Noise-induced optical multistability

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This paper for the first time predicts that optical multistability can, in principle, be induced by noise in a system of noninteracting two-level atoms. In the semiclassical approximation in a ring-cavity geometry we derive an equation describing the evolution of the amplitude of an electromagnetic field with a multiplicative source of noise in the form of a generalized Gaussian delta-correlated random process. A stochastic differential equation in the sense of Stratonovich (or Ito) and the respective Fokker–Planck equation are also derived. The behavior of the random process at the boundary of the variation region is studied and the exact stationary solution for the probability density is found. We show that, depending on the intensity of noise and the value of the cooperative constant in the theory, stable, bistable, and multistable optical phase transitions can be induced by noise in a system of two-level atoms.

1. INTRODUCTION

Optical bistability and multistability are the most vivid examples of self-organization of systems far from thermodynamic equilibrium. These phenomena have formed a separate field of research in nonlinear physics. The great interest is primarily due to the possibility of applications in optical data processing and the employment of bistable elements in such devices as optical memory cells, amplifiers, multivibrators, power clippers, and transistors.

Optical bistability (multistability) is a manifestation is a nonlinear effect in light waves. A luminous flux that has passed through a medium may prove to be a multivalued function of the incident luminous flux, that is, exhibit a hysteresis-loop dependence of the transmitted luminous flux on the incident flux. An important aspect in the emergence of such a dependence in the presence of feedback and a nonlinear dependence of the refractive index or the absorption coefficient on the light's intensity. The most complete description of optical bistability and multistability can be found in the encyclopedic monograph of Gibbs.[1] He formulates the theoretical basis of these phenomena, describes bistable materials and devices, considers optical switches, and analyzes instabilities and related self-pulsations and the like. In addition, a fairly complete review of optical bistability in semiconductors can be found in Refs 2 and 3. The first to build a theory of optical bistability in the excitonic part of the spectrum were Elesin and Kopaev,[4] and further developments can be found in Refs. 5–7.

Although the phenomena of optical bistability and multistability are undergoing intensive investigation both theoretically and experimentally, there is a certain lack of rigor in the studies of the effect of fluctuations and noise on bistability (multistability), optical switching, and periodic and stochastic self-pulsations of bistable and multistable optical systems. It is well known, however, that the internal fluctuations of the medium and external noise in highly nonequilibrium and nonlinear systems not only have no disorganization effects but in certain conditions lead to noise-induced phase transitions that have no deterministic analogs.[8]

The first study of noise-induced phase transitions in bistable systems was done by Bulsara, Schieve, and Gragg,[9] who showed that when the intensity of the noise is fairly high, optical bistability is possible for all values of the optical-bistability cooperative constant.

Below we undertake a rigorous study of the phenomenon. We show that intense noise may induce not only optical bistability but also optical multistability.

2. OPTICAL BISTABILITY IN A RING CAVITY

For the simplest model we take a substance consisting of isolated (noninteracting) fixed two-level atoms or molecules placed in an external electric field.

In the semiclassical approximation the electromagnetic field is considered a classical quantity but the material system is described quantum mechanically via the equation for the density matrix,

$$i\hbar\frac{d\rho}{dt}=[\mathscr{H},\rho]+\text{relaxation terms}, \qquad (1)$$

where the Hamiltonian $\mathscr{H}$ of the problem consists of the Hamiltonian of a free atom (or molecule, which we will still call an atom) $\mathscr{H}_0$ with eigenvalues $\hbar\omega_1$ and $\hbar\omega_2$, and the Hamiltonian describing the interaction between the atom and the field,

$$\mathscr{H}'=-\mu E(\mathbf{r},t). \qquad (2)$$

Here μ is the dipole moment of the atom, and $E(\mathbf{r},t)$ the electric field at the point where the dipole is located.

