Quantum limit on force measurements

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During continuous measurements of the coordinate of a test oscillator, the inverse fluctuation effect does not impose a lower limit on forces which can be observed: There is no standard quantum limit for forces. The error of force measurements decreases with increasing pump level. The pump can be in a coherent state; there is no need for a squeezed, energetic, or other nonclassical state. Dissipation in a mechanical oscillator (at absolute zero) leads to the existence of an optimum power, at which the minimum force is given by a quantum Nyquist formula, in which the energy of zero-point fluctuations, $\hbar\Omega/2$, is replaced by $\hbar\Omega$.

1. INTRODUCTION

The quantum noise of a displacement sensor is a key problem in an interference gravitational-radiation antenna (we have the LIGO project in mind) as well as in several other fundamental experiments. The sensitivity of the sensor is determined by the inverse fluctuation effect on the object.[1–3] In the case of *continuous* measurements, this fluctuation effect leads to a standard quantum limit, $\Delta X_{\rm SQL}=(\hbar/2m\omega_M)^{1/2}$ for an oscillator (m and ω_M are the mass and frequency of the oscillator) and $\Delta X_{\rm SQL}=(\hbar\tau/m)^{1/2}$ for a free mass (τ is the observation time). The standard quantum limit is reached at an optimum pump power level. The standard quantum limit is believed to also determine a minimum observable force (again, during *continuous* measurements of the coordinate).

In the present paper we show that during *continuous* measurements of a coordinate the inverse fluctuation effect does not impose a lower limit on forces which can be observed: There is no standard quantum limit for forces. We use the example of a very simple optical sensor. The error of force measurements decreases with increasing pump level. The pump can be in a coherent state; there is no need for a squeezed, energetic, or other nonclassical state. We also show that dissipation in a mechanical oscillator (at absolute zero) leads to the existence of an optimum power, at which the minimum force is governed by a quantum Nyquist formula [Eq. (13) below] in which the energy of zero-point fluctuations, $\hbar\Omega/2$, is replaced by $\hbar\Omega$.

To see how the standard quantum limit is obtained, we consider the example of the optical sensor in Fig. 1. The displacement X of a mechanical oscillator changes the phase of the reflected wave by an amount $\Delta\phi=2\omega_0X/c$ (ω_0 is the average pump frequency; c is the velocity of light). If the incident wave is in a coherent state (with an average energy $\hbar\omega_0/n$ and a phase uncertainty $\Delta\phi_{\rm coh}\simeq 1/2\sqrt{n}$, $n\gg1$), the error of a coordinate measurement is $\Delta X_{\rm meas}\simeq c/(4\omega_0\sqrt{n})\sim 1/\sqrt{n}$. The inverse effect in this case reduces to a fluctuation of the radiation-pressure force acting on the oscillator: $F_{LP}\sim\sim\sqrt{n}$. *This fluctuation causes a coordinate perturbation* $\Delta X_{\rm inf}\simeq 2\hbar\omega_0\sqrt{n}/cm\omega_M$. An optimization of the sum $(\Delta X_{\rm meas})^2+(\Delta X_{\rm inf})^2$ with respect to n then gives us $\Delta X_{\rm SQL}$ at the optimum pump level $n=n_{\rm opt}$.

It would seem that an increase in the pump level ($n>n_{\rm opt}$) would lead to simply a strong perturbation, which would "paint over" the useful signal. However, this is not the case. Let us assume that the incident wave is in a coherent state and that the fluctuations of the amplitude and phase are uncorrelated. The radiation-pressure force is proportional to the amplitude, so the amplitude fluctuations convert into phase fluctuations in the reflected wave: An amplitude-phase correlation arises. This result means that the reflected wave is in a *squeezed* state (Fig. 2). The very same radiation-pressure mechanism which perturbs the oscillator thus leads to a *squeezed* reflected wave. If the external agent acting on the oscillator now corresponds to a phase shift $\Delta\phi_{\rm coh}$ of the reflected wave (as shown by the dashed lines in Fig. 2), this agent will be observable through observation of a squeezed quadrature component B_ψ (not of the phase). The latter can be measured destructively in a balanced homodyne arrangement. Detection of the squeezed quadrature component allows the instrument to "not see" the perturbation which it introduces and to detect the external force within an error determined by $\Delta X_{\rm meas}$.

