The WKB method for resonances

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(Submitted 7 April 1993)
Zh. Eksp. Teor. Fiz. **104**, 2293–2313 (July 1993)

We establish the analytic continuation of the Bohr–Sommerfeld quantization rule for potentials with a barrier into the range of above-the-barrier resonances. The derived equations are valid for an arbitrary analytic potential satisfying quasi-classical conditions and determine both the position E_r and width Γ of the resonance. The results are illustrated by a number of model potentials and also for the Stark effect in a strong field. We find the asymptotic behavior of the resonance energy in the tight-binding approximation.

1. INTRODUCTION

The quasiclassical approximation, or the WKB method, is one of the most effective approximate methods of quantum mechanics and theoretical physics (see, e.g., Refs. 1–14). In contrast to perturbation theory, this approximation is not linked to the weakness of interaction and has therefore a broader range of applications, making it possible to study the qualitative laws of the behavior and properties of quantum-mechanical systems. Usually the WKB method is used in the case of a discrete spectrum and somewhat more rarely to calculate the wave functions of the continuous spectrum and in scattering theory.[1–5,14] However, in various physical problems one encounters potentials with a barrier for which the energy levels are quasistationary. For instance, atomic physics exhaustively studies resonances with large quantum numbers (Rydberg states), including those that lie above the classical ionization threshold ($E_r > U_m$, with U_m the top of the potential barrier).

Below we formulate a generalization of the Bohr–Sommerfeld quantization rule to the case of quasistationary states (resonances) with a complex-valued energy $E=E_r-i\Gamma/2$. The derived equations determine both the position of a resonance E_r and resonance's width Γ, and simplify considerably the calculation of these quantities. For one thing, they allow finding the asymptotic behavior of the resonance energy in the tight-binding approximation, for instance, for atomic levels in strong external fields.

Here is the plan of the paper. Section 2 considers the analytic continuation of the Bohr–Sommerfeld quantization conditions from the discrete spectrum range to above-the-barrier energy range. The main result is Eqs. (8) and (12), which determine the spectrum of quasistationary states. Sections 3 and 4 study the simplest model of an anharmonic oscillator (one-dimensional and three-dimensional) with a power-law nonlinearity. The quantization rule is used to establish an asymptotic expansion for the energies of quasistationary levels in the tight-binding approximation, $g\to\infty$ [g is the coupling constant; see Eq. (13)]. Comparison with the results of a numerical solution[15] shows that the range of applicability of this asymptotic result (formally valid for $g\gg 1$) extends down to values of g of the order of unity even for small quantum numbers, $n=0$ and 1. Section 5 studies resonances in the potential (28) which is a generalization of the spherical model of the Stark effect. In Sec. 6 we derive similar equations for the energies of Stark resonances in the hydrogen atom, consider their analytic continuation to above-the-barrier region, compare results with the experimental data, and obtain the asymptotic behavior of resonance energies in a strong electric field. Appendices A, B, and C are devoted to auxiliary questions, including the method of analytical calculation of the coefficients of the asymptotic series (17) for an anharmonic oscillator.

A remark concerning the history of the problem is in order. The quantization rule (8) for quasistationary states (without allowing, however, for corrections proportional to $\hbar^2$ and $\hbar^4$) was, apparently, first used in Refs. 15–17; namely, in the problem of a cubic anharmonic oscillator,[15]

$$H=\tfrac{1}{2}(p^2+kx^2)+gx^3, \quad -\infty<x<\infty, \tag{1}$$

and in calculations of the Stark shifts and of the widths of the $(n-1,0,0)$ and $(0,n-1,0)$ states of a hydrogen atom in a strong electric field.[16,17] In both cases we demonstrate the effectiveness of this quantization condition in the tight-binding approximation. As far as we know, there is no detailed derivation of Eq. (8) in the literature.

2. ANALYTIC CONTINUATION OF THE BOHR–SOMMERFELD QUANTIZATION RULE[1)]

As is known (see, e.g., Refs. 5–7), the quantization rule for states from the discrete spectrum has the form

$$J(E)=\frac{1}{\pi}\int_{x_0}^{x_1}p(x,E)dx=n+\frac{1}{2}, \quad n=0,1,\dots . \tag{2}$$

A generalization of this rule for a finite barrier factor was obtained in Refs. 18–20:

$$J(E)=n+\frac{1}{2}-\frac{1}{2\pi}\varphi(a), \tag{3}$$

where $\hbar=m=1$,

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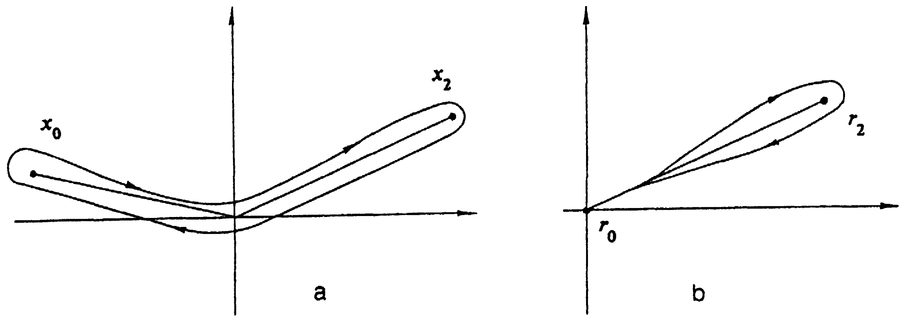

FIG. 1. Integration contour C in Eqs. (8) and (12) for (a) the one-dimensional case, and (b) a spherically symmetric three-dimensional potential ($l=0$). The cuts of the function $p(x,E)$ are also shown.

$$\varphi(a)=\frac{1}{2i}\ln\left\{\frac{\Gamma\left(\frac{1}{2}+ia\right)}{\Gamma\left(\frac{1}{2}-ia\right)[1+\exp(-2\pi a)]}\right\}+a(1-\ln a), \tag{3'}$$

with $\Gamma(\varphi)$ the gamma function and

$$a=\frac{1}{\pi}\int_{x_1}^{x_2}dx\sqrt{-p^2(x,E)}, \tag{4}$$

$p(x,E)=\sqrt{2(E-U(x))}$ is the quasiclassical momentum, and x_i the turning points. Here $x_0<x<x_1$ and $x>x_2$ are the classically allowed regions, $x_1<x<x_2$ is the subbarrier region in which $p^2(x)<0$, and for $x>x_2$ the particle goes to infinity (see Fig. 1 in Ref. 19).

Near the top of the barrier, $x\approx x_m$, we use the parabolic approximation to get

$$a=\frac{U_m-E}{\sqrt{-U''(x_m)}}=\frac{U_m-E_r}{\omega}+i\frac{\Gamma}{2\omega}, \tag{5}$$

where $U_m=U(x_m)$, and $\omega=\sqrt{-U''(x_m)}$ is the frequency of a "flipped oscillator." In Eq. (4) this parameter is written in a more general form applicable for $|a|\gg 1$, too. For complex-valued resonance energies both the parameter a and the turning points are complex-valued.

For the function $\varphi(a)$ introduced in Eq. (3′) the following asymptotic behavior (as $|a|\to\infty$) holds true:

$$\varphi(a)\approx\frac{1}{24a}+\frac{7}{2880a^3}+...+\frac{i}{2}e^{-2\pi a}\quad\text{if}\quad -\frac{\pi}{2}<\arg a<\frac{\pi}{2}, \tag{6}$$

$$\varphi(a)\approx-2\pi ia+\frac{1}{24a}+...\quad\text{if}\quad\frac{\pi}{2}<\arg a<\pi \tag{7}$$

(for this function the ray $a=\pi/2$ is a Stokes' line).

