Formation of extreme (squeezed) states for laser pulses and beams in Bragg diffraction of light in a spatially periodic nonlinear medium

A. P. Alodzhants and S. M. Arakelyan

Erevan State University
(Submitted 23 March 1992; resubmitted 3 September 1992)
Zh. Eksp. Teor. Fiz. **103**, 910–941 (March 1993)

We solve for the first time ever the problem of obtaining squeezed light for space-time wave packets in dynamic diffraction under Bragg resonance conditions. We consider the space-time behavior of the fluctuations of laser pulses and beams propagating in such a nonlinear medium. We obtain a complicated law for the redistribution of the fluctuations along the beam width and along the pulse length, which accounts for oscillations that depend on the control parameters of the problem, with a possibility to optimize noise and to obtain squeezed states. The use in the scheme considered of liquid crystals as the nonlinear inhomogeneous medium makes it possible to hope for an experimental observation of the quantum states of light in the field of relatively low-power lasers with high coherence properties and to obtain in this way laser pulses with extremal characteristics.

INTRODUCTION

At the present time the possibility of forming quantum squeezed states of light during dynamic scattering of radiation in a spatially periodic medium under Bragg resonance conditions has been established in principle.[1] The physics of this effect is determined by interference and energy exchange between two coupled phase-conjugated waves. As the result there is a transfer (redistribution) of fluctuations between different components of the field, and for well defined values of the control parameter of the problem (e.g., the parameters of the spatial lattice in the medium) it becomes possible to suppress the fluctuations in one of the field quadratures when it leaves the system as compared to their level at the entrance. Of course, in this case the Heisenberg uncertainty relations are conserved[2] so that the suppression of the fluctuations in one of the field quadratures is accompanied by their growth in another one.

Moreover, when frequency-modulated light (laser) pulses propagate in such a medium their duration changes; if it decreases one can speak of temporal squeezing—compression,[3] which is usually considered to be a classical effect. The spatial periodicity of the medium then fulfills the role of a temporal delay line used in traditional schemes for obtaining ultrashort laser pulses.

Both these processes take place simultaneously when light pulses propagate in a spatially periodic medium and they in fact affect each other; the wave aspect with its energy exchange is for them the determining factor.

In the present paper we first solve the problem of obtaining pulsed squeezed light for (as the result of phase modulation) frequency-modulated laser pulses in a spatially periodic nonlinear medium under Bragg resonance conditions. We show the conditions for the appearance of this effect and of the temporal compression effect. We consider the spectral-temporal aspect of the problem. The analysis is carried out for the most convenient scattering geometry with two co-propagating waves (Laue scheme). The results obtained are universal in nature for the pulsed wave process and are applicable, in particular, for diffraction of x-ray radiation by a solid crystalline lattice (cf. Ref. 3). Moreover, they are also applicable when laser beams of spatially limited transverse cross-section propagate in such a spatially periodic medium; thanks to the space-time analogy which occurs, one succeeds in describing in a single manner these nonlinear wave processes for light pulses and beams.

Squeezed light for pulsed systems has often been studied theoretically before (see, e.g., Ref. 4 and the literature given there). However, the case considered by us has two principal differences.

Firstly, we have solved a new problem for an optical system under Bragg resonance conditions when it is necessary to take spatial (instead of temporal) dispersion into account.

Secondly, one usually considers the propagation of solitons, i.e., the shape of the light pulse is assumed not to change when they interact. Nowhere do we introduce restrictions on the shape of the propagating pulse (and, hence, its spectrum) which thanks to the nonlinearity of the medium undergoes considerable phase modulation (chirp)—the main features of the results obtained are just connected with this. Of course, to find the actual form of the solution the initial shape of the pulse incident upon the medium must be defined; we assume it to be Gaussian [see Sec. 3 below and Eq. (A2) in Appendix 1] although this assumption is not a matter of principle in the general approach considered by us on the basis of a self-similar substitution.

The spatially periodic structures discussed can occur in optics both when laser radiation in an (initially uniform) optical medium induces a lattice of the refractive index or—in the case of an amplifying medium—of the amplification coefficient,[5–8] and when one uses nonlinear media which initially (when there is no laser radiation) have a spatial periodicity of the material parameter, i.e., of naturally nonuniform media.[9] Especially promising in this respect are liquid crystals (LC) and photorefractive materials with a large nonlinearity, which makes it possible to work with high-contrast lattices in the field of relatively low-power continuous lasers (including even He–Ne lasers).[9,10] This is of principal importance from an experimental point of view since it enables us to form extreme states of light radiation

 1063-7761/93/030445-14$10.00

(in particular, squeezed light) in the field of highly stable coherent laser beams with a low level of characteristic noise.[1]

Both these cases (lattices induced by light and naturally periodic media) can be represented in a single way from the point of view of the physics of the phenomena considered.