Suppose that a laser field

$$E(x,t)=\tilde{E}(x,t)\exp(i\omega t-iqx-i\varphi)+\text{c.c.} \qquad (3)$$

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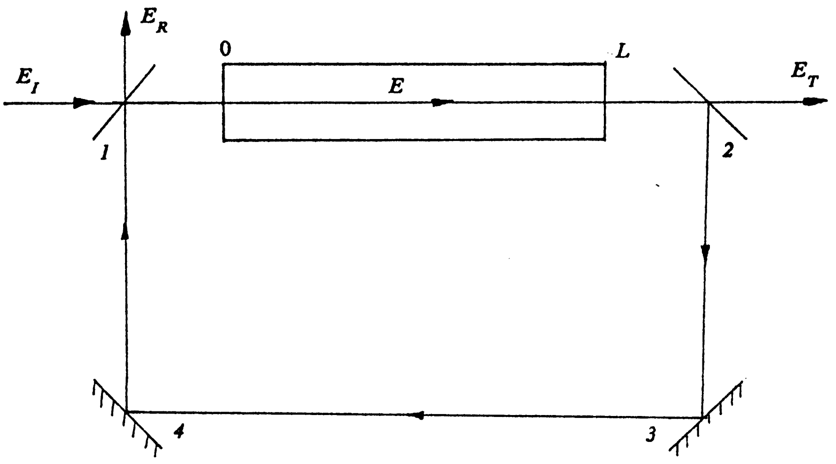

FIG. 1. Diagram of the ring cavity. Mirrors *1* and *2* have the transmission coefficient *T*, mirrors *3* and *4* are ideally reflecting, and *L* is the length of the cavity.

of frequency ω, phase φ, and wave vector q acts on a system of two-level atoms in the direction x. Then Eq. (1) without relaxation terms assumes the form

$$\begin{aligned}
\frac{d\rho_{11}}{dt}&=i\kappa\tilde{E}\cos(\omega t-qx-\varphi)(\rho_{21}-\rho_{12}),\\
\frac{d\rho_{22}}{dt}&=i\kappa\tilde{E}\cos(\omega t-qx-\varphi)(\rho_{12}-\rho_{21}),\\
\frac{d\rho_{12}}{dt}&=-i\omega_{12}\rho_{12}+i\kappa\tilde{E}\cos(\omega t-qx-\varphi)(\rho_{22}-\rho_{11}),\\
\frac{d\rho_{21}}{dt}&=-i\omega_{21}\rho_{21}+i\kappa\tilde{E}\cos(\omega t-qx-\varphi)(\rho_{11}-\rho_{22}),
\end{aligned}\tag{4}$$

where $\omega_{21}=\omega_2-\omega_1=\omega_{12}$ is the angular frequency of transition between the levels, and $\kappa=2\mu/\hbar$. Introducing the notions of longitudinal and transverse relaxation times T_1 and T_2, assuming that the equilibrium density matrix for particles in the absence of a field contains no off-diagonal matrix elements, and using the substitution

$$\rho_{12}\to\rho_{12}\exp[i(\omega t-qx-\varphi)]$$

and Eq. (4), we arrive at the following system of equations for the diagonal and off-diagonal elements of the density matrix in the rotating-wave approximation:

$$\begin{aligned}
\frac{d\rho_{11}}{dt}&=\frac{i\kappa\tilde{E}}{2}(\tilde{\rho}_{21}-\tilde{\rho}_{12})-\frac{1}{T_1}(\rho_{11}-\rho_{11}^0),\\
\frac{d\rho_{22}}{dt}&=\frac{i\kappa\tilde{E}}{2}(\tilde{\rho}_{12}-\tilde{\rho}_{21})-\frac{1}{T_1}(\rho_{22}-\rho_{22}^0),\\
\frac{d\tilde{\rho}_{12}}{dt}&=i\tilde{\rho}_{12}\Delta\omega+\frac{i\kappa\tilde{E}}{2}(\rho_{22}-\rho_{11})-\frac{\tilde{\rho}_{12}}{T_2},\\
\frac{d\tilde{\rho}_{21}}{dt}&=-i\tilde{\rho}_{21}\Delta\omega-\frac{i\kappa\tilde{E}}{2}(\rho_{22}-\rho_{11})-\frac{\tilde{\rho}_{21}}{T_2},
\end{aligned}\tag{5}$$

where $\Delta\omega=\omega_{21}-\omega$ is the detuning from resonance. Now it is convenient to go over to the variables

$$u=\tilde{\rho}_{12}-\tilde{\rho}_{21},\quad v=i(\tilde{\rho}_{12}-\tilde{\rho}_{21}),\quad w=\rho_{22}-\rho_{11}.\tag{6}$$

We assume that in the absence of a field only the lower level is occupied. Then, combining (6) with (5), we can easily get

$$\begin{aligned}
\frac{du}{dt}&=v\Delta\omega-\frac{u}{T_2},\\
\frac{dv}{dt}&=-u\Delta\omega-\kappa\tilde{E}w-\frac{v}{T_2},\\
\frac{dw}{dt}&=-\kappa\tilde{E}v-\frac{1+w}{T_1}.
\end{aligned}\tag{7}$$