For deep-lying levels ($E<U_m$ and $a\gg 1$) we can easily verify, using expansion (6), that Eq. (3) transforms into the ordinary Bohr–Sommerfeld quantization rule and its imaginary part yields the Gamow formula for the level width.

The turning points x_1 and x_2 move closer as $E\to U_m$; there is a (narrow) energy region in which the quasiclassical approximation is invalid. However, as E_r grows, these points move farther apart and into the complex plane, and the conditions necessary for the WKB method to apply begin to be met once more. For $|a|\gg 1$ we arrive, taking (7) into account at the following:[2)]

$$\int_{x_0}^{x_1}pdx=(n+\tfrac{1}{2}+ia)\pi=(n+\tfrac{1}{2})\pi+i\int_{x_1}^{x_2}dx\sqrt{-p^2}.$$

Noting that $\sqrt{-p^2}=ip(x)$, we can write this equation in the final form:

$$\oint_C dx\sqrt{p^2(x,E)}=2\pi(n+\tfrac{1}{2}), \tag{8}$$

where the integration contour encompasses the complex-valued turning points x_0 and x_2 (see Fig. 1a). For a discrete spectrum the quasiclassical quantization rule can also be written in the form (8) but with an integration contour C that encompasses the points x_0 and x_1, both of which lie on the real axis.[6,7,13] Thus, the integration contour becomes "re-engaged" as we move from the subbarrier region to the above-the-barrier one.

Equation (8) gives the analytic continuation of the Bohr–Sommerfeld quantization rule into the above-the-barrier region $E_r>U_m$. The continuation can also be performed via the formal substitution

$$a\to ae^{-2\pi i} \tag{9}$$

directly in the quantization integral $I(E)$. Indeed, if the level's energy approaches the top of the barrier, $a\to 0$ and the quantization integral acquires a logarithmic singularity with a coefficient independent of the shape of the potential:

$$J(E)=J_0+\frac{1}{2\pi}a\ln a-J_1a+J_2a^2+... \tag{10}$$

[in contrast to the other coefficients J_k, which are determined by the potential $U(x)$; explicit formulas for these are given in Ref. 22]. Note that this singularity is due to the divergence of the oscillation period $T(E)$ as $E\to U_m$:

$$\omega T=-\ln a+(2\pi J_1-1)-2\pi J_2a+...,$$

[with ω defined in (5)]. If in the quantization rule (2) we perform the substitution (9), the rule, as (10) clearly shows, coincides with Eq. (3), provided that in the latter we allow for the asymptotic behavior (7) of the function

$\varphi(a)$. Thus, the substitution (9) transforms Eq. (2), valid for discrete spectra, to Eq. (8) for above-the-barrier resonances. The condition $|a| \gg 1$ used in the derivation was necessary so that the quasiclassical approximation could be applied [actually, as Eq. (7) shows, a less stringent condition is needed, namely, $|a| \gg 1/\sqrt{48\pi} \approx 0.08$].

As is known,[1,2] $\hbar^2$ is the formal parameter in the quasiclassical expansion. Higher-order corrections (of the order of $\hbar^2$, $\hbar^4$, etc.) have been calculated by various authors,[7-10] who considered only the discrete spectrum. Here we use the quantization condition in a form suggested by Kesarwani and Varshni:[9]

$$\sum_{k=0}^{\infty} J_{2k} = n + \tfrac{1}{2}, \tag{11}$$

where

$$J_0 = \frac{1}{\pi} \int_{x_0}^{x_1} p\,dx, \quad J_2 = -\frac{1}{12\pi} \frac{d}{dE} \int_{x_0}^{x_1} \frac{U''}{p}\,dx,$$

$$J_4 = -\frac{1}{2880\pi} \frac{d^3}{dE^3} \int_{x_0}^{x_1} \frac{5U'''U' - 7U''^2}{p}\,dx,$$

etc. (in Ref. 9 corrections up to the order of $\hbar^8$ inclusive are listed). Differentiation with respect to the energy variable E can be done explicitly if we go over to contour integrals and allow for the fact that

$$\frac{d^2}{dE^s}\left(\frac{1}{p}\right) = \frac{(-1)^s (2s-1)!!}{p^{2s+1}}.$$

As a result we get the following:

$$\frac{1}{2\pi} \oint_C dx \left\{ p + \frac{\hbar^2 m}{24 p^3} U'' + \frac{\hbar^4 m^2}{384 p^7} (5U'''U' - 7U''^2) - \frac{\hbar^6 m^2}{256 p^9} U'''^2 + \frac{\hbar^6 m^3}{1536 p^{11}} (35 U^{(4)} U'^2 - 224 U''' U'' U' + 93 U''^3) + O(\hbar^8) \right\} = \left(n + \frac{1}{2}\right)\hbar, \tag{12}$$

where the contour C encompasses the turning points x_1 and x_2 for a discrete spectrum and the points x_0 and x_2 for above-the-barrier resonances (we have re-established Planck's constant $\hbar$ and mass m explicitly). Equations (8) and (12) can be used for an arbitrary differentiable (analytic) potential $U(x)$ that allows continuation into the complex plane. The quantization condition is quite useful for calculations if contour integrals are employed, which is what we do below. If ordinary integrals are used in calculations of the corrections J_{2k}, Eq. (12) is invalid because of singularities at the turning points, and we must resort to (11).

We illustrate the application of Eqs. (8) and (12) by examples that employ a number of model potentials for which the calculations can be done analytically.

3. ANHARMONIC OSCILLATOR

Alvarez[15] studied in detail the behavior of resonances (as a function of the coupling constant g) for the one-dimensional oscillator (1) and found the asymptotic behavior of the energies $E_n(g)$ both for weak coupling and for $g \to \infty$. Generalizing somewhat this example, let us consider the potential

$$U(x) = \tfrac{1}{2} k x^2 - g x^N \tag{13}$$

with odd $N = 3, 5, \ldots$ (in what follows $k = 1$), a potential that has a maximum at $x = x_m$:

$$x_m = (Ng)^{-1/(N-2)}, \quad U_m = \frac{N-2}{2N} (Ng)^{-2/(N-2)} \tag{13'}$$

[the frequency $\omega = \sqrt{N-2}$ introduced in (5) is independent of g]. Note that the Schrödinger equation with potential (13) can serve as a standard equation for the theory of quasistationary states.

The equation $U(x_i) = E(g)$ determines the positions of N complex-valued turning points and can be solved explicitly in two limiting cases: $g \to 0$ and $g \to \infty$. In the latter we can ignore the oscillator potential in comparison to gx^N in Eq. (13). Hence,[3)]

$$x_0 \approx -\left(\frac{E}{g}\right)^{1/N}, \quad x_2 \approx \left(\frac{E}{g}\right)^{1/N} e^{i\pi/N},$$

and Eq. (8) assumes the form

$$\oint_{x_0}^{x_2} dx \sqrt{2(E_n + g x^N)} = 2\pi\left(n + \tfrac{1}{2}\right). \tag{14}$$

The integral can easily be evaluated if we take for the cut connecting the turning points two segments, one from x_0 to 0 and the other from 0 to x_2. As a result we arrive at an asymptotic formula for the energies of quasistationary states in the tight-binding mode:

$$E_n(g) \approx \tilde{E}_n(g) = C_N \exp\left(-\frac{i\pi}{N+2}\right) \left[\left(n + \frac{1}{2}\right)^N g\right]^{2/(N+2)}, \tag{15}$$

where

$$C_N = \left\{ \sqrt{\frac{\pi}{8}} (N+2) \Gamma\left(\frac{N+2}{2N}\right) \Big/ \Gamma\left(\frac{1}{N}\right) \cos\frac{\pi}{2N} \right\}^{2N/(N+2)} \tag{16}$$

(Fig. 2). In connection with this formula we note the following:

(a) The $E_n(g) \propto g^{2/(N+2)}$ dependence follows already from scaling considerations ($x \to \mu x$ in the Schrödinger equation with a power-law potential gx^N and a suitable choice of the scaling factor μ). In this respect see Simanchik's remark cited on page 85 of Simon's paper.[23] Note, however, that the given statement now refers not to a discrete spectrum but to quasistationary states.

