A system of unsteady-state dynamical equations for a superconductor

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A system of equations for treating the spatial and temporal dynamics of a superconductor is presented and unifies the order parameter, the free quasiparticles, the phonons and the applied electromagnetic field within a single dynamic complex.

1. It has been shown[1,2] that a superconductor may, in some sense, be thought of as a laser generating a Bose condensate of Cooper pairs. There is a heuristic significance in viewing a superconductor in this way because the dynamic operating regimes of the laser are extremely diverse in character and have been studied in great detail both theoretically and experimentally. Depending on the parameters, one can observe an aperiodic or pulsed transition to steady-state lasing; undamped periodic pulsed operation; or chaotic strange-attractor lasing. In multimode operation, pulsed power radiation corresponding to locking of individual modes can be observed.[3–7]

In superconductors, a gamut of regimes at least as wide as that may be expected. The theoretical treatment of nonlinear superconductor dynamics requires a well-developed system of dynamic equations including those for the order parameter, phonons, external electromagnetic field, and the density of quasiparticles. So far, however, no *ab initio* system of this kind has been obtained.

In analyzing unsteady-state regimes, various quasi-steady approximations are used. One of these is based on a free-quasiparticle kinetic equation[8,9] of the form

$$\frac{\partial n_{\mathbf{p}}}{\partial t} + \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}} = -S_{\mathbf{p}}^{R} + S_{\mathbf{p}}^{c} + Q_{\mathbf{p}}. \tag{1}$$

In this equation, n_p is the concentration of quasiparticles of energy $\varepsilon(\mathbf{p})$; $S_{\mathbf{p}}^{R}$ the recombination integral related to the creation or recombination of free quasiparticles due to the breaking or formation of Cooper pairs; $S_{\mathbf{p}}^{c}$ the sum of collision integrals unrelated to the creation or recombination of quasiparticles; $Q_{\mathbf{p}}$ the external source of free quasiparticles.

The recombination integral $S_{\mathbf{p}}^{R}$ plays an important role in unsteady dynamic processes. It has been shown[8,9] that

$$S_{\mathbf{p}}^{R} = \int U_n(\varepsilon, \varepsilon', \Psi^2)[n_{\mathbf{p}}n_{\mathbf{p}'}(N+1)_{\mathbf{q}} - (1-n)_{\mathbf{p}}(1-n)_{\mathbf{p}'}N_{\mathbf{q}}]d^3p, \tag{2}$$

where $U_n(\varepsilon,\varepsilon',\Psi^2)$ is a function proportional to the coupling between the charged particles and phonons (see Refs. 8 and 9); $N_{\mathbf{q}}$ the density of phonons with momentum $\mathbf{q}$; $\psi = \Psi e^{i\varphi}$ is the order parameter, with a modulus Ψ proportional to the width of the superconducting energy gap (Ψ^2 being proportional to the Cooper-pair density). The first term in the bracket describes the process in which two quasiparticles recombine into a Cooper pair by emitting a phonon of energy $\hbar\Omega$; the second term represents the breaking of a Cooper pair and creation of two quasiparticles by absorbing a phonon of the same energy. In these processes, the following energy and momentum conservation laws must be obeyed:

$$\hbar\Omega = \varepsilon(\mathbf{p}, \Psi) + \varepsilon(\mathbf{p}', \Psi), \qquad \varepsilon(\mathbf{p}', \Psi) \equiv \varepsilon', \tag{3}$$

$$\mathbf{p} + \mathbf{p}' = \mathbf{q}. \tag{4}$$

The quasiparticle energy $\varepsilon(\mathbf{p},\Psi)$ is measured from the Fermi energy ε_F. To determine the dependence of $\varepsilon(\mathbf{p},\Psi)$ on its arguments (the momentum of a quasiparticle and the modulus of the order parameter, respectively), a microscopic theory is needed. The Bardeen–Cooper–Schrieffer (BCS) theory gives, for example,

$$\varepsilon(\mathbf{p}, \Psi^2) = \Delta_0(\xi^2 + \Psi^2)^{1/2}, \qquad \xi = v_F(\mathbf{p} - \mathbf{p}_F)/\Delta_0, \qquad \Psi = \Delta/\Delta_0. \tag{5}$$

Here v_F and $\mathbf{p}_F$ are the velocity and momentum of the Fermi surface electrons; Δ is the superconducting energy gap at nonzero temperature; Δ_0 is the same at absolute zero temperature.