Indeed, when a refractive-index lattice is induced (by means of light) in a medium we are dealing with a degenerate (in frequency) four-wave process of wave mixing.[5] The nonlinear polarization of the medium recorded for the wave $\mathbf{E}_4$ which is produced in the interaction process is in that case determined by the equation

$$P(\mathbf{E}_4) = \chi^{(3)}\mathbf{E}_1^*\mathbf{E}_2\mathbf{E}_3 + \tilde{\chi}^{(3)}\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3^*, \qquad (1)$$

where $\chi^{(3)}$ and $\tilde{\chi}^{(3)}$ are components of the cubic susceptibility of the medium while the geometry of the interaction was chosen as follows: $\mathbf{E}_{1,2}$ are two counterpropagating (collinear—the direction of propagation is z) waves and the wave $\mathbf{E}_3$ is directed at an angle $\tilde{\delta}' \ll 1$ to z, and the wave $\mathbf{E}_4$ which is its conjugate is produced as the result of the nonlinear interaction and propagates in the opposite direction. The first term on the right-hand side of (1) characterizes the distributed feedback (DFB) in the system and the effective interaction (the production of the wave $\mathbf{E}_4$) is determined by the Bragg resonance condition for the wave vectors $\mathbf{k}_4 = \mathbf{k}_3 - 2\mathbf{k}_1$, where we have used the fact that $\mathbf{k}_1 = -\mathbf{k}_2$; the second term is the generation of the phase-conjugate wave (the scheme of wavefront formation) for which the resonance (synchronism) condition $\mathbf{k}_4 = -\mathbf{k}_3$ is always satisfied. It is convenient to regard both these cases as diffraction (scattering) of a wave (usually $\mathbf{E}_2$ into $\mathbf{E}_4$ or $\mathbf{E}_1$ into $\mathbf{E}_4$) by a periodic refractive-index lattice formed as the result of the interference in the nonlinear medium of two other waves ($\mathbf{E}_1$ and $\mathbf{E}_3$—lattice by "transmission," or $\mathbf{E}_2$ and $\mathbf{E}_3$—by "reflection" [1)]).[5] One then assumes that no other waves appear in the nonlinear interaction, i.e., higher diffraction orders (propagating at different angles) which correspond to a thin lattice (Raman–Nath diffraction) are quenched by interference (bulk lattice, Bragg diffraction[5]).

For initially spatially periodic lattices Eq. (1) for the wave $\mathbf{E}_4$ generated in the diffraction process can also be retained, but in this case we have only one incident (propagating) wave ($\mathbf{E}_1$ or $\mathbf{E}_2$) which is scattered (diffracted) in the medium with dielectric susceptibilities $\chi^{(3)}_{\rm eff}$ and $\tilde{\chi}^{(3)}_{\rm eff}$ which in (1) formally replace the combinations $\chi^{(3)}\mathbf{E}_2\mathbf{E}_3$ and $\tilde{\chi}^{(3)}\mathbf{E}_2\mathbf{E}_3^*$ or $\chi^{(3)}\mathbf{E}_1^*\mathbf{E}_3$ and $\tilde{\chi}^{(3)}\mathbf{E}_1\mathbf{E}_3^*$, respectively, and which take into account the initial periodicity of the medium [we shall write down exact relations in Sec. 3—see (13)[2)]]. It is customary to denote the pumping wave $\mathbf{E}_1$ or $\mathbf{E}_2$ in this case by $\mathbf{E}_0$ and the generated (scattered) wave $\mathbf{E}_4$ by $\mathbf{E}_h$—the two-wave approximation of the dynamic diffraction theory (neglecting the birefringence of each of these waves which is unimportant for the present problem).

In the present paper we shall be interested in just this last case and only for a geometry with DFB—the first term in (1) where, e.g., a cholesteric LC (CLC) may serve as the initially periodic medium.

The material of the present paper is distributed as follows.

In Sec. 1 we consider the approach to the problem and the procedure for considering quantum squeezed states of light in a spatially periodic medium.