The error of force measurements depends on only the *initial* uncertainty of the phase in the incident wave. If the pump is in an ultracoherent state ($\Delta\phi\ll\Delta\phi_{\rm coh}$), the accuracy is better.

If a disturbance of the initial uncertainties in the coordinate and momentum of the oscillator is to be avoided, the frequency of the external force, ω_F, must not be equal to the resonant frequency of the oscillator: $|\omega_F-\omega_M|\tau\gg1$ (for a free mass, $\omega_F\tau\gg1$).

For the optimum detection of an external force within an error below the standard quantum limit, the noise of the instrument must be correlated in a special way.[3] But this requirement means that the pump is in a squeezed, not coherent, state.[4] In this paper we show that a preliminary squeezing is not necessary.

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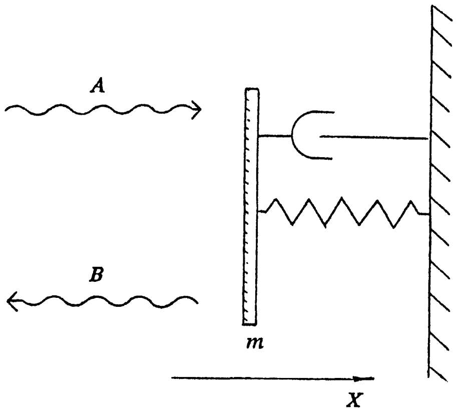

FIG. 1. Optical instrument for detecting mechanical displacements. A plane electromagnetic wave is incident normally on the surface of an ideal plane mirror. The external force changes the mirror coordinate X. This change causes a change $\Delta\phi_S=2\omega_0X/c$ in the phase of the reflected wave (ω_0 is the average pump frequency, and c is the velocity of light). This change is to be measured.

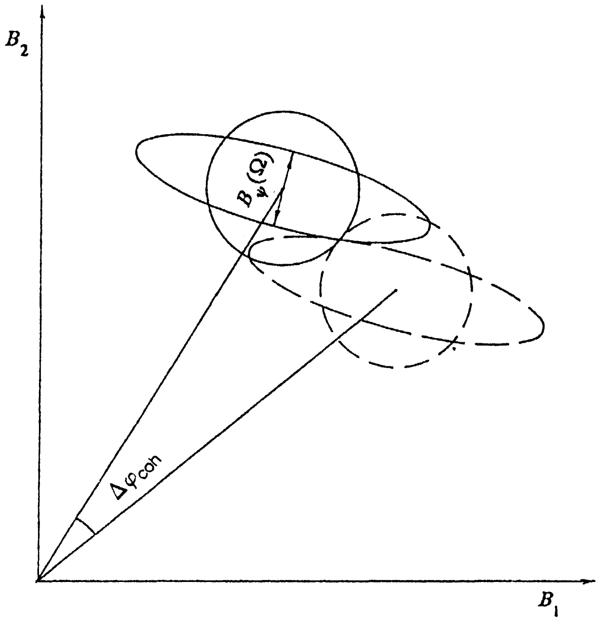

FIG. 2. If the incident wave is in a coherent state, its fluctuations will be described on this phase diagram by a circular spot which is revolving at a distance $\sqrt{n}$ from the center. The dispersions of the quadrature components $\langle\Delta^2B_1\rangle$ and $\langle\Delta^2B_2\rangle$ are equal. Because of the ponderomotive nonlinearity (the radiation pressure), a phase-amplitude correlation arises in the reflected wave. This correlation corresponds to a squeezing of the quadrature component B_ψ; the circle becomes an ellipse.