Now equation (1) is obviously an unclosed one and must be supplemented by a relation between Ψ and $n_{\mathbf{p}}$. For this purpose, it is customary[8,9] to use the BCS equation[9]

$$\Lambda \int [(1 - 2n_{\mathbf{p}})/\varepsilon(\mathbf{p}, \Psi)]\, d^3p = 1, \tag{6}$$

where Λ is the lattice-electron coupling constant.

The BCS equation determines the steady-state value of the order parameter. For a time-dependent $n_{\mathbf{p}}$, this equation is only adequate if both $n_{\mathbf{p}}$ and Ψ vary sufficiently slowly. This is why Eqs. (1) and (6) represent a quasi-steady-state approximation to the dynamical description of a superconductor.

An alternative approach to the nonlinear dynamics of a superconductor is to write the order parameter equation

$$\frac{\partial \psi}{\partial t} - \frac{1}{\tau}(\alpha - \beta\Psi^2)\psi + D\left(i\nabla - \frac{2e}{\hbar c}\mathbf{A}\right)^2\psi = 0, \tag{7}$$

known as the nonstationary Ginzburg-Landau equation.[10,11] In this equation α and β are ψ- and $\mathbf{A}$-independent constants; $\mathbf{A}$ is the vector potential of the electromagnetic field; D is the diffusion coefficient of Cooper pairs. Although Eq. (7) is closed and so cannot be used together with (1), we can regard it as a quasi-steady-state version of a more complete—but as yet unavailable—system of equations from which both quasiparticles and phonons are adiabatically excluded as dynamic variables. Formally, such a system is obtainable by a microscopic approach similar to the one used in Ref. 11 to justify Eq. (7). This, however is a difficult as yet unsurmountable task.

In Ref. 12 the author employed the laser analogy to

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postulate a system of equations permitting a unified dynamic description of Cooper pairs, quasiparticles, and phonons. It is the purpose of the present paper to provide a justification for that system.

2. We use the law of conservation of the total number of particles as a basis for the unsteady-state system of equations.

The dynamics of a superconductor are determined by three major subsystems of charge particles: Cooper pairs, whose density proprotional to Ψ^2; $n_{\rm p}$ free quasiparticles of energy $\varepsilon > \Delta$ ('n' subsystem); and $m_{\rm p}$ quasiparticles with energies near the Fermi energy ε_F ("m" subsystem).

It is known[13] that even at absolute zero temperature the conduction electrons do not all form Cooper pairs. We shall denote by n_S the total number of the electrons paired at absolute zero. Now suppose that the temperature is close to 0 K and that we apply a magnetic field to the superconductor. Suppose further that the magnetic field is increased adiabatically slowly. But then it is known[14] that the field partially penetrates into the sample and that it decreases the modulus of the order parameter in the penetration region. As a result of this, the density of the paired electrons decreases and becomes less than n_S. Where have these Cooper pairs gone? Since $T = 0$, clearly these pairs cannot transform into free quasiparticles with $\varepsilon > \Delta$. It follows then that they must go into the subsystem of unpaired electrons with energies close to ε_F (subsystem "m" in our notation). Clearly for $T > 0$ the above three subsystems will all participate in the dynamical processes in the superconductor—but subject to the conservation law

$$m + n + n_s\Psi^2 = n_s, \tag{8}$$

where n is the total number of the free quasiparticles.

Using the conservation of the number of particles we write

$$\frac{\partial\Psi^2}{\partial t} = \Sigma_m + \Sigma_n - 2D\,{\rm Re}\left[\psi^*\left(i\nabla - \frac{2e}{\hbar c}{\rm A}\right)^2\psi\right]. \tag{9}$$

In this equation $\Sigma_n = 0.5 \int S^R_{\rm p} d^3p$, and Σ^R_m is the recombination integral for transitions between the Cooper-pair and m subsystems. The third term on the right accounts for the motion of the Cooper pairs and their interaction with the field. The form of this term is selected such as to fully correspond to its counterpart in the Ginzburg-Landau Eq. (7).