(b) The equality $\operatorname{Im} E_n(g) = -\Gamma_n/2$ has, according to (15), the correct sign, and for large values of N the level is fairly narrow: $\Gamma/E_r \propto N^{-1}$. The smallness is due to the fact that as $N \to \infty$ the potential becomes very sharp (see the discussion at the end of Appendix A).

(c) For a cubic anharmonic potential ($N=3$), Eq. (15) coincides with formula (5) of Alvarez's paper.[15]

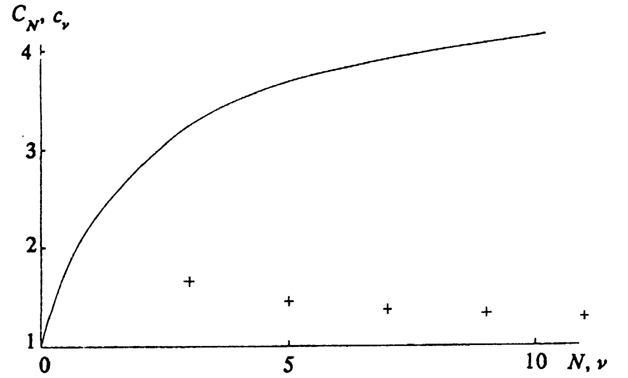

FIG. 2. Asymptotic coefficients C_N (+) and c_ν (solid curve) for the anharmonic oscillator.

(d) An interesting question is the structure of the asymptotic expansion of $E_n(g)$ for large values of g. If in (13) we regard $kx^2/2$ as a perturbation, we can show that

$$E_n(g)\approx\widetilde{E}_n(g)\sum_{k=0}^{\infty} d_k\lambda^{-k\alpha}, \tag{17}$$

where $\alpha=4/(N+2)$,

$$d_k=d_k^{(0)}+d_k^{(2)}(n+\tfrac{1}{2})^{-2}+d_k^{(4)}(n+\tfrac{1}{2})^{-4}+..., \tag{18}$$

with $d_0^{(0)}=1$, and $\lambda=g(n+\frac{1}{2})^{(N-2)/2}$ is the effective coupling constant for high-lying levels (see Appendix A). The coefficients $d_k^{(j)}$ can be calculated analytically by a procedure described in Appendix B. For one thing,

$$d_0^{(2)}=\frac{N(N-1)\cos(\pi/N)}{6\pi(N+2)^2\tan(\pi/2N)},$$

$$d_1^{(0)}=\frac{\Gamma(3/N)\Gamma((N+2)/2N)}{\Gamma(1/N)\Gamma((N+6)/2N)}\,C_N^{-(N-2)/N}\left(\cos\frac{\pi}{N}-\frac{1}{2}\right)$$
$$\times\exp\left(\frac{4\pi i}{N(N+2)}\right) \tag{18$'$}$$

(see Fig. 3; note that all the coefficients $d_0^{(j)}$ are real).

The potential (13) with $N=3$ (the cubic oscillator) was studied by Alvarez,[15] who obtained with a very high accuracy the numerical values of $E_n(g)$ for $n=0$ and 1 and $g\leqslant 100$ by employing the method of complex-valued rotations well-known in atomic physics (see, e.g., Ref. 24). In this case the few first terms in the expansion (17) and (18) have the following values:

$$C_3=[5\sqrt{\pi/6}\Gamma(\tfrac{5}{6})/\Gamma(\tfrac{1}{3})]^{6/5}=1.658602...,$$

$$d_1^{(0)}=0,\quad d_2^{(0)}=-\frac{C_3\sqrt{3}}{25\pi}e^{-i\pi/5}=-0.03658e^{-i\pi/5}, \tag{19}$$

$$d_3^{(0)}=0.00558e^{i\pi/5},\quad d_0^{(2)}=0.01103,....$$

The coefficients $d_k^{(j)}$ rapidly decrease as j and k grow. We keep therefore only three terms in (17):

$$E_n^{(\mathrm{as})}=\widetilde{E}_n(g)[1+d_0^{(2)}(n+\tfrac{1}{2})^{-2}+d_2^{(0)}\lambda^{-8/5}]. \tag{20}$$

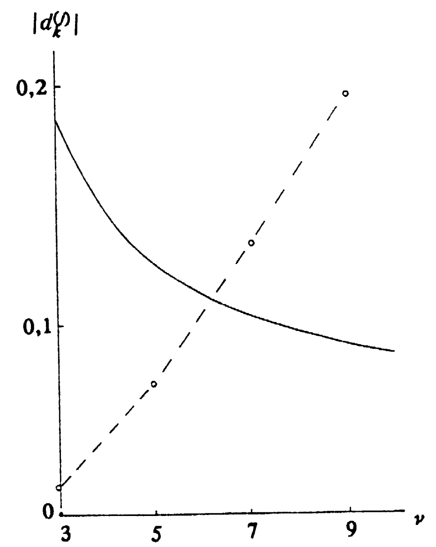

FIG. 3. Power-law corrections to the leading term of the asymptote of $\widetilde{E}_n(g)$. The coefficients $|d_k^{(j)}|$ for the anharmonic oscillator are depicted as functions of the exponent N (the one-dimensional case: ○, $d_0^{(2)}$; ●, $d_1^{(0)}$) and ν (the three-dimensional case: solid curve, $d_1^{(0)}$).

This formula and the numerical results given in Ref. 15 are compared in Table I (see also Fig. 1 in Ref. 21). This required the use of the scaling relation

$$E_n(k,g)=\sqrt{k}E_n(1,gk^{-(N+2)/4})$$

(we used $k=1$, while in Alvarez's paper $k=\frac{1}{4}$). Table I shows that the range of applicability of the quasiclassical asymptotics, formally valid for $g\gg 1$, extends down to values of g of the order of unity even for the ground level $n=0$. An especially high accuracy is achieved for the ratio

$$\xi=-\frac{\mathrm{Im}\,E_n(g)}{\mathrm{Re}\,E_n(g)}, \tag{21}$$

which in the limit $g\to\infty$ is independent of n and g and equal to $\xi=\tan\pi/(N+2)$. Remarkably, such a simple approximation as (20) describes the resonance energies in the region $g\gtrsim$ not only qualitatively but also quantitatively

TABLE I.

	$n=0$		$n=1$	
g	Re E_r	ξ	Re E_r	ξ
100	3.8940	0.726541	13.8388	0.726542
	3.8478	0.726534	13.8399	0.726539
10	1.5502	0.72646	5.5093	0.72650
	1.5708	0.72620	5.5098	0.72639
5	1.1747	0.72630	4.1751	0.72643
	1.1905	0.7255	4.1757	0.7261
1	0.6165	0.72323	2.1921	0.72502
	0.6098	0.7128	2.1935	0.7206
0.5	0.4664	0.71613	1.6596	0.72182
	0.4622	0.684	1.6623	0.714

The first line (for fixed n and g) contains the results of numerical calculations,[15] and the second the results of calculations by the asymptotic formula (20). The limiting value $\xi_\infty=\tan\pi/5=0.726\,5425...$.