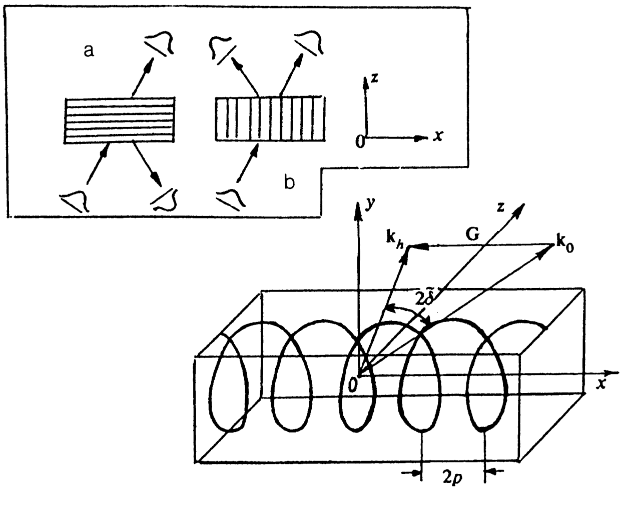

FIG. 1. Geometry of the dynamic scattering of laser pulses for copropagating waves in a DFB system. We show in the insert the Bragg (a) and the Laue (b) schemes. $\mathbf{G}$ is a spatial lattice wave vector; the Bragg resonance condition is $\mathbf{k}_h = \mathbf{k}_0 + \mathbf{G}$ where the $\mathbf{k}_{0,h}$ are the wave vectors of the incident and the diffracted (scattered) waves, $G = 2\pi/p$, and p is the lattice period. At the entrance into the medium a (temporally and spatially) Gaussian wave packet propagates in the 0-channel and a vacuum mode (not shown in the figure) in the h-channel.

In the following Secs. 2, 3 we write down the basic equations and solve the problem of the formation of squeezed light for space-time wave packets (in the Heisenberg representation).

In Sec. 4 we consider the solution of the self-similar problem in the Schrödinger representation.

In the Appendix we discuss a number of auxiliary problems.

1. PHYSICAL PICTURE AND CALCULATION METHOD

In the case considered by us of Bragg diffraction-dynamic scattering there are thus in an initially spatially periodic nonlinear medium two coupled waves—a transmitted one [with complex amplitude $\mathbf{A}_0(\mathbf{r},t)$ and wave vector $\mathbf{k}_0$] and a scattered one—$\mathbf{A}_h(\mathbf{r},t)$, $\mathbf{k}_h$ (under Bragg resonance conditions $\mathbf{k}_h = \mathbf{k}_0 + \mathbf{G}$, where $\mathbf{G}$ is a reciprocal lattice vector), between which there is an efficient exchange of energy (see Fig. 1).[12]

The qualitative picture of the phenomena considered here is determined by the fact that owing to different scattering angles for the different spectral components of the transmitted pulse (the distribution of which under its envelope is given by phase modulation) their optical path depends on the frequency.

When we change to a quantal description the complex amplitudes $\mathbf{A}_{0,h}(\mathbf{r},t)$ and $\mathbf{A}^*_{0,h}(\mathbf{r},t)$ are, respectively, annihilation and creation operators, $\hat{\mathbf{A}}_{0,h}(\mathbf{r},t)$ and $\hat{\mathbf{A}}^+_{0,h}(r,t)$, satisfying the commutation relation $[\hat{\mathbf{A}}_{0,h}(\mathbf{r},t),\hat{\mathbf{A}}^+_{0,h}(\mathbf{r},t)] = \hat{C}$, where $\hat{C}$ in the general case is a positive-definite operator[13] the form of which must be specified in each problem which we consider.[3)] Finding the interaction Hamiltonian for the operators of the wave packets in the medium enables us to solve the problem either in the Schrödinger[14,15] or in the Heisenberg[15,16] representation.

The connection between the creation and annihilation operators for the usual case of photons ($\hat{a}_{0,h}, \hat{a}^+_{0,h}$) and for the wave packets $(\hat{\mathbf{A}}_{0,h}(\mathbf{r},t),\hat{\mathbf{A}}^+_{0,h}(\mathbf{r},t))$ is in such an approach established on the basis of the quantization procedure discussed here, taking into account the specific features of the problem (see Sec. 2);[16,18] in the case of wave packets it is customary to speak of the quantum properties of the macrofields.[17,20] The spatially periodic medium is then also characterized by an averaged macroscopic quantity—the permittivity (cf. Ref. 18).