This system of equations must be augmented with the system of the Maxwell equations for the electromagnetic field, from which we can easily derive the wave equation for $\tilde{E}$ (for waves propagating along the x axis),

$$\frac{\partial^2\tilde{E}}{\partial x^2}-\frac{n_0}{c^2}\frac{\partial^2\tilde{E}}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2},\tag{8}$$

where n_0 is the linear refractive index, P is the average value of the medium's polarization,

$$P=N\,\mathrm{Tr}\,\rho\mu,\tag{9}$$

and N is the number of two-level particles per unit volume.

When we combine (3) and (9), in the approximation of a slowly varying amplitude Eq. (8) yields

$$\begin{aligned}
\frac{\partial\tilde{E}}{\partial x}+\frac{n_0}{c}\frac{\partial n_0}{\partial t}&=-\frac{2\pi\omega}{n_0 c}N\mu v,\\
\tilde{E}\left(\frac{\partial\varphi}{\partial x}+\frac{n_0}{c}\frac{\partial\varphi}{\partial t}\right)&=\frac{2\pi\omega}{n_0 c}N\mu u.
\end{aligned}\tag{10}$$

Equations (7) and (10) comprise the system of equations of bistability theory in the model of two-level atoms and form the base for further consideration.

Now let us consider optical hysteresis in the case where the two-level medium is placed in a ring cavity of length L. Two mirrors of the cavity have the same transmission coefficient T, while the other two are assumed ideal. We use the well-known boundary conditions for the field amplitudes at points 0 and L:

$$\tilde{E}(0)=\sqrt{T}\tilde{E}_1+R\tilde{E}(L),$$
$$\tilde{E}_T=\sqrt{T}\tilde{E}(L), \qquad (11)$$
$$\tilde{E}_R=\sqrt{RT}\tilde{E}(L)-\sqrt{R}\tilde{E}_I,$$

where $\tilde{E}_I$, $\tilde{E}_T$, and $\tilde{E}_R$ are, respectively, the amplitudes of the field incident on the entrance mirror, of the field exiting from the cavity, and of the reflected field, and R is the reflection coefficient of the mirrors (Fig. 1).

In a cavity with a high Q-factor the atoms instantly "tune in" to the field. In this case we can adiabatically exclude the atomic variables by putting the time derivatives in (7) to zero. At exact resonance we have

$$u_0=0, \quad v_0=\frac{F}{1+F^2}\sqrt{\frac{\gamma_1}{\gamma_2}}, \quad w_0=-\frac{1}{1+F^2}, \qquad (12)$$

where

$$\gamma_1=T_1^{-1}, \quad \gamma_2=T_2^{-1}, \quad F=\frac{\sqrt{\gamma_1\gamma_2}}{\kappa}\tilde{E}. \qquad (13)$$

The equation for the field in the cavity that allows for (10) and (12) has the form

$$\frac{\partial F}{\partial t}+\frac{c}{n_0}\frac{\partial F}{\partial x}=-\alpha\frac{c}{n_0}\frac{F}{1+F^2}, \qquad (14)$$

where α is the linear absorption coefficient specified by the formula

$$\alpha=\frac{4\pi\omega\mu^2}{\hbar c n_0 \gamma_1}N. \qquad (15)$$

From (14) in the mean-field approximation and with allowance for the boundary conditions (11) we can easily arrive at the equation for the evolution of the field amplitude:

$$\frac{dX}{d\tau}=Y-X-\frac{2CX}{1+X}, \qquad (16)$$

where X and Y are the normalized entrance and exit field amplitudes, τ is the dimensionless time, and C is a constant in the optical-bistability theory; these are determined by the following expressions:

$$X=\frac{\kappa E_T}{\sqrt{\gamma_1\gamma_2 T}}=\frac{F_T}{\sqrt{T}}, \quad Y=\frac{\kappa E_I}{\sqrt{\gamma_1\gamma_2 T}},$$
$$\tau=\frac{t}{\tau_C}, \quad \tau_C=\frac{L}{CT}, \quad C=\frac{\alpha L}{2T}. \qquad (17)$$

In the stationary case Eq. (16) yields the equation of state familiar from the theory of optical bistability for two-level media in the mean-field approximation, derived by Bonifacio and Lugiato:[10]