(for excited states with $n \gg 1$ its accuracy is even higher). This shows that power-law asymptotes of the type (17) can be of considerable interest.

4. SPHERICAL OSCILLATOR

Let us now consider the potential

$$U(r)=\tfrac{1}{2}r^2-gr^\nu,\quad l=0, \tag{22}$$

with an arbitrary $\nu>2$, which is, for ns-states, the three-dimensional analog of (13). The left turning point is fixed, $r_0=0$, the integration contour C is shown in Fig. 1b, and in the quantization rule $n+\frac{1}{2}$ must be replaced by $n-\frac{1}{4}$ (see Refs. 2 and 11). We restrict our discussion to the tight-binding approximation. The asymptotic expansion of $E_n(g)$ has the form

$$E_n(g)\approx c_\nu \exp\left(-\frac{2\pi i}{\nu+2}\right)[g(n-\tfrac{1}{4})^\nu]^{2/(\nu+2)}[1+d_0^{(2)}(1\tfrac{1}{4})^{-2}+d_1^{(0)}(\lambda e^{-i\pi})^{-\alpha}+d_2^{(0)}(\lambda e^{-i\pi})^{-2\nu}+...], \tag{23}$$

where now $\alpha=4/(\nu+2)$,

$$c_\nu=\left[\sqrt{\frac{\pi}{2}}(\nu+2)\Gamma\left(\frac{\nu+2}{2\nu}\right)\Big/\Gamma\left(\frac{1}{\nu}\right)\right]^{2\nu/(\nu+2)}, \tag{23'}$$

and

$$d_1^{(0)}=\frac{\Gamma(3/\nu)\Gamma((\nu+2)/2\nu)}{2\Gamma(1/\nu)\Gamma((\nu+6)/2\nu)}c_\nu^{-(\nu-2)/\nu},... \tag{23''}$$

[a generalization of the (23″) is given in (A8)]. The most important difference between the asymptotes (17) and (23) is in the phase factor; for instance, now $\xi_\infty=\tan 2\pi/(\nu+2)$. The dependence of the coefficients $d_1^{(0)}$ and $d_0^{(2)}$ on the exponent (N or ν) is illustrated by Fig. 3, which shows that power-law corrections to the leading term in the asymptotic expansion of $\tilde{E}_n(g)$ are small.

Let us consider in detail the case $\nu=4$. Here (22) coincides, to within an inessential energy shift, with the potential

$$U(r)=-\frac{\omega^2}{8R^2}(r^2-R^2)^2 \tag{24}$$

[$\omega=\sqrt{2}$ and $r_m=R=(1/2)\sqrt{g}$], which (with the opposite sign) is often encountered in quantum field theory (the potential for Higgs bosons in the Weinberg–Salam model). The turning points are

$$r_{1,2}=r_m\sqrt{1\mp\sqrt{1-\epsilon}}, \tag{25}$$

where $\epsilon=E/U_m=16gE$. The quantization integral (2) can be calculated analytically here and gives the following equation for the energy of ns-levels:

$$G(\epsilon)=32\lambda, \tag{26}$$

where $\lambda=g(n-\frac{1}{4})$ and $G(z)=zF(1/4,3/4;2;z)$, a function whose properties are discussed in Appendix C. The turning points (25) collide at $\epsilon=1$, or $\lambda=\lambda_*=(2^{3/2}3\pi)^{-1}=0.0375$. The analytic continuation of Eq. (26) into the above-the-barrier region ($\epsilon>1$ and $\lambda>\lambda_*$) has the form

$$G(\epsilon)-i\sqrt{2}G(1-\epsilon)=32\lambda. \tag{27}$$

In the limit as $\lambda\to\infty$, this equation leads, as expected, to the asymptotic formula (23) with $\nu=4$ (for more details see Appendix C).

5. A GENERALIZED STARK MODEL

Our last example, which we briefly consider, is a generalization of the spherical model of the Stark effect in the hydrogen atom (into which it transforms at $\alpha=1/2$):

$$U(r)=-(r^{-2\alpha}+gr^{2\alpha}),\quad 0<\alpha<1. \tag{28}$$

In this case ($l=0$), $r_m=g^{-1/4\alpha}$, $U_m=-2\sqrt{g}$, and the turning points are

$$r_{1,2}=r_m\left[\frac{1}{\sqrt{z}}(1\mp\sqrt{1-z})\right]^{1/2\alpha},\quad z=\left(\frac{U_m}{E}\right)^2. \tag{29}$$

For levels with $E<U_m$ the variable z is real ($0<z<1$), provided that we ignore the level width. The quantization rule (2) assumes the form

$$A_\alpha z^{(1-\alpha)/4\alpha}F\left(\frac{1-\alpha}{4\alpha},\frac{1+\alpha}{4\alpha};1+\frac{1+2\alpha}{2\alpha};z\right)=\lambda^{(1-\alpha)/4\alpha}, \tag{30}$$

where $\lambda=gn^{4\alpha/(1-\alpha)}$, and

$$A_\alpha=\Gamma\left(\frac{1-\alpha}{2\alpha}\right)\Big/2^{1/2\alpha}\sqrt{\pi}\,\Gamma\left(\frac{1}{2\alpha}\right). \tag{30'}$$

The solution to this equation remains real up to $z=1$, which corresponds to an effective coupling constant

$$\lambda=\lambda_*=\left[\frac{2^{3/2}\alpha}{\pi(1-\alpha^2)}\right]^{4\alpha/(1-\alpha)}$$

For $\lambda>\lambda_*$ the analytic continuation of Eq. (30) has the form

$$A_\alpha z^{(1-\alpha)/4\alpha}F\left(\frac{1-\alpha}{4\alpha},\frac{1+\alpha}{4\alpha};1+\frac{1}{2\alpha};z\right)-\frac{i(1-z)}{2^{3/2}\alpha}z^{-(1+\alpha)/4\alpha}\times F\left(\frac{3\alpha-1}{4\alpha},\frac{5\alpha-1}{4\alpha};2;1-z\right)=\lambda^{(1-\alpha)/4\alpha}. \tag{31}$$

This equation determines (in the quasiclassical approximation) the energies of above-the-barrier resonances for model (28): $E=-2\sqrt{g/z}$. In particular, as $g\to\infty$ we have $z\propto g^{-(1-\alpha)/2(1+\alpha)}\to 0$, and on the left-hand side of Eq. (31) the second term is predominant. Hence the asymptotic formula

$$E_n(g)\approx c_{2\alpha}n^{-2\alpha/(1-\alpha)}(\lambda e^{-i\pi})^{1/(1+\alpha)}, \tag{32}$$

with $c_{2\alpha}$ the same coefficient as in Eq. (23′) at $\nu=2\alpha$. Several remarks are in order.

(a) Under the substitution $g\to -g$ the potential (28) becomes a cutoff potential, in which there is only a discrete spectrum.[4] Here Eq. (30) remains valid if we substitute $-z$ for z, and the asymptote (32) becomes real.