2. ANALYSIS OF A DISPLACEMENT SENSOR

The boundary condition which relates the incident and reflected waves is written

$$A(t)\exp(i\omega_0X/c)+\int_0^\infty\sqrt{\hbar\omega/Sc}\,d\omega a(\omega)\times\exp(-i(\omega-\omega_0)t+i\omega X/c)=-B(t)\exp(-i\omega_0X/c)-\int_0^\infty\sqrt{\hbar\omega/Sc}\,d\omega b(\omega)\times\exp(-i(\omega-\omega_0)t-i\omega X/c),\tag{1}$$

where $2A(t)$ and $2B(t)$ are the mean complex electric field amplitudes, $a(\omega)$ and $b(\omega)$ are annihilation operators of, respectively, the incident and reflected waves describing quantum fluctuations [their commutation relations are $[a(\omega)a^+(\omega')]=[b(\omega)b^+(\omega')]=\delta(\omega-\omega')$; their expectation values are $\langle a^+(\omega)a(\omega)\rangle=b^+(\omega)b(\omega')\rangle=0$], and S is the area of the mirror. We assume $\omega_0X/c\ll1$, and we expand the exponential function in a series. In zeroth order in the small quantities we then have $A(t)=-B(t)$. In first order we have

$$b(\omega_0+\Omega)=-a(\omega_0+\Omega)-2i(\omega_0/c)X(\Omega)\sqrt{W/\hbar\omega_0},\tag{2}$$

where $X(\Omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}X(t)\exp(i\Omega t)dt$ is the Fourier transform of X, and $W=AA^*Sc/2\pi$ is the average power of the incident wave. For simplicity we set $A=A^*$ at this point. In taking this approach we are ignoring the Doppler effect (radiation friction and parametric frequency conversion); its incorporation leads to terms smaller by a factor of ω_M/ω_0 than those which are written out.

A mechanical oscillator is subjected to a force $F=F_{LP}+F_S+F_{FL}$, where F_{LP} is the radiation-pressure force, given by

$$F_{LP}=\frac{2W}{c}+\sqrt{\frac{2W\hbar\omega_0}{\pi c^2}}\int_{-\omega_0}^{\infty}\sqrt{1+\Omega/\omega_0}\,d\Omega a^+(\omega_0+\Omega)\times\exp(i\Omega t)+\text{H.a.}\tag{3}$$

(below we drop the first term and assume $\sqrt{1+\Omega/\omega_0}\simeq1$), and F_S is the signal force, which is to be determined. We assume

$$F_S=\begin{cases}F_0\sin(\omega_Ft) & \text{for } -\tau/2\leqslant t\leqslant\tau/2,\\ 0 \quad \text{for } t<-\tau/2 & \text{and } t>\tau/2.\end{cases}\tag{4}$$

The quantity F_{FL} is the fluctuation force, which is related to a damping δ:

$$F_{FL}=\int_0^\infty\sqrt{\frac{m2\delta\hbar\Omega}{\pi}}\,d\Omega e(\Omega)\exp(-i\Omega t)+\text{H.a.}\tag{5}$$

The operators $e(\Omega)$ and $e^+(\Omega)$ here describe the heat reservoir of the mechanical oscillator; their commutation relation is $[e(\Omega)e^+(\Omega')]=\delta(\Omega-\Omega')$, and their expectation values are $\langle e^+(\Omega)e(\Omega')\rangle=\delta(\Omega-\Omega')n_T(\Omega)$, $n_T(\Omega)=(\exp(\hbar\Omega/k_BT)-1)^{-1}$, where T is the reservoir temperature, and k_B is the Boltzmann constant.