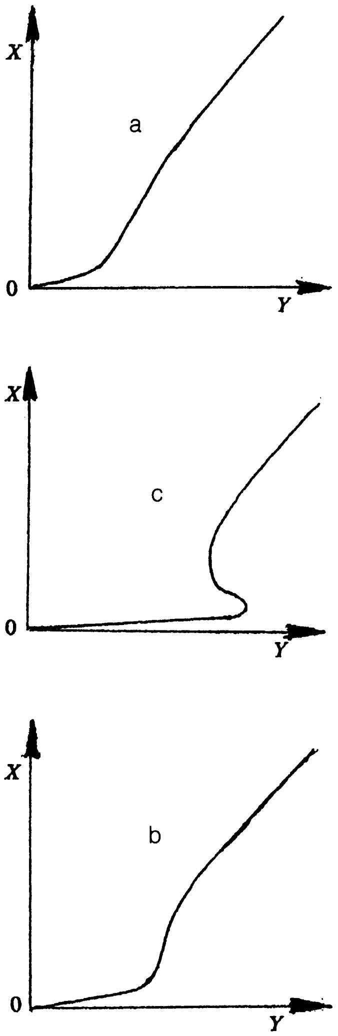

FIG. 2. Amplitude of the radiation leaving the cavity as a function of the amplitude of the incident radiation for different values of the optical-bistability constant: (a) $C=2$, (b) $C=4$, and (c) $C=10$.

$$Y=X+\frac{2CX}{1+X^2}. \qquad (18)$$

For $C<4$ the field amplitude Y is a monotone function of the variable X and bistability does not occur. For $C>4$ the curve representing the X vs Y dependence undergoes marked changes: there appears a region where one value of Y has three values of X corresponding to it (Fig. 2). Analysis shows that on the section of the curve with a negative slope the solution is unstable. Thus, in the deterministic case optical bistability manifests itself in the dependence of the transmitted light intensity on the incident light intensity.

In real conditions, however, the problem considered here has different sources of noise, both additive noise and multiplicative. As we will shortly show, allowing for the latter type of noise leads to the induction of optical phase transitions.

3. NOISE-INDUCED NONEQUILIBRIUM PHASE TRANSITIONS

One source of external noise with given characteristics, which can be controlled fairly well in experiments, is the fluctuations of the number density of atoms in the cavity. This leads to fluctuations of the constant C in the theory of optical bistability.

In our further study of the system we assume that the total external noise is a result of adding an infinitely large number of independent infinitely small perturbations, which cause high-frequency random oscillations of the cooperative parameter C with a negligible correlation time, $\tau_{\rm cor}\to 0$. We describe the fluctuations of the parameter C by Gaussian white noise of intensity σ:

$$C_t=C+\sigma\xi_t, \tag{19}$$

where C_t is the random value of the cooperative parameter at time t, C is the average value of C_t, $C=\bar{E}C_t$, and ξ_t is the Gaussian generalized delta-correlated random process with zero average, $\bar{E}\xi_t=0$, with $\bar{E}(\xi_t,\xi_\rho)=\delta(t-\rho)$.

The process ξ_t is the generalized derivative of the Wiener process W_t, that is, a stochastic process originating at zero, characterized by stationary independent increments whose distribution is Gaussian,

$$\bar{E}(W_t-W_s)=0, \quad \bar{E}(W_t-W_s)^2=|t-s|, \tag{20}$$

and most probably having continuous trajectories.

Allowing for the fact that $\xi_t=W_t$, replacing $\xi_t dt$ with the differential dW_t of the Wiener process W_t, and employing (19), from Eq. (16) we get the stochastic differential equation

$$dX=\left(Y-X_t-2C\frac{X_t}{1+X_t}\right)dt+\sigma\frac{X_t}{1+X_t^2}dW_t, \tag{21}$$

where for the sake of convenience we have introduced the notation σ instead of 2σ and W_t instead of $-W_t$. The integral form of the stochastic differential equation (21) is

$$X_t=X_0+\int_0^t\left(Y-X_\rho-2C\frac{X_\rho}{1+X_s^2}\right)ds+\sigma\int_0^t\frac{X_s}{1+X_s^2}dW_s. \tag{22}$$