(b) The case $\alpha=1/2$ corresponds to the spherical model of the Stark effect. Here $A_\alpha=1/2$, $g\equiv\mathscr{E}$ is the electric field, and $z=4\mathscr{E}/E^2=16F/\epsilon^2$, with $\lambda\equiv F$ and ϵ variables defined in (36). The field dependence of the reduced energy $\epsilon=2n^2E$ is determined by equations that follow from (30):

$$z^{-3/4}G(z)=2F^{1/4}, \quad \epsilon=-4\sqrt{\frac{F}{z}}, \tag{33}$$

where $G(z)$ is the same function as in (26). The value $z=1$ corresponds to the "classical" ionization threshold $F_*=2^{10}(3\pi)^{-4}=0.1298$, (see Refs. 25 and 26), and for $F>F_*$ the resonance energy and width are determined by the equation

$$z^{-3/4}[G(z)-i\sqrt{2}G(1-z)]=2F^{1/4}, \tag{34}$$

which can easily be solved numerically.

(c) For $\alpha=\frac{1}{2}$ (and only in this case), the same hypergeometric function enters into Eqs. (30) and (31), in view of which the quasiclassical equations simplify considerably.

(d) The case $\alpha=1$ corresponds to the exactly solvable potential $U(r)=-(r^{-2}+gr^2)$ considered in Ref. 20.

6. THE ASYMPTOTES OF STARK RESONANCES

The quasiclassical quantization rules in the below-barrier region were obtained in a convenient form in Refs. 17 and 25. They can be written as follows:

$$\left(\frac{\beta_1}{z_1}\right)^{3/4} G(-z_1)=2\nu_1F^{1/4},$$
$$\left(\frac{\beta_2}{z_2}\right)^{3/4} G(z_2)=2\nu_2F^{1/4}, \quad \beta_1+\beta_2=1 \tag{35}$$

[for Rydberg states $(n_1,n_2,0)$ with $n=n_1+n_2+1\gg1$, where n_1, n_2, and m the parabolic quantum numbers], where $z_i=16\beta_iF/\epsilon^2$, the β_i are the decoupling factors, and ϵ, F, and ν_i are the "reduced" variables

$$\epsilon=2n^2\left(E_r-\frac{i\Gamma}{2}\right), \quad F=n^4\mathscr{E}, \quad \nu_i=\frac{1}{n}\left(n_i+\frac{1}{2}\right) \tag{36}$$

(we use atomic units). In (35) we ignored corrections of the order of $\hbar^2$; allowance for these is no problem.[27]

The variable z increases monotonically with F. At $z=1$ the quasiclassical energy ϵ acquires a singularity; this corresponds to an electric field strength equal to F_*, the classical ionization threshold.[5)] The parameter a in (4) in this case can also be calculated analytically,

$$a=\frac{n}{4F}\left(-\frac{\epsilon}{2}\right)^{3/2} G(1-z_2), \tag{37}$$

with $a(F)\to0$ as $F\to F_*$:

$$a(F)=na_*(1-z_2)+..., \quad a_*=\begin{cases}2^{-13/2}3\pi & \text{if } \nu_1=0,\\ \frac{3}{8}(1-\nu_1) & \text{as } \nu_1\to1.\end{cases} \tag{38}$$

If we combine this with (9), we see that the analytic continuation of Eq. (35) into the above-the-barrier region $(F>F_*)$ can be done via the substitution

TABLE II.

n_1,n_2,m	$-E_r$, cm^{-1} Theory	Expt.[28]	$\Gamma/2$, cm^{-1} Theory	Expt.[28]	F	F_*
16,1,0	106.9	103.8	9.0	9.0	0.343	0.265
15,1,0	167.8	167.9	0.8	2.1	0.273	0.263
15,0,0	196.5	198.5		1.1×10^{-4}	0.214	0.308
14,2,0	212.1	210.1	5.4	6.6	0.273	0.236
13,2,0	273.6	275.4		0.23	0.214	0.233
12,3,0	313.3	314.8		1.6	0.214	0.214
11,4,0	353.8	351.4	2.5	3.0	0.214	0.200
11,3,0	384.2	386.3		1.8×10^{-3}	0.165	0.211
10,4,0	418.7	419.2		3.2×10^{-2}	0.165	0.197

Here n_1, n_2, and m are parabolic quantum numbers of a Stark resonance, $F=n^4\mathscr{E}$ the reduced electric field strength, and F_* the classical ionization threshold[22] (the case $F>F_*$ corresponds to above-the-barrier resonances, and the case $F<F_*$ to below-barrier resonances).

$$1-z_2\to(1-z_2)e^{-2\pi i}, \tag{39}$$

under which the function $G(z_2)$ in the second equation in (35) is replaced by $\tilde{G}(z_2)$ [see Eq. (C4)], while the first equation remains unchanged.

The derived equations can easily be solved by numerical methods (the authors are grateful to A. V. Sergeev who performed the necessary calculations). In Table II the calculated values of E_r are compared with the positions of the peaks in the photoionization cross section established in experiments with hydrogen atoms (at $\mathscr{E}=16.8$ kV cm^{-1}; see Ref. 28). Evidently the quasiclassical results agree with the experimental data to within the accuracy of the latter (1–2 cm^{-1}; see Ref. 28). Since for $F<F_*$ the solution to Eqs. (35) is real and does not determine the resonance width, there are corresponding gaps in Table II. In this case more exact equations[19,20] must be used that will allow for the finite value of the barrier factor.

In conclusion let us briefly discuss the strong-field limit. Analysis of the equations shows that $\epsilon\propto F^{2/3}$ and $\beta_i\propto F^{1/3}$ as $F\to\infty$. We consider two limiting cases: the longest-lived states $(n-1,0,0)$ (for a fixed field $\mathscr{E}$), and the short-lived states $(0,n-1,0)$. It can be demonstrated that in the first $\beta=1$ and $\beta=0$ for any value of F, and in the second $\beta=0$ and $\beta=1$. Hence, the system (36) simplifies.[6)] The asymptotic expansion of the reduced energy ϵ has a somewhat more complicated form than (17) and contains logarithms in addition to power corrections proportional to $F^{-k/3}$ to the leading term in the asymptotic expansion.[7)] Using (C1), we get

$$\epsilon=(3\pi f)^{2/3}[1-k_1f^{-1/3}\ln f-k_2f^{-1/3} + O(f^{-2/3}\ln^2 f)]. \tag{40}$$

Here

$$k_1=\tfrac{2}{3}(3\pi)^{-4/3}=0.0335,$$

$$k_2=(3+6\ln2+4\ln3\pi)k_1=0.5402,$$

and the variable f is equal to F in case (a) and to $Fe^{-i\pi}$ in case (b). The reason why F is replaced by $Fe^{-i\pi}$ when we go from (a) to (b) is that the equations for the states

(n_1,n_2,m) and (n_2,n_1,m) of the hydrogen atom are interchanged by the substitutions $F\to -F$ and $\beta_1\rightleftharpoons\beta_2$. Note that (40) has an imaginary part $\epsilon''=-\mathrm{Im}\,\epsilon(F)$ only in case (b). The reason is that in case (a) $\epsilon''(F)$ is of extra order of smallness in the parameter $1/n$ (see Ref. 26).

Qualitatively, the asymptotic expansion (40) agrees with the results of numerical calculations of Rydberg states at $F\sim 1$, but its accuracy is insufficient for quantitative calculations. Here, as in the case of an anharmonic oscillator, we must allow for second-order corrections $\propto f^{-2/3}$, which raises no serious difficulties. At present such calculations are done.

We would like to thank P. É. Volkovitskiĭ, N. B. Delone, B. M. Karnakov, and A. V. Sergeev for useful discussions and helpful remarks.

APPENDIX A

Let us consider the effective coupling constant and the expansion parameter in (17) and (18).

