Phi=\Phi_r(\theta_{qp};q,p)+\varepsilon\cos[(p^2-s^2)y]\exp[(s-q)\theta_{sq}]$$
$$+\varepsilon\sqrt{\frac{q}{p}}\frac{s+p}{s+q}\cos[(q^2-s^2)y]\exp[(s-p)\theta_{sp}]$$
$$+\varepsilon^2\frac{(s+p)^2}{4ps}\exp[(s-p)\theta_{sp}+(s-q)\theta_{sq}]. \qquad (3.1)$$

As is easily seen, the soliton chain $\Phi_r(\theta_{qp};q,p)$ which is asymptotically free for $p<s<q$ as $t\to-\infty$, begins to decay at $t\sim0$ into two chains $\Phi_r(\theta_{sp};s,p)$ and $\Phi_r(\theta_{qs};q,s)$ which are free again at $t\to+\infty$. The dynamics of this process is shown in Fig. 3.

The original chain decays on account of the growing mode of the discrete spectrum, which was first found by Burfsev.[8] This mode can be found from (3.1) by neglecting terms $\sim O(\varepsilon^2)$, and is a two-periodic perturbation of 2D soliton chain with a certain phase ratio which distinguishes it from the 2-mode perturbation (2.9). The transverse wave numbers of this perturbation determine the periods of the chains that form in the decay process. From (3.1) it follows that these wave numbers are related by

$$k_{y3}=k_{y1}+k_{y2} \tag{3.2}$$

because

$$k_{y1}=s^2-p^2, \quad k_{y2}=q^2-s^2, \quad k_{y3}=q^2-p^2.$$

The relation (3.2) corresponds to the conservation law for the spatial density of the 2D solitons of which the original nonlinear wave consists. As a result, the breakdown of the soliton chain is not accompanied by merging of individual 2D solitons (Fig. 3).

The relation (3.2) is known in the theory of weak nonlinear waves[17] as the synchronism condition for the y-projection of the wave vector of the quasimonochromatic waves which are assumed to be of the form $\exp[i(k_x x+k_y y-\omega_{ln}t)]$. Together with analogous conditions for k_x and $\omega_{ln}(k_x,k_y)$ they determine the wave resonance at which a wave with parameters $(\omega_{ln3},k_{x3},k_{y3})$ decays into two other waves $(\omega_{ln1},k_{x1},k_{y1})$ and $(\omega_{ln2},k_{x2},k_{y2})$, or alternatively, two waves merge into the third wave.

The dispersion relation of a linear perturbation for the KP1 equation (2.1) is

$$\omega_{ln}(k_x,k_y)=-\frac{k_x^4+3k_y^2}{4k_x}. \tag{3.3}$$

This is a decaying dispersion relation,[7] i.e., the conditions of three-wave synchronism are met. A similar property also holds for strongly nonlinear periodic waves.

Specifically, introducing the "x-projection of the wave number" k_x and the "frequency" $\omega_{nl}(k_x,k_y)$ in the exponential phase of the soliton chains $\Phi_r(\theta_{qp};q,p)$, $\Phi_r(\theta_{sp};s,p)$ and $\Phi_r(\theta_{qs};q,s)$ in the form

$$k_{x1}=s-p, \quad k_{x2}=q-s, \quad k_{x3}=q-p;$$

$$\omega_{nl1}=s^3-p^3, \quad \omega_{nl2}=q^3-s^3, \quad \omega_{nl3}=q^3-p^3, \tag{3.4}$$

we can easily verify that the parameters $k_x,k_y,\omega_{nl}(k_x,k_y)$ are related by the same formula (3.3), with the replacement $\omega_{ln}\rightarrow-\omega_{nl}$. Consequently, for the decay of a soliton chain, the parameters of the original and two new chains obey the same conditions of three-way synchronism as in the linear case:

$$\omega_{nl}(\mathbf{k}_1+\mathbf{k}_2)=\omega_{nl}(\mathbf{k}_1)+\omega_{nl}(\mathbf{k}_2). \tag{3.5}$$

So, all decay processes which occur as a result of the quasilinear wave interaction within the KP1 equation are also observed for nonlinear solitary waves.