The first integral on the right-hand side of (22) is an ordinary Riemann integral, and the second is a stochastic integral in the sense of Ito[11] or Stratonovich.[12] Equation (21) satisfies the condition of existence and uniqueness of a strong solution. Indeed, the coefficients

$$b(X)=Y-X-\frac{2CX}{1+X^2}, \quad \sigma(x)=\frac{\sigma X}{1+X^2} \tag{23}$$

have bounded derivatives,

$$|b'(X)|=\left|1+2C\frac{1-X^2}{(1+X^2)^2}\right|\leqslant 1+2C,$$
$$|\sigma'(X)|=\sigma\left|\frac{1-X^2}{(1+X^2)^2}\right|\leqslant\sigma, \tag{24}$$

and satisfy the Lipschitz condition

$$|b(X_1)-b(X_2)|+|\sigma(X_1)-\sigma(X_2)|$$
$$\leqslant(1+2C+\sigma)|X_1-X_2|. \tag{25}$$

Also,

$$|b(X)|^2=\left|Y-X-2C\frac{X}{1+X^2}\right|^2$$
$$\leqslant 3\left(\left|Y\right|^2+\left|X\right|^2+4C\frac{X^2}{(1+X^2)^2}\right)$$
$$\leqslant K(1+|X|^2),$$
$$|\sigma(X)|^2=\sigma^2\frac{X^2}{(1+X^2)^2}<\sigma^2(1+X^2), \tag{26}$$

where $K=3\times\max(1,|Y|^2+4C^2)$.

Thus, the stochastic differential equation (21) has a unique strong solution in the sense of Ito.[13,14] Using the link between the stochastic differential equation in Ito's interpretation and that in Stratonovich's, we can easily show that in the latter the conditions for existence and uniqueness of a strong solution are also met.

The solution X_t for $t\leqslant 0$ of Eq. (21) is a homogeneous Markov process. Let us assume that X_t possesses a transition probability density $p(t,x,y)$ with the continuous partial derivatives $p'_t(t,x,y)$, $p'_y(t,x,y)$, and $p''_{yy}(t,x,y)$. Then the probability density satisfies the Fokker–Planck equation

$$\frac{\partial}{\partial t}p(t,x,y)=-\frac{\partial}{\partial y}\left[\left(Y-y-\frac{2C_y}{1+y^2}\right)p(t,x,y)\right]$$
$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\frac{\sigma^2y^2}{(1+y^2)^2}p(t,x,y)\right]. \tag{27}$$

In the stationary case Eq. (27) yields

$$\frac{dJ(x)}{dx}=0, \quad J(x)=J_0=\text{const}, \tag{28}$$

where

$$J(x)=\left(Y-x-\frac{2Cx}{1+x^2}\right)p(x)-\frac{\sigma^2}{2}\frac{d}{dx}\left[\frac{x^2}{(1+x^2)^2}p(x)\right]. \tag{29}$$

To determine the constant J_0, we examine the behavior of the random process X_t, $t\geqslant 0$, at the boundary of the variation region. Note that the points $r_1=0$ and $r_2=\infty$ are the inner boundaries of the process X_t, since $\sigma(x)=\sigma x/(1+x^2)>0$ on the set $(0,\infty)$, and at the boundary of the region $(0,\infty)$ we have

$$\sigma(0)=\sigma(\infty)=0, \quad b(0)=Y\geqslant 0, \quad b(\infty)<0.$$

In this case the solution X_t, $t\geqslant 0$, of Eq. (21) satisfies, for any initial value $x\in(0,\infty)$, the inequalities $0<X_t<+\infty$ for almost for every positive value of t (see Ref. 13).

Let $\beta>r_1$. According to Ref. 13, we introduce the constant

$$L_1=\int_{r_1}^{\beta}\exp\left\{-\int_\beta^x\frac{2b(z)}{\sigma^2(z)}dz\right\}dx. \tag{30}$$

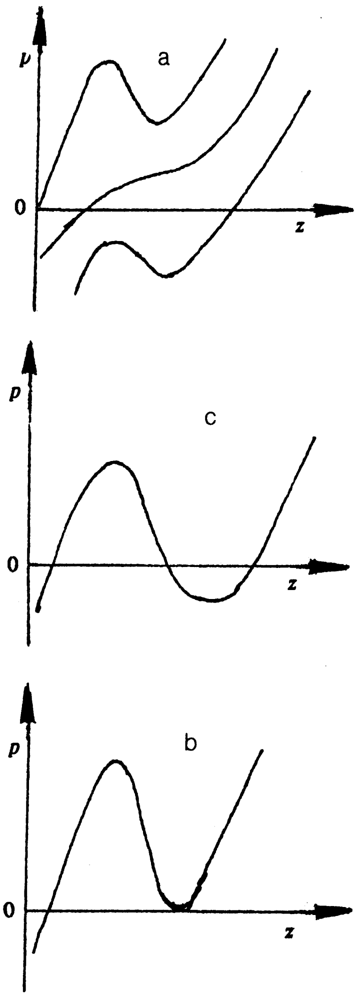

FIG. 3. Plots of $p(z,k)$ vs. z for different values of k: (a) $k<k_{cr}$, (b) $k=k_{cr}$, and (c) $k>k_{cr}$.