The energies of highly excited levels $(n\gg 1)$ in the potential

$$U(r)=\delta^{-1}r^{\delta}-gr^{\nu},\quad \nu>\delta>-2, \tag{A1}$$

which transforms at $\delta=2$ into (22), can be determined from the quantization rule (2). Introducing the scaled parameters

$$\varepsilon=2n^{-2\delta/(\delta+2)}E_{nl},\quad \lambda=gn^{2(\nu-\delta)/(\delta+2)},$$
$$\rho=n^{-1}(l+\tfrac{1}{2}),\quad x=rn^{2/(\delta+2)}, \tag{A2}$$

where l is the orbital angular momentum, we obtain for the reduced energy $\varepsilon=\varepsilon(\lambda)$ an equation containing no large parameters when $n\to\infty$:

$$\int_{x_0}^{x_1}(\varepsilon-\rho x^{-2}-2\delta^{-1}x^{\delta}+2\lambda x^{\nu})^{1/2}dx=\pi. \tag{A3}$$

If the coupling constant g tends to zero, the level's energy is determined by an attraction $\propto r^{\delta}$. For one thing, for ns-levels at $g=0$ we get

$$E_{ns}\underset{b\to\infty}{\approx}\begin{cases}-An^{-2\sigma/(2-\sigma)} & \text{if } \delta=-\sigma<0,\\ A'n^{2\delta/(\delta+2)} & \text{if } \delta>0,\end{cases} \tag{A4}$$

where $n\gg 1$, and the constants A and A' are independent of n; for instance,

$$A=(2\pi)^{-\sigma/(2-\sigma)}\sigma^{-2/(2-\sigma)}\left[\Gamma\left(\frac{2-\sigma}{2\sigma}\right)\Big/\Gamma(\sigma^{-1})\right]^{2\sigma/(2-\sigma)} \tag{A5}$$

$[0<\sigma<2$; the coefficient A is approximately equal to σ^{-1} as $\sigma\to 0$ and is proportional to $(2-\sigma)^{4/(2-\sigma)}$ as $\sigma\to 2]$. From (A2) and (A3) it follows that $\langle x\rangle\sim 1$, that is, the average radius of a bound state, $\langle r\rangle$, is proportional to $n^{2/(\delta+2)}$. For instance, $\delta=-1$ corresponds to the Coulomb potential, with $A=\frac{1}{2}$ and $\langle r\rangle=\frac{1}{2}(3-\rho^2)n^2$.

As g increases, repulsion becomes stronger (we assume that $\nu>0$), and when $\lambda\sim 1$ it balances the attraction at characteristic distances $r\sim\langle r\rangle$. Here both terms in potential (A1) are of the same order of magnitude; from the condition that $r^{\delta}\sim gr^{\nu}$ at $r\sim\langle r\rangle$ we get

$$gn^{2(\nu-\delta)/(\delta+2)}\sim 1.$$

This suggests that for excited states λ is the effective coupling constant rather than g.

Putting $U(r)=v(r)-gr^{\nu}$ in Eq. (8), we find that as $g\to\infty$,

$$\int_{r_0}^{r_2}p_0dr=n\pi(1+\tfrac{1}{2}\Delta+...), \tag{A6}$$

where

$$p_0=\sqrt{2(E+gr^{\nu})},\quad \Delta=\int_{r_0}^{r_2}\frac{vdr}{p_0}\Big/\int_{r_0}^{r_2}\frac{dr}{p_0}.$$

This leads to the estimate $\Delta\sim r_2^{\delta}E^{-1}$ or, allowing for (15),

$$\Delta\sim(g^{-\delta}E^{\delta-\nu})^{1/\nu}$$
$$\sim n^{2(\delta-\nu)/(\nu+2)}g^{-(\delta+2)/(\nu+2)}\sim\lambda^{-(\delta+2)/(\nu+2)}. \tag{A7}$$

Thus, $\lambda^{-\alpha}$ is the expansion parameter in (17); for one thing, for the potential (22) we have $\delta=2$ and $\alpha=4/(\nu+2)$.

From Eq. (A3) it also follows that the leading term $\widetilde{E}_n(g)$ in the tight-binding mode does not depend on δ, that is, on the shape of the attractive potential that binds the particle at small distances. However, such a dependence does exist in the next terms of the asymptotic expansion. For instance, for the potential (A1) we have $\alpha=(\delta+2)/(\nu+2)$ and

$$\delta_1^{(0)}=\frac{\Gamma((\delta+1)/\nu)\Gamma(\nu^{-1}+\frac{1}{2})}{\delta\Gamma(\nu^{-1})\Gamma\nu^{-1}(\delta+1)+\frac{1}{2})}c^{(\delta-\nu)/\nu}, \tag{A8}$$

where c_ν is the coefficient from (23'). In particular, for $\nu\gg 1$ we have

$$c_\nu=\frac{\pi^2}{2},\quad d_1^{(0)}=\frac{2}{\pi^2\delta(\delta+1)}, \tag{A9}$$

and also in (17) as $N\to\infty$,

$$C_N=\frac{\pi^2}{8},\quad d_1^{(0)}=\frac{1}{3\pi^2},\quad d_0^{(2)}=\frac{N}{3\pi^2}. \tag{A10}$$

Since $r_0=0$ for s-states, δ must be greater than -1 for the integral in (A6) to have a finite value. The case of $\delta=-1$ corresponds to a Coulomb singularity appearing in $U(r)$ when $r\to 0$. Here the integral $\int_0^{r_2}vp_0^{-1}dr$ is divergent at the lower limit, and expansion (23) transforms into

$$E_n(g)\approx\widetilde{E}_n[1+\alpha_1'(\lambda e^{-i\pi})^{-\alpha}\ln(\lambda e^{-i\pi})+d_1(\lambda e^{-i\pi})^{-\alpha}$$
$$+O(\lambda^{-2\alpha}\ln^2\lambda)], \tag{A11}$$

where $\alpha=\alpha_0=1/(\nu+2)$, and $\lambda=\widetilde{g}n^{2\nu+2}$. As $\delta\to -1$, the coefficient (A8) diverges. Allowing for the fact that

$$\lim_{\delta\to -1}\Gamma\left(\frac{\delta+1}{\nu}\right)(\lambda^{-\alpha}-\lambda^{-\alpha_0})=-\frac{1}{\nu+2}\lambda^{-\alpha_0}\ln\lambda,$$

we get

$$d_1'=-\frac{\sqrt{2/c_\nu}}{\pi(\nu+2)^2}.$$

The case where $\nu=1$ corresponds to the Stark effect in the hydrogen atom; here

$$\alpha=\tfrac{1}{3},\quad c_1=\tfrac{1}{2}(3\pi)^{2/3},\quad d_1'=-\tfrac{2}{3}(3\pi)^{-4/3}, \tag{A12}$$

which reproduces the first terms in expansion (40).

Let us now discuss resonance widths for potential (13) with $N\gg 1$. As noted in Sec. 3, in the tight-binding mode $\Gamma/E_r\propto N^{-1}\to 0$. The reason for this smallness (related to the fact that the potential becomes very sharp when $N\to\infty$) can be understood by employing the simple model[8]

$$U(x)=\begin{cases}+\infty & \text{if } x<0,\\ 0 & \text{if } 0<x<R,\\ -U_0 & \text{if } x>R,\end{cases} \tag{A13}$$

which admits of an exact solution. The energies of quasistationary states, $E_n=\frac{1}{2}k_n^2$, can be found by solving the equation

$$k\cot kR=i\sqrt{K_0^2+k^2},\quad K_0=\hbar^{-1}\sqrt{2mU_0}. \tag{A14}$$

A potential with a sharp edge corresponds to the case $\varkappa\equiv(K_0R)^{-1}\ll 1$, when

$$k_nR=n\pi\left\{1-\varkappa^2-i\left[\varkappa-\left(\frac{n^2\pi^2}{6}+1\right)\varkappa^3\right]+\ldots\right\}, \tag{A15}$$

and $\Gamma/E_r\approx 2\varkappa$. The relative smallness of the widths Γ_n and the fact that momentum k_n is close to $n\pi/R$ can, obviously, be attributed to the interference of the waves reflected by the sharp edges at $x=0$ and $x=R$ of the potential.