4. THE RESONANCE OF PLANE SOLITONS AS A CAUSE OF THEIR DECAY

The projection of the wave vector k_y might seem to be equal to zero for plane solitons parallel to the y-axis. Consequently, the synchronism condition in the form (3.2) does not hold for interactions between plane solitons and 2D soliton chains. Actually, however, in this case nonlinear wave synchronism is possible too, but it has some peculiarities. We shall study this process by considering the interaction of one plane soliton and one steady-state nonlinear wave periodic along y. For simplicity the fronts of both waves are supposed to be parallel to the y-axis. The corresponding solution can be obtained from the general formulas (2.2) and (2.4) with $N=3$:

$$\begin{aligned}\Phi=&1+\exp[2(p_0x-\omega_0t)]+2\cos(k_yy)\exp[(p_0+p_1)x\\&-(\omega_0+\omega_1)t]+a\exp[2(p_1x-\omega_1t)]\\&+2b\cos(k_yy)\exp[-(p_2x-\omega_2t)]\\&+ab^2\exp[-2(p_2x-\omega_2t)],\end{aligned} \tag{4.2}$$

where

$$\omega_0=p_0^3, \quad \omega_2=\frac{p_2^4+3k_y^2}{4p_2}, \quad p_1=p_0-p_2, \quad \omega_1=\omega_0-\omega_2,$$

$$a=\frac{k_y^2}{k_y^2-p_2^4}, \quad \text{and} \quad b=\frac{k_y^2-p_2^2(2p_0-p_2)^2}{k_y^2-p_2^2(2p_0+p_2)^2}.$$

For $0<b<1$ this solution describes the collision between a plane soliton with the parameters p_0,ω_0 and a chain of 2D solitons with the parameters p_2,k_y,ω_2. After interaction at $t\rightarrow+\infty$ both the waves retain their initial amplitudes and velocities, but acquire the phase shift $\pm\delta=\ln b$.

The infinite increase in the phase shift, as $b\rightarrow0$, is due to instability of the plane soliton and its decay before interactly closely with the 2D soliton chain. This decay gives rise to a new soliton chain displaced relative to the original one by $+\delta$. Besides, a "virtual"[18] plane soliton also occurs with parameters p_1,ω_1 which do not obey the "dispersion relation" for plane solitons ($\omega_1\neq p_1^3$). As some time, it begins to interact with the original soliton chain, which results in their merging and in the formation of a plane soliton with parameters p_0,ω_0, but phase shifted by $-\delta$.

The process of plane soliton decay in a pure form, which was analyzed in 2.1, is observed at $b=0$. In this case we have $\omega_1=p_1^3$, i.e., the "virtual" soliton becomes a real plane soliton of the wave field and the original soliton chain with the parameters p_2,k_y,ω_2 is located at infinite distance from plane soliton. Hence, the condition of wave resonance in this case reduces to the following: Besides the known relation for "frequencies" ω and "wave numbers" p,

$$\omega_0=\omega_1+\omega_2, \quad p_0=p_1+p_2, \tag{4.3}$$

the additional condition for an infinite phase shift,

$$k_y=p_0^2-p_1^2 \tag{4.4}$$

must be met too.

This is a specific feature of resonant processes in the interaction of two-dimensional structures of different kinds (plane solitons and 2D soliton chains). Formally, as is clear from the analysis presented in Sec. 2, we can consider a plane soliton as a particular case of a general 2D soliton chain if the transverse wave number k_y depends on the longitudinal wave number p as $k_y=p^2$. Then relation (4.4) coincides exactly with the synchronism condition (3.2) for soliton chains. Apparently, this formal similarity allows us to treat all decaying processes of quasiplane nonlinear waves in positive-dispersion media in terms of the concept of resonance of 2D soliton chains.

5. CONCLUDING REMARKS

The analysis of decaying processes of the simplest solitary waves presented in our paper allows us to understand in novel way the problem of plane-wave instability in positive-dispersion media. As nonlinear quasiplanar structures decay, the energy is not lost on small oscillations of the medium. Instead, it is condensed in some two-dimensional and plane solitons. Since the considered model is conservative, these processes are reversible, i.e., soliton merging is possible, too.

The appearance of a 2D soliton lattice as a result of plane perturbation self-focusing enables us to relate the decaying character of the spectrum of linear and nonlinear waves to the existence of a solitary solution in the form of a two-dimensional soliton. It can serve a convenient criterion for the existence of such solutions. In the description of 2D nonlinear perturbation dynamics these solitons will probably play the same role as solitons in one-dimensional evolution equations.

Although all the results obtained in this paper are based on the integrability of the KP1 equation we believe that for other similar models with positive dispersion[19–21] where the instability of plane soliton relative to self-focusing processes was discovered, a similar plane soliton decay may be observed. However, the question of the origin of small oscillations of the wave medium which can accompany soliton decay needs independent investigation.

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