If $L_1=\infty$, the process X_t reaches point β with a probability equal to unity earlier than it does point r_1 for any initial condition $x\in(r_1,\beta)$. In this case the probability that $X_t\to r_1$ as $t\to\infty$ is zero. Thus, the boundary r_1 is inaccessible.

The point $r_1=0$ is the natural boundary of process X_t, $t\geqslant 0$. Indeed,

$$\Phi(x)=\int_\beta^x \frac{2b(z)}{\sigma^2(z)}dz=\frac{2}{\sigma^2}\left[-\frac{Y}{x}-(1+2C)\ln x-Q(x)\right], \tag{31}$$

where

$$Q(x)=\frac{x^4}{4}-\frac{Yx^3}{3}+(1+C)x^2-2Yx+M,$$

$$M=\frac{4}{\beta}-(1+2C)\ln\beta+2Y\beta-(1+C)\beta^2+\frac{4\beta^2}{3}-\frac{\beta^4}{4}.$$

Then for L_1 we obtain

$$L_1=\int_0^\beta \exp\{-\Phi(x)\}dx$$
$$=\int_0^\beta \exp\left\{\frac{2Y}{x}\right\}x^{2(1+2C)/\sigma^2}\exp\left\{\frac{2Q(x)}{\sigma^2}\right\}dx. \tag{32}$$

Since the polynomial $Q(x)$ is bounded in the neighborhood of zero, we can easily see that $L_1=\infty$. Thus, the boundary $r_1=0$ is the natural boundary of the process X_t (see Ref. 13).

Reasoning along the same lines, we can prove that the boundary point $r_2=\infty$ is natural, too. Since the boundaries $r_1=0$ and $r_2=\infty$ are natural, the process X_t does not reach the boundaries. In this case the stationary probability flux defined in (29) is zero at the boundaries: $J(0)=J(\infty)=0$. Thus, we arrive at the following equation for the stationary probability density $p(x)$:

$$\left(Y-x-\frac{2Cx}{1+x^2}\right)p(x)-\frac{\sigma^2}{2}\frac{d}{dx}\left[\frac{x^2}{(1+x^2)^2}p(x)\right]=0, \tag{33}$$

whose solution has the form

$$p(x)=N\frac{(1+x^2)^2}{x^2}\exp\left\{-\frac{2}{\sigma^2}\left[\frac{Y}{x}+(1+2C)\ln x+\frac{x^4}{4}\right.\right.$$
$$\left.\left.-\frac{x^3}{3}+(1+C)x^2-2Yx\right]\right\}, \tag{34}$$

where $\bar{N}$ is the normalization constant determined for the relation

$$\int_0^\infty p(x)dx=1.$$

The solution to (27) can be written as

$$p(x)=\bar{N}\exp\left\{-\frac{2}{\sigma^2}V(x)\right\}, \tag{35}$$

where $V(x)$ is a stochastic potential determined by the following expression:

$$V(x)=\frac{Y}{x}+(1+2C+\sigma^2)\ln x-\sigma^2\ln(1+x^2)+\frac{x^4}{4}$$
$$-\frac{Yx^3}{3}+(1+C)x^2-2Yx. \tag{36}$$

In contrast to the deterministic case, in which the stationary states were found from the condition that $\dot{X}=0$, stationary states in the stochastic case must be interpreted as the points $\bar{X}$ of the system's state space at which the stationary probability density $p(x)$ takes on extremal values. The maximum points of the function $p(x)$ correspond to the most probable states of the system, and the minimum points to the least probable. Equation (36) shows that the extrema of $p(x)$ coincide with the extrema of the