Although the integrals Σ^R_m and Σ^R_n should be analogous in form, there is a difference between then in that the momenta of the paired m particles are equal in magnitude and opposite in direction. Thus

$$\Sigma^R_m = 0{,}5\int U_m({\rm p}, \Psi^2)[m_{\rm p}m_{-{\rm p}} - (1 - m)_{\rm p}(1 - m)_{-{\rm p}}]d^3p. \tag{10}$$

The factor 0.5 arises from the fact that the recombination of two m or n particles produces only one Cooper pair. As in the integral (2), the quantity $U_m(\mathbf{p},\Psi^2)$ is proportional to the square of the Cooper pair/m-particle interaction matrix element. In what follows, we will express U_m in terms of the phenomenological parameters of the superconductor, so that the specific form of U_m is of no importance here. Once should only recognize that $U_m(\mathbf{p},\Psi^2)$ is a function of Ψ^2 and that it tends to zero as $\Psi \to 0$—as indeed it must because of the Ψ^2 behavior of the square of the matrix element for the transition to the state with Ψ^2 bosons. Thus we may write $U_m(\mathbf{p},\Psi^2) = \Psi^2 V_m(\mathbf{p})$. Now since the integration region in (10) is close to the Fermi momentum $\mathbf{p}_F$, we have

$$\Sigma^R_m \approx \Psi^2 V_m({\rm p}_F)\int (m_{\rm p} - 0{,}5)d^3p. \tag{11}$$

The integral $\int (m_{\rm p} - 0.5)d^3p$ is equal to the total number m, of particles in the subsystem and from the conservation law (8) we have

$$\int (m_{\rm p} - 0{,}5)d^3p = n_s - n - n_s\Psi^2. \tag{12}$$

Substituting (12) into (11) we find that

$$\Sigma^R_m = \frac{2}{\tau}\left(1 - \frac{n}{n_s} - \Psi^2\right)\Psi^2, \quad \frac{2}{\tau} = n_s V_m({\rm p}_F), \tag{13}$$

and hence Eq. (9) may be rewritten as

$$\frac{\partial\Psi^2}{\partial t} = \frac{2}{\tau}\left(1 - \frac{n}{n_s} - \Psi^2\right)\Psi^2 - 2D\,{\rm Re}\left[\psi^*\left(i\nabla - \frac{2e}{\hbar c}{\rm A}\right)^2\psi\right] + \Sigma^R_n. \tag{14}$$

To determine the complex order parameter ψ, an equation corresponding to the conservation law (14) may now be written. For this purpose we represent Σ^R_n as a product of two factors,

$$\Sigma^R_n = 2\sigma_R(\Psi, n)\Psi^2. \tag{15}$$

The equation for the complex order parameter ψ then takes the form

$$\frac{\partial\psi}{\partial t} = \frac{1}{\tau}\left(1 - \frac{n}{n_s} - \Psi^2\right)\psi - D\left(i\nabla - \frac{2e}{\hbar c}{\rm A}\right)^2\psi + \sigma_R(\Psi, n)\psi. \tag{16}$$

This equation forms a joint system with the free quasiparticle kinetic equation (1). It should only be kept in mind that if fields are present, it is necessary to consider the energy ε as dependent in a canonical fashion on the fields.[15]

As usual, the description of electromagnetic field dynamics requires the Maxwell equations be used; these are related to Eqs. (1) and (16) through the superconducting-current density $\mathbf{j}_S$ given by[10]

$$\mathbf{j}_s = \frac{e\hbar}{m_e}n_s\Psi^2\left(\nabla\varphi - \frac{2e}{\hbar c}{\rm A}\right). \tag{17}$$

Both the recombination and collision integrals are dependent on the density of phonons in the superconducting sample. The nonequilibrium dynamics of phonons may be described by a kinetic phonon equation, for which one of the simplest possible forms is[9]

$$\frac{\partial N_{\rm q}}{\partial t} = -\frac{1}{\tau_{ph}}(N_{\rm q} - N_{{\rm q}0}) + S^R_{\rm q} + S^c_{\rm q} + I_{\rm q}, \tag{18}$$

Here $N_{\rm q}$ is the density of phonons of momentum $\mathbf{q}$; $N_{{\rm q}_0}$ the equilibrium distribution function for phonons; $I_{\rm q}$ the intensity of the external source of phonons; $S^R_{\rm q}$ the recombination integral as expressed in terms of phonon momenta; $S^c_{\rm q}$ the free particle–phonon collision integral.

Substitution of the equilibrium quasiparticle distribution function into (16) causes the recombination integral to vanish and the total concentration of quasiparticles, n, becomes a function of temperature alone; Eq. (16) then goes over into the classical Ginzburg–Landau equation.

By appropriately introducing a recombination integral, it is easy to extend the system (1) and (16) to incorporate the effects of the excitation and/or recombination of free quasiparticles due to the applied electromagnetic field.

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