In the framework of such an (essentially macroscopic and fluctuationless) model of the medium a simple change from the temporal problem to the spatial problem is possible—it reduces to a formal substitution of the derivatives $\partial/\partial t \rightarrow (\omega/k^2)\mathbf{k}\nabla$ in the Heisenberg equations of motion.[18] We apply this procedure of deriving the spatial equations for propagating waves from the quantum-mechanical (temporal) equations but for our case of optical pulses (beams).[1] As a result we are led to a nonlinear Schrödinger equation[21] the solution of which we shall look for in the self-similar substitution approximation (cf. Ref. 12).

2. BASIC QUANTUM EQUATIONS AND THEIR SOLUTION

The task of the present section is the derivation of the quantum equations for the wave-packet operators $\hat{\mathbf{A}}_{0,h}(\mathbf{r},t)$ and $\hat{\mathbf{A}}^+_{0,h}(\mathbf{r},t)$. We do this in a rather general form for an arbitrary operator $\hat{a}(y,\tau)(\hat{a}^+(y,\tau))$ which is simultaneously a function of the space and the time coordinates (in particular, it may be the same as $\hat{\mathbf{A}}_{0,h}(\mathbf{r},t)(\hat{\mathbf{A}}^+_{0,h}(\mathbf{r},t))$); we shall here assume y to be a transverse and τ to be a "traveling" longitudinal coordinate (it plays the role of time). The Heisenberg equation of motion,

$$\frac{\partial \hat{a}(y,\tau)}{\partial \tau} = -\frac{i}{\hbar}[\hat{a}(y,\tau);\hat{H}_{\text{int}}]$$

[the partial derivative on the left defines the derivative of $a(y,\tau)$ with respect to τ] can be written in a rather universal standard form which in form is the same as the quantum nonlinear Schrödinger equation (cf. Ref. 21):

$$i\frac{\partial}{\partial \tau}\hat{a}(y,\tau) = -\frac{\partial^2}{\partial y^2}\hat{a}(y,\tau) + 2\varkappa\hat{a}^+(y,\tau)\hat{a}^2(y,\tau), \qquad (2)$$

where $\varkappa > 0$ is the nonlinearity parameter, and the interaction Hamiltonian is (cf. Ref. 14)

$$H_{\text{int}} = \hbar\int \frac{\partial}{\partial y}\hat{a}^+(y,\tau)\frac{\partial}{\partial y}\hat{a}(y,\tau)dy + \varkappa\int \hat{a}^+(y,\tau)\hat{a}^+(y,\tau)\hat{a}(y,\tau)\hat{a}(y,\tau)dy. \qquad (3)$$

We have dropped in the Hamiltonian (3) Hermitian terms of the type $\hat{a}^{+2}(y,\tau) + \hat{a}^2(y,\tau)$ [see later (6) and (14)] which, however, lead to fast oscillating terms (they correspond, e.g., to emission at the doubled frequency); we shall return to this problem below. The following commutation relations are then satisfied:[2]

$$[\hat{a}(y,\tau);\quad \hat{a}^+(y',\tau')] = \delta(y-y')\delta(\tau-\tau'),$$
$$[\hat{a}(y,\tau);\quad \hat{a}(y,\tau)] = 0, \qquad (4)$$

in which the operators $\hat{a}(y,\tau)$, $\hat{a}^+(y,\tau)$ must be associated with field operators (we drop in what follows the operator sign).

We look for the solution of Eq. (2) in the aberrationless approximation in the form of a self-similar substitution:[22]

$$a(y,\tau) = \frac{1}{f^m(\tau)}\exp\{F_1(y,\tau) + i\tilde{\Phi}_1(y,\tau) + i\tilde{\Phi}_2(\tau)\}a, \qquad (5)$$

where $m = 1$ for a three-dimensional spherical wavefront, and $m = \frac{1}{2}$ for a two-dimensional cylindrical wave front (cf. Ref. 12), $a \equiv a(y,\tau)|_{\tau=0}$; the operators f, F_1, and $\tilde{\Phi}_{1,2}$ depend on $\varkappa$. The operators $f(\tau)$ and $F_1(\tau,y)$ determining the longitudinal and transverse distribution of the light emission depend also on the number of photons; the operators $\tilde{\Phi}_{1,2}$ characterize the nonlinear phase distribution (only $\tilde{\Phi}_1(y,\tau)$ corresponds to the transverse distribution).