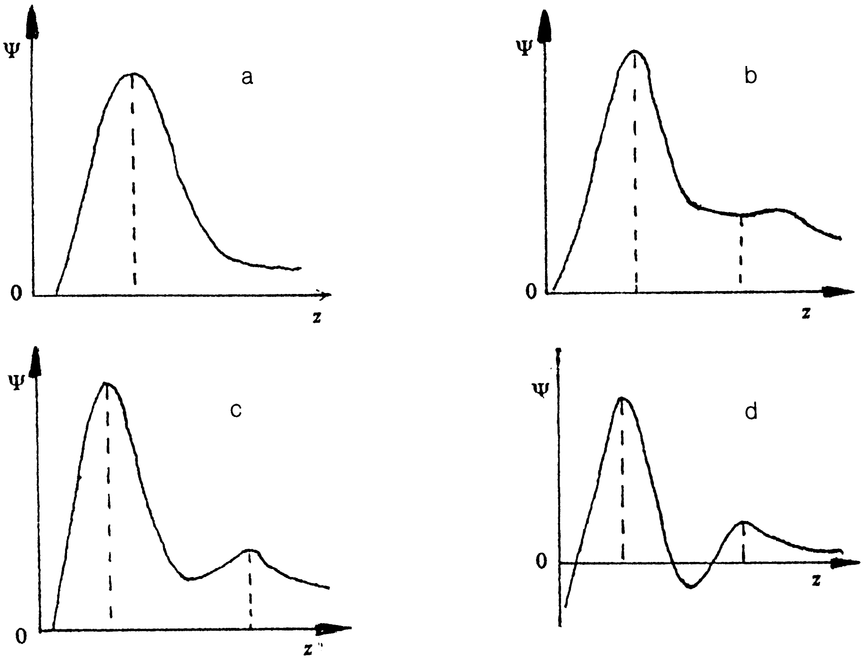

FIG. 4. Plots of $\Psi(z,k)$ vs. z for different values of k: (a) $k<k_{cr}$, (b) $k=k_{cr}$, and (c) $k>k_{cr}$.

potential $V(x)$, while the maximum points of $p(x)$ correspond to the minimum points of $V(x)$, and vice versa.

To determine the stationary states of the system we put $V'(x)$ to zero, as a result of which we get

$$Y=f(x,\sigma^2,C), \tag{37}$$

where

$$f(x,\sigma^2,C)=x+\frac{2Cx}{1+x^2}+\sigma^2\frac{x(1+x^2)}{(1+x^2)^3}. \tag{38}$$

To find the solutions to Eq. (37) we examine the function $f(x,\sigma^2,C)$. Clearly,

$$f'(x,\sigma^2,C)=1-2C\Psi(x^2,k),\quad k=\frac{\sigma^2}{2C}, \tag{39}$$

where

$$\Psi(z,k)=\frac{z^2-(3k-1)z^2+8(k-1)-k-1}{(1+z)^4},\quad z=x^2.$$

Also,

$$\Psi'(z,k)=-\frac{z^3-(6k+1)z^2+(30k-5)z-(12k+3)}{(1+z)^5}. \tag{40}$$

The polynomial $z^3-(6k+1)z^2+(30k-5)z-(12k+3)$ has one positive root for $k<k_{cr}=2.596$, one simple and one double positive root for $k=k_{cr}$, and three different positive roots for $k>k_{cr}$ (Fig. 3). Since $\Psi_z'(0,k) = 12k + 3 > 0$, $\Psi(0,k)=-(k+1)<0$, and $\Psi(z,k)>0$ hold for fairly large values of z, and $\Psi(z,k)\to 0$ as $z\to\infty$, the function $\Psi(z,k)$ has either one maximum or two maxima and one minimum. In Fig. 4 the function $\Psi(z,k)$ is depicted for different values of k. Thus, the equation $f'(x)=0$, equivalent to the equation $\Psi(x^2,k)=1/2C$, can have no roots or one, two, or three roots depending on the values of the parameters k and C. Moreover, $f'(x)(0,\sigma^2,C)>0$ and $f(x,\sigma^2,C)-x\to 0$ as $x\to\infty$. Hence, depending on the values of the cooperative parameter C and the intensity of external noise, $f(x,\sigma^2,C)$ is either a strictly increasing function or has one maximum and one minimum or two maxima and two minima.