Finally, a remark concerning the energy spectrum for potential (22) with $\nu>2$. Here the Hamiltonian is a Hermitian rather than self-adjoint operator,[15] and selecting a self-adjoint extension requires fixing an additional boundary condition at infinity.[29] Physically this means that the time τ that it takes a classical particle to get to infinity is, for $\nu>2$, finite:

$$\tau=\int_{x_2}^{\infty}p^{-1}dx\sim\int_{x_2}^{\infty}x^{-\nu/2}dx<\infty,$$

whence in quantum mechanics a particle can be "reflected" at infinity with an arbitrary phase. However, for the questions of interest here large distances are not essential, since the quantization integral contains only the segment from x_0 to x_2, and turning points $x_i\sim(E/g)^{1/\nu}\sim g^{-1/(\nu+2)}\ll 1$ when $g\to\infty$. The authors are grateful to B. M. Karnakov for discussions of this problem.

APPENDIX B

Let us calculate the coefficients $d_k^{(j)}$. The asymptotic expansion (17) can be arrived at in the following way: we regard the term $\frac{1}{2}x^2$ in $U(x)=\frac{1}{2}x^2+gx^N$ as a perturbation, expand the quasiclassical momentum in a series,

$$p(x)=\sum_{k=0}^{\infty}a_kx^{2k}p_0^{1-2k},\quad p_0=\sqrt{2(E-gx^N)}, \tag{B1}$$

with $a_0=1$ and $a_k=-(2k-3)!!/2^kk!$ for $k\geqslant 1$, and integrate (8) term by term. As $g\to\infty$, the terms of the resulting series rapidly decrease as number k increases, which follows from the estimate

$$\oint x^m(E-gx^N)^{1/2-k}dx\propto x_0^{m+1}E^{1/2-k}\propto g^{-(2k+m)/(N+2)} \tag{B2}$$

[we have allowed for the asymptotic expansion (15) of $E(g)$]. As a result we arrive at the equation

$$E^{(N+2)/2N}g^{(N+2)/N(N-2)}\sum_{k=0}^{\infty}2^{-k}a_kS_k(E^{N-2}g^2)^{-k/N}$$
$$=2^{-1/2}\pi\lambda^{2/(N-2)},$$

in which

$$S_k=\oint_{t_2}^{t_0}t^{2k}(1-t^N)^{1/2-k}dt,\quad t=\frac{x}{x_0},\quad x_0=\left(\frac{E}{g}\right)^{1/N} \tag{B3}$$

($t_0=1$ and $t_2=-e^{i\pi/N}$, with N odd). Assuming that $z=(E^{N-2}g^2)^{1/N}$, we get

$$z\left(\sum_{k=0}^{\infty}b_kz^{-k}\right)^{-\beta}=B\lambda^\alpha,\quad z\to\infty, \tag{B4}$$

where

$$\alpha=\frac{4}{N+2},\quad \beta=\frac{2N-4}{N+2},\quad B=\left(\frac{\pi}{S_0\sqrt{s}}\right)^\beta, \tag{B5}$$

and

$$b_0=1,\quad b_k=a_k\frac{S_k}{S^{-k}S_0}\quad\text{for } k\geqslant 1.$$

We seek the solution to (B4) as $\lambda\to\infty$ in the form of a power series in $\lambda^{-\beta}$,

$$z=B\lambda^\beta\left\{1-\frac{\beta b_1}{B}\lambda^{-\beta}-\frac{\beta}{B^2}\left[b_2+\frac{1}{2}(\beta-1)b_1^2\right]\lambda^{-2\beta}+\ldots\right\},$$

from which there follows expansion (17) for the energy,

$$E_n(g)=g^{-2/(N-2)}z^{N/(N-2)},$$

in which

$$d_0^{(0)}=1,\quad d_1^{(0)}=-\frac{2Nb_1}{(N+2)B},$$

$$d_2^{(0)}=-\frac{2N}{(N+2)B^2}\left(b_2+\frac{N-10}{2N+4}b_1^2\right),\ldots,$$

$$B^{N/(N-2)}=C_N\exp\left(-\frac{i\pi}{N+2}\right), \tag{B6}$$

with C_N the coefficient defined in (16).

Let us now study the integrals in (B3). At $k=0$ and $k=1$ these can be evaluated in the same way as in (14):

$$S_0=2\sqrt{\pi}(1+e^{i\pi/N})\Gamma\left(\frac{1}{N}\right)\Big/(N+2)\Gamma\left(\frac{N+2}{2N}\right), \quad \text{(B7)}$$

$$S_1=2\sqrt{\pi}(1+e^{3\pi i\pi/N})\Gamma\left(\frac{3}{N}\right)\Big/N\Gamma\left(\frac{N+2}{2N}\right). \quad \text{(B8)}$$

To generalize this result to arbitrary k it is convenient to examine somewhat more general integrals by introducing a parameter $M>0$:

$$T_{km}(\mu)=\oint_C t^m(\mu-t^N)^{1/2-k}dt=\mu^{1/2-k+(m+1)/N}T_{km} \quad \text{(B9)}$$

(the contour C encircles in positive direction the branch points $t_0=1$ and t_2). Taking the derivative of (B9) with respect to μ and then putting $\mu=1$, we arrive at the recurrence formula

$$T_{k+1,m}=\left[1-\frac{2(m+1)}{(2k-1)N}\right]T_{km}. \quad \text{(B10)}$$

On the other hand, a direct calculation yields

$$T_{0m}=2\sqrt{\pi}(1-t_2^{m+1})\Gamma\left(\frac{m+1}{N}\right)\Big/(N+2m+2)\times\Gamma\left(\frac{N+2m+2}{2N}\right). \quad \text{(B11)}$$

From this it follows that at $k=1,2,3,...,$

$$T_{km}=(-1)^{k-1}(1-t_2^{m+1})\frac{2^k\sqrt{\pi}}{(2k-3)!!}\Gamma\times\left(\frac{m+1}{N}\right)\Big/N\Gamma\left(\frac{m+1}{N}+\frac{3}{2}-k\right) \quad \text{(B12)}$$

[here $(-1)!!=1$]. If $m\equiv-1(\text{mod } N)$, then $t_2^{m+1}=1$ and

$$T_{km}=0, \quad m=N-1,2N-1,... \quad \text{(B13)}$$

[this result follows also from the fact that $t^{N-1}(1-t^N)^{1/2-k}$ is a total differential]. Bearing in mind that

$$S_k=T_{k,2k}, \quad 1-t_2^{m+1}=1+(-1)^m\exp\left[\frac{i\pi(m+1)}{N}\right],$$

we finally find that

$$S_k=(-1)^k\frac{2^k\sqrt{\pi}}{(2k-3)!!N}\left[1+\exp\left(\frac{i\pi(2k+1)}{N}\right)\right]\times\Gamma\left(\frac{2k+1}{N}\right)\Big/\Gamma\left(\frac{1}{2N}[3N+2-(2N-4)k]\right), \quad \text{(B14)}$$

which fully determines the coefficients b_k and $d_k^{(0)}$.