It is important to note that although in the general case the operators f, F_1, and $\tilde{\Phi}_{1,2}$ are not Hermitian and themselves satisfy some operator equations, in the self-consistent field approximation, however, and in the framework of perturbation theory one can require Hermiticity properties and determine their explicit form by expanding these operators in series in the small nonlinearity parameter $\varkappa a^+ a$, neglecting terms of order higher than second. The required solution of Eq. (2) then reduces to the following:

$$a(y,\tau) = (K_1 + K_2a^+a + K_3(a^+a)^2)\times\exp(\varkappa_1 + \varkappa_2a^+a + \varkappa_3(a^+a)^2)a, \qquad (6)$$

where $K_i \equiv K_i(\tau)$ and $\varkappa_i \equiv \varkappa_i(y,\tau)$ are, respectively, real and complex numerical functions; here K_1 and $\varkappa_1$ are independent of $\varkappa$, K_2, and $\varkappa_2$ are proportional to $\varkappa$, and K_3 and $\varkappa_3$ are proportional to $\varkappa^2$. We emphasize that Eq. (6) is a consequence of the above-mentioned important assumptions defining the approximation in which the present problem is considered; it is obtained from (5) just for this case (its form is not at all obvious *a priori* in the self-similar substitution method). Moreover, from this we might formulate additional arguments which justify the absence of the already mentioned terms of the type $a^{+2}(y,\tau) + a^2(y,\tau)$ in the relations given here.

Indeed, the general expression (5) must in the limiting case of plane waves [i.e., for $f(\tau) \equiv 1$, $F_1(y,\tau) \equiv 0$, and $\tilde{\Phi}_1(y,\tau) \equiv 0$] go over into the form describing the usual phase modulation determining the number of photons $(\sim \varkappa a^+(\tau)a(\tau))$, which is the only one considered by us in the present problem. This is guaranteed by (5) just for $\tilde{\Phi}_2(\tau) \sim a^+(\tau)a(\tau)$ and when the above-mentioned additional terms are not there. Similarly, in the classical limit the parameters f, F_1, and $\tilde{\Phi}_{1,2}$ in (5) are functions only of the light intensity $(\sim aa^* \equiv |a|^2)$ and should not depend on the initial phases of the fields at the entrance into the medium [i.e., on the terms $\sim a^{*2}(y,\tau)$, $a^2(y,\tau)$; cf. the quasiclassical limit in Sec. 4].

On the basis of Eq. (6) one easily obtains Hermitian quadratures $Q(y,\tau)$ and $P(y,\tau)$ from a combination of the operators $a(y,\tau)$ and $a^+(y,\tau)$:

$$Q(y,\tau) = a(y,\tau) + a^+(y,\tau),$$
$$P(y,\tau) = i(a^+(y,\tau) - a(y,\tau)), \qquad (7)$$

by means of which the squeezed-light states can be realized.[2] To do this we determine the state vector (in the Heisenberg representation) when the wave functions are independent of τ (i.e., for $\tau = 0$). It will correspond to the radiation entering the medium in a coherent state: $a \equiv a(y,\tau)|_{\tau=0}$ so that (cf. Ref. 16)

$$a|\alpha(y,0)\rangle = \alpha(y,0)|\alpha(y,0)\rangle, \tag{8}$$

where $|\alpha(y,0)\rangle$ is the wave function of the initial ($\tau = 0$) state with eigenvalue (the envelope of the wave packet) $\alpha(y,0) \equiv \alpha$.

We introduce next the mean square fluctuations

$$\langle \Delta Q^2(y,\tau)\rangle \equiv \langle Q^2(y,\tau)\rangle - \langle Q(y,\tau)\rangle^2,$$
$$\langle \Delta P^2(y,\tau)\rangle \equiv \langle P^2(y,\tau)\rangle - \langle P(y,\tau)\rangle^2. \tag{9}$$

We use here the normally ordered form of writing (we denote it by $\langle : \; : \rangle$) which fixes the constant noise level for the initial coherent state:

$$\langle :\Delta Q^2(y,0): \rangle = \langle :\Delta P^2(y,0): \rangle = 0. \tag{10}$$