Substituting (3) and (5) into the equation for the mechanical oscillator, finding an expression for the Fourier harmonic $X(\Omega)$, and substituting this expression into (2), we find the following result for the reflected wave:

$$b(\omega_0+\Omega)=a(\omega_0+\Omega)(K_1+iK_2-1) +a^+(\omega_0-\Omega)(K_1+iK_2) -ie(\Omega)\sqrt{2K_1}|Z(\Omega)|/Z(\Omega) -i\sqrt{\frac{4W\omega_0}{\hbar}}\frac{F_S(\Omega)}{mcZ(\Omega)}, \quad (6)$$

$$K_1+iK_2=\frac{-4i\omega_0 W}{mc^2 Z(\Omega)} =\frac{4\omega_0 W}{mc^2|Z(\Omega)|^2}[2\delta\Omega+i(\Omega^2-\omega_M^2)], \quad (7)$$

where $Z(\Omega)=\omega_M^2-\Omega^2-2i\delta\Omega$. Introducing the quadrature component in the standard way,

$$B_\psi(\Omega)=[b(\omega_0+\Omega)\exp(i\psi)+b^+(\omega_0-\Omega) \times\exp(-i\psi)] \quad (8)$$

[the second quadrature component is $B_{\psi+\pi/2}(\Omega)$], we find

$$B_\psi(\Omega)=\cos(\psi+\gamma)e^{i\phi}[a(\omega_0+\Omega)e^{i\gamma}C_-+a^+ \times(\omega_0-\Omega)e^{-i\gamma}C_+]-i\sin(\psi+\gamma)e^{i\phi} \times[a(\omega_0+\Omega)e^{i\gamma}(2K+C_-)+a^+(\omega_0-\Omega)e^{-i\gamma} \times(2K+C_+)]+2e(\Omega)\sin(\psi)\sqrt{2K_1} \times|Z(\Omega)|/Z(\Omega)+B_S(\Omega), \quad (9)$$

where $K=|K_1+iK_2|$, $\phi=\mathrm{arctg}(K_2/K_1)$, $\gamma\simeq\sin\phi/K$, $C_+=(1+2K_1)/K$, $C_-=(1-2K_1)/K$, and

$$B_S(\Omega)=-\sqrt{\frac{W\omega_0}{\hbar}}\frac{4F_S(\Omega)}{mcZ(\Omega)}\sin\psi.$$

Expressions (9) were derived under the condition $K\gg1$, which corresponds to a strong inverse fluctuation effect. In the case $\psi+\gamma=0$ we have a squeezing of the component $B_\psi(\Omega)$.

Expressions (6)–(9) hold under the condition $n=W\tau/\hbar\omega_0\ll n_{\max}=(mc^2\omega_M/\hbar\omega_0{}^2)^2$ (the major axis of the ellipse in Fig. 2 is much smaller than the mean amplitude).

3. FORCE LIMIT FOR A LOSSLESS MECHANICAL OSCILLATOR

In this case we have $\delta=0$, $K_1=0$, $K_2=4\omega_0 W/[mc^2(\omega_M^2-\Omega^2)]$.

We assume that the external force has a narrow spectrum: $\tau\omega_F\gg1$. The observation condition $2|B_S(\Omega)|\Delta\Omega \gtrsim \sqrt{2\langle\Delta B_\psi(\Omega)\Delta B_\psi(-\Omega)\rangle\Delta\Omega}$ can then be written

$$X_S=\frac{F_0}{m(\omega_M^2-\omega_F^2)}\gtrsim\frac{c}{4\omega_0\sqrt{n}}. \quad (10)$$

Here $n=\pi W/\hbar\omega_0\Delta\Omega$, $\Delta\Omega\simeq2\pi/\tau$. We see that the error in the measurement of the external force falls off with increasing pump level n. The quantity X_S is essentially the same as $\Delta X_{\rm meas}$. The derivation of (10) ignored the noise due to the uncertainty in the initial values of the coordinate and momentum of the oscillator. The spectrum of this noise occupies a band $\Delta\omega$ near the frequency ω_M, and this noise does not affect the measurement error if $|\omega_M-\omega_F|\gg\Delta\Omega$.