To determine the diagrams of C vs σ describing the behavior of the function $f(x,\sigma^2,C)$ and, hence, the behavior of the stationary states of the system, we find the critical values of C and σ from the conditions $f_x' - f_{xx}'' = 0$, equivalent to the following system of equations:

$$1-2C\Psi(x^2,k)=0,$$
$$\Psi'(x^2,k)=0. \tag{41}$$

This readily yields the following expressions for the parameters C and σ:

$$C=\frac{3(1+z)^4(z^2-5z+2)}{3z^5-13z^4-18z^3+6z^2-z-9},$$
$$\sigma^2=\frac{(1+z)^6(z-3)}{3z^5-13z^4-18z^3+6z^2-z-9}. \tag{42}$$

Since both C and σ^2 are nonnegative, the set $z\in[z_1,3]\cup(z_2,\infty)$ serves as the domain of z, where $z_1 = (5 - \sqrt{17})/2$ and z_2 is the only positive root of the polynomial $3z^5-13z^4-18z^3+6z^2-z-9$, or $z_2\approx 5.39$.

Formulas (42) specify the parametric equation of the separatrix in the (C,σ) plane (Fig. 5), which divides the range of variation of the parameters into three subsets *A*, *B*, and *C*, in which the behavior of the function $f(x,\sigma^2,C)$

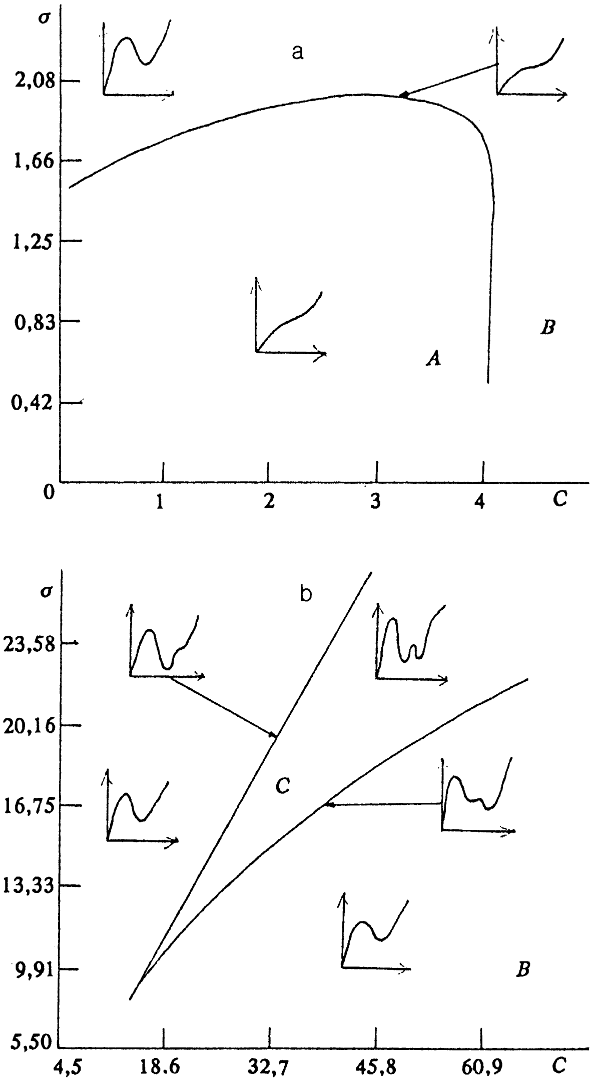

FIG. 5. The phase state diagram describing the behavior of $f(x,\sigma^2,C)$. The function is depicted (a) in the regions A and B and at the boundary separating the two regions, and (b) in the regions B and C and at the boundary separating the two regions.

and, hence, the dependence of the stationary states of the system on the intensity Y of the incident field are markedly different. When the point (C,σ) crosses the boundaries of the subsets A, B, and C, a phase transition occurs. For $(c,\sigma)\in A$ the stationary probability density $p(x)$ is unimodal and the system possesses a single stable stationary state. When the point crosses the boundary separating A and B, bifurcation takes place. Depending on the intensity Y of the incident field, the stationary probability density $p(x)$ is either unimodal or bimodal, that is, the system has either one stable stationary state or three stationary states, of which two are stable and one unstable. This is the case where noise-induced optical bistability sets in. When the point crosses the boundary separating B and C, a new phase transition occurs. In this case, depending on the intensity of the field, the stationary probability density $p(x)$ is unimodal or bimodal or trimodal, that is, the system has one (stable) state or two states (two stable and one unstable) or five states (three stable and two unstable) (Fig. 5).

Thus, external noise can induce in a system not only optical bistability but also optical tristability.

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