In this formalism squeezed light corresponds to a negative value (from -1 to zero) of these fluctuations.[2] Using (6) to (9) we then have

$$\left.\begin{matrix}\langle :\Delta Q^2: \rangle \\ \langle :\Delta P^2: \rangle\end{matrix}\right\} = 2\exp(2C_1 + 2|\alpha|^2 C_2)\{\pm [K_1^2\{C_2^2 - U_2^2\}\cos\Psi_0$$
$$- 2U_2C_2\sin\Psi_0\}|\alpha|^4 + K_1^2\{C_2\cos\Psi_0 - U_2\sin\Psi_0\}|\alpha|^2$$
$$+ 2K_1|\alpha|^4\{C_3\cos\Psi_0 - U_3\sin\Psi_0\} + \{K_1K_2|\alpha|^2$$
$$+ K_2^2|\alpha|^4 + 2K_1K_3|\alpha|^4\}\cos\Psi_0$$
$$+ 4K_1K_2|\alpha|^4\{C_2\cos\Psi_0 - U_2\sin\Psi_0\}]$$
$$+ K_1^2|\alpha|^4(C_2^2 + U_2^2) + (2K_1K_2C_2 + K_2^2)|\alpha|^4\}, \tag{11}$$

where the C_i and U_i are functions of the $\varkappa_i$, $\Psi_0 = 2\theta_0 + 2U_1 + 2|\alpha|^2 U_2$, θ_0 is the phase of the quantity α ($\alpha \equiv |\alpha|\exp(i\theta_0)$), and we have neglected the contribution to the fluctuations from terms proportional to $(a^+a)^2$.

In the next section this calculation scheme planned by us will be applied for DFB systems.

3. QUANTUM FLUCTUATIONS OF WAVE PACKETS AND BEAMS FOR NONLINEAR DYNAMICAL DIFFRACTION. HEISENBERG REPRESENTATION

Turning to the actual problem considered by us of dynamical diffraction in a DFB system we write the total field operator $\mathbf{E}$ inside the medium as a superposition of two modes $\tilde{\mathbf{A}}_0(\mathbf{r},t)$ and $\mathbf{A}_h(\mathbf{r},t)$:

$$\mathbf{E} = e(\omega)[\mathbf{l}_0A_0(\mathbf{r},t)\exp(i\mathbf{k}_0\mathbf{r})$$
$$+ \mathbf{l}_hA_h(\mathbf{r},t)\exp(i\mathbf{k}_h\mathbf{r})\exp(-i\omega t) + \text{h.c.}, \tag{12}$$

where $e(\omega)$ is a complex mode function (for a narrow-band signal it may be assumed in what follows to be constant[4]) and the $\mathbf{l}_{0,h}$ are polarization unit vectors ($\mathbf{l}_0\mathbf{l}_h = 1$).

We write the expression for the electrical induction vector $\mathbf{D}$ of the light field in the case of a nonlinear (cubic susceptibility $\chi^{(3)}$; we assume the response to be instantaneous) spatially periodic medium in the form (cf. Refs. 22 and 23):

$$\mathbf{D} = 8\pi\{\chi_1(\cos\mathbf{G}\mathbf{r})\mathbf{E} + (0{,}5\chi_2^{(3)}$$
$$+ \chi_3^{(3)}\cos\mathbf{G}\mathbf{r} + \chi_4^{(3)}\cos 2\mathbf{G}\mathbf{r})\mathbf{E}^+\mathbf{E}^2\}, \tag{13}$$

where χ_1 and $\chi_{2,3,4}^{(3)}$ are the components of, respectively, the linear and the nonlinear susceptibility, $G = 2\pi/p$, and p is the lattice period.

Using (4), (12), and (13) we then have for the interaction Hamiltonian H_{int} [for exact Bragg resonance ($\mathbf{k}_h = \mathbf{k}_0 + \mathbf{G}$) and dropping higher spatial harmonics]:

$$H_{\text{int}} = \sum_{i,j}\int dx\{4\pi\tilde{c}^2 dA_i(x,z,t)A_j^+(x,z,t)\chi_1$$
$$+ 12\pi\tilde{c}^4 d[\chi_2^{(3)}(A_i^{+2}(x,z,t)A_i^2(x,z,t)$$
$$+ 2A_i^+(x,z,t)A_i(x,z,t)A_j^+(x,z,t)A_j(x,z,t))$$
$$+ \chi_3^{(3)}(2A_i^{+2}(x,z,t)A_i(x,z,t)A_j(x,z,t)$$
$$+ 2A_i^+(x,z,t)A_j^+(x,z,t)A_j^2(x,z,t))$$
$$+ \chi_4^{(3)}(A_i^{+2}(x,z,t)A_j^2(x,z,t))]\}, \tag{14}$$

where $\tilde{c} = (\hbar\omega/2\varepsilon_0 d)^{1/2}$, d is the thickness of the medium, and $i,j = 0, h$ ($i \neq j$).