In the detection of gravitational waves, one is interested in detecting a broad-band signal, i.e., in the case $\tau=2\pi/\omega_F$. If the bulk of the signal spectrum is far from resonance ($\omega_F\gg\omega_M$; the free-mass case), condition (10) remains valid, with one reservation: The angle ψ depends on Ω, so experimentally it is a more difficult task to distinguish the quadrature component $\int B_{\psi(\Omega)}(\Omega)d\Omega$ over a broad band.

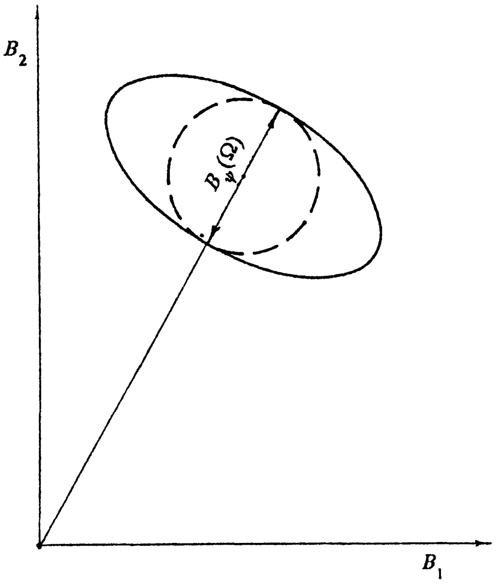

FIG. 3. For a mechanical oscillator with dissipation, the fluctuations in the reflected wave are described at resonance ($\Omega=\omega_M$) by an ellipse. The minor semiaxis of this ellipse is equal to the diameter of the circle for the incident wave; there is no squeezing. The area of the ellipse is larger than that of the circle because of the dissipation.

4. FORCE LIMIT FOR A MECHANICAL OSCILLATOR WITH DISSIPATION

In this case the dispersion of the squeezed component is

$$\left\langle\left(\int\Delta B_\psi(\Omega)e^{-i\Omega t}d\Omega\right)^2\right\rangle =\frac{2mc^2}{W\omega_0}\left[\left|F_S(\Omega)\Delta\Omega\right|^2\sin^2\phi+\hbar\Omega\delta m\Delta\Omega \times\left[\sin^2\phi(2n_T+1)+\frac{1}{2}\left(\frac{1}{2K_1}+2K_1\right)\right]\right]. \quad (11)$$

We see that at resonance ($\omega_M=\omega_F$, $K_2=0$, $K_1=K\gg1$, $\sin\phi=0$) the sensitivity drops to zero. In this case there is essentially no squeezing (Fig. 3). Far from mechanical resonance, under the conditions $K_2\gg K_1$, $\sin\phi\simeq a$, we find the following expression for the amplitude of the minimum detectable force:

$$F^2_{o\,\min} \simeq \frac{\hbar\omega_F 2m\delta\Delta\Omega}{\pi}\left[(2n_T+1)+\frac{1}{2}\left(\frac{1}{2K_1}+2K_1\right)\right]. \qquad (12)$$

It can be seen from (12) that there is no monotonic increase in the sensitivity with increasing pump level. There exists an optimum pump level $n(2K_1=1)$, at which the maximum sensitivity is achieved:

$$F^2_{o\,\min} \simeq \frac{\hbar\omega_F 4m\delta\Delta\Omega}{\pi}\left[n_T+\frac{1}{2}+\frac{1}{2}\right]. \qquad (13)$$

We see that this is simply the Nyquist formula; the last 1/2 in the brackets corresponds to the quantum noise of the instrument.

5. CONCLUSION

In the absence of mechanical dissipation, the sensitivity in (10) thus increases with the pump level. We do not obtain information on the total mechanical coordinate—only on the part of this coordinate which is caused by the external force. The validity of this comment is not restricted to the detection of external forces. In quantum nondemolition measurements of energies on the basis of the shift of the dielectric constant of a nonlinear dielectric,[5,6] for example, the fluctuation self-effect of the test mode again imposes no limit on the sensitivity.

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