Let us consider the contributions of higher-order corrections to the quasiclassical approximation. Allowing for the fact that when $g\to\infty$ the term $x^2/2$ in the expression for $p(x,E)$ can be discarded, we find from Eqs. (12), (13′), and (15) that

$$J_2=\frac{N(N-1)gx_0^{N-1}}{24\pi(2E)^{3/2}}T_{2,N-2},$$

$$\frac{gx_0^{N-2}}{E^2}=e^{i\pi/N}C_N^{(N+2)/N}\frac{1}{n+1/2},$$

from which it follows that

$$d_0^{(2)}=-e^{i\pi}\frac{N^2(N-1)}{48(N+2)}C_N^{-(N+2)/N}\frac{T_{2,N-2}}{T_{00}}, \quad \text{(B15)}$$

$$d_0^{(4)}=e^{2i\pi/N}\frac{N^3(N-1)(N+3/2)}{1536(N+2)}C_N^{-(2N+4)/N}\frac{T_{4,N-4}}{T_{00}},$$

etc. Using Eqs. (16) and (B12), we obtain explicitly, for one thing, Eq. (18) for $d_0^{(2)}$ (note that gamma functions are absent from this formula owing to the identity $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$). For any value of l we have

$$d_0^{(2l)}\propto\exp\left(\frac{i\pi l}{N}\right)\frac{T_{l,(N-2)l}}{T_{00}}, \quad \text{(B16)}$$

and from (B12) it follows that (N is odd) the phase of $T_{l,(N-2)l}/T_{00}$ coincides with the phase of

$$\frac{1+\exp[-i\pi(2l-1)/N]}{1+\exp(i\pi/N)},$$

and this cancels out $e^{i\pi l/N}$ completely. Thus, all the coefficients $d_0^{(j)}$ in (18) are real.

Finally, in the case of a cubic oscillator ($N=3$) we have $b_1=b_4=...=0$, and

$$b_2=\tfrac{5}{12}[\Gamma(\tfrac{2}{3})\Gamma(\tfrac{5}{6})/\Gamma(\tfrac{1}{3})\Gamma(\tfrac{1}{6})]e^{-i\pi/3}, \quad b_3=-\tfrac{5}{648},$$

from which we obtain

$$d_2^{(0)}=-\frac{6b_2}{5B^2}=-\frac{1}{25\pi\sqrt{3}}C_3e^{-i\pi/5},$$

$$d_3^{(0)}=-\frac{6b_3}{5B^3}=-\frac{1}{108C_3}e^{i\pi/5}, \quad d_4^{(0)}=\frac{11}{12}(d_2^{(0)})^2, \quad \text{(B17)}$$

$$d_0^{(2)}=\frac{\sqrt{3}}{50\pi}, \quad d_1^{(2)}=0,...,$$

where C_3 is the constant of Eq. (16). The numerical values of these coefficients are listed in (19).

APPENDIX C

Let us list the properties of the function $G(z)$ introduced in (26). As $z\to0$,

$$G(z)+z+\tfrac{3}{32}z^2+...,$$

and at $z=1$ the function has a singularity:

$$G(z)=\frac{2^{7/2}}{3\pi}\left[1+\frac{3}{16}G(t)\ln t-\sum_{n=1}^{\infty}g_nt^n\right], \quad \text{(C1)}$$

where

$$g_1=\tfrac{9}{8}(\ln 2+\tfrac{1}{6}), \quad g_2=\tfrac{27}{256}(\ln 2-\tfrac{17}{36}),...,$$

and $t=1-z\to 0$. This function has a cut $1<x<\infty$, the jump at which is

$$\Delta G(x)=\frac{1}{2i}\left[G(x+i0)-G(x-i0)\right]=-\frac{1}{\sqrt{2}}G(1-x), \tag{C2}$$

with $\Delta G=(x-1)/\sqrt{2}+...$ as $x\to 1$, and $\Delta G\approx cx^{3/4}(c>0)$ as $x\to\infty$.

According to rule (39), upon bypassing the branch point the function undergoes the following transformation:

$$G(z)\to\tilde{G}(z)=G(z)-i\sqrt{2}G(1-z) \tag{C3}$$

[see formula 2.10(12) in Ref. 30]. Hence,

$$\tilde{G}(z)=-\frac{16i}{3\pi}\left[1+\frac{3}{16}G(z)(\ln z+i\pi)-\sum_{n=1}^{\infty}g_n z^n\right],\quad z\to 0. \tag{C4}$$

Note that in (34) $z\propto F^{-1/3}\to 0$ as $F\to\infty$, whence (C4) implies the asymptote (40). Finally, when $z\to\infty$,

$$G(z)=B_1e^{-i\pi/4}z^{3/4}-B_2e^{-3\pi i/4}z^{1/4}+O(z^{-1/4}),$$

$$B_1=2^{3/2}\frac{\Gamma(1/4)}{3\sqrt{\pi}\Gamma(3/4)},\quad B_2=\frac{16}{3\pi B_1}. \tag{C5}$$

Here we have allowed for the fact that $-z=ze^{i\pi}$, since in our case ε acts as z, and

$$\tilde{G}(\varepsilon)=D\varepsilon^{3/4}\sum_{k=0}^{\infty}h_k\varepsilon^{-k/2}=32\lambda, \tag{C6}$$

where

$$D=B_1e^{i\pi/4},\quad h_0=1,$$

$$h_1=\frac{iB_2}{B_1},\quad h_2=\frac{3}{8}B_1(1-e^{-i\pi/4}),\ ...$$

(all the other coefficients h_k can also be written explicitly). This implies that $|\varepsilon|\propto g^{4/3}$, and at $E_n=\varepsilon_n/16g$ we arrive at expansion (23) in which $\alpha=\frac{2}{3}$ and

$$c_4=\left(\frac{3\pi}{4}B_2\right)^{1/3}=\left[3\sqrt{2\pi}\Gamma\left(\frac{3}{4}\right)\bigg/\Gamma\left(\frac{3}{4}\right)\right]^{4/3}. \tag{C7}$$

which fully agrees with formula (23′) obtained directly from the quantization rule (8).

1)The results of this section were announced in Refs. 20 and 21.

2)What is important here is that for above-the-barrier resonances the parameter a lies in the second quadrant of the complex plane. As Eq. (5) shows, in the below-barrier region we have Re $a>0$ and the imaginary part of a is exponentially small. As the level 'touches' the top of the barrier ($E_r=U_m$), point a crosses the imaginary axes, and after that $\arg a>\pi/2$.

3)Since for a quasistationary state $-\pi<\arg E(g)<0$, the turning point x_2 lies in the upper half-plane and the point x_1 in the lower. This can be shown to be true also for finite values of g.

4)For one thing, at $\alpha=1/2$ we arrive at the funnel potential often used in QCD.

5)The numerical value of F_* depends on ν_1 and ν_2 and changes from 0.1298 at $\nu_1=0$, that is, for states $(0,n-1,0)$ with $n\to\infty$, to 0.3834 at $\nu=1$ [states $(n-1,0,0)$]; see Refs. 25–27.

6)For levels $(0,n-1,0)$ with $n\gg 1$ these equations coincide with (34). Thus, the spherical model has a direct physical meaning.

7)This is due to the Coulomb singularity in the potential as $r\to 0$ (for states with $m=0$, i.e., without a centrifugal barrier). In this respect see Eq. (A10). Note, however, that numerically k_1 is small.

8)We have P. É. Volkovitskiĭ to thank for this example.

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Translated by Eugene Yankovsky

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