Using (14) and the scheme of Sec. 2 we can easily obtain equations of motion which we write down at once for the case $L_{\text{nl}} > L_{\text{exc}}$ (the parameters L_{nl} and L_{exc} are defined below) when the structure of the field in a linear RFB system is assumed to have been formed up to the point where the nonlinearity starts to affect it (see Ref. 22):[4)]

$$\left(2i\frac{\partial}{\partial\eta} \pm L_{\text{exc}}\text{tg}^2\tilde{\delta}\frac{\partial^2}{\partial x^2}\right)a_{1,2}(\eta,\xi,x)$$
$$= p_{1,2}a_{1,2}^+(\eta,\xi,x)a_{1,2}^2(\eta,\xi,x)$$
$$+ la_{2,1}^+(\eta,\xi,x)a_{2,1}(\eta,\xi,x)a_{1,2}(\eta,\xi,x), \tag{15}$$

where $p_{1,2} = \beta(3 \pm 4m_3 + m_4)$, $l = 2\beta(1 - m_4)$, $L_{\text{exc}} = (\hbar\omega\cos\tilde{\delta})/(4\pi|\tilde{c}|^2\chi_1 kd)$ is the exciton length characterizing the effective spatial length scale for energy exchange between the transmitted and scattered waves—the length of "pendulum" beats,[12] $\beta^{-1}|\alpha|^{-2} \equiv L_{\text{nl}} = (\hbar\omega\cos\tilde{\delta})/24\pi \times |\tilde{c}|^4\chi_2^{(3)}|\alpha|^2 kd$ is the analogous parameter for taking nonlinearity into account (nonlinear length); the normalized nonlinearity parameters are $m_{3,4} \equiv \chi_{3,4}^{(3)}/\chi_2^{(3)}$, $\chi_2^{(3)} \neq 0$, $|\alpha|^2$ is the average number of photons in the mode (it is determined by the radiation intensity at the entrance into the medium), the scattering angle is $\tilde{\delta} = \pi/2 - \tilde{\delta}'$, η and ξ are "running" coordinates connected with z and t through the transformation: $\eta = z$, $\xi = t - z/v\cos\tilde{\delta}$, and v is the propagation velocity of the wave packet along the z axis. We have introduced in (15) partial amplitudes $a_{1,2}(\eta,\xi,x)$ which determine the form of the solution:

$$A_{0,h}(\eta,\xi,x) = \frac{1}{\sqrt{2}}\{a_1(\eta,\xi,x)\exp(-iL_{\text{exc}}^{-1}\eta)$$
$$\pm a_2(\eta,\xi,x)\exp(iL_{\text{exc}}^{-1}\eta)\}, \tag{16}$$

and the following commutation relations hold:

$$[a_i(\eta,\xi,x); a_j^+(\eta',\xi',x')] = \delta_{ij}\delta(\eta-\eta')\delta(\xi-\xi')\delta(x-x'),$$
$$i,j = 1,2. \tag{17}$$

We shall in what follows for the sake of simplicity assume that $l = 0$ (condition on the nonlinearity of the medium) and one can easily check that Eqs. (15) can be reduced to the dimensionless form (2). To do this we must multiply both sides of (15) by the quantity $1/\xi^2 L_{\text{exc}}\tan^2\tilde{\delta}$ where

$\xi = 1/(\pi L'_{\rm exc} L_{\rm exc})^{1/2} \tan\tilde{\delta}$ while the parameter $L'_{\rm exc} = r_0^2/L_{\rm exc} \sin^2\tilde{\delta}$ r_0 is the beam radius) determines the linear spatial drift of the 0- and h-beams relative to one another, and also go over to new variables by the substitution

$$\tau = \eta/2\pi L'_{\rm exc}, \quad y = \xi x, \quad \pi L'_{\rm exc} p_{1,2} = \pm 2\varkappa. \qquad (18)$$

The equations for $a_{1,2}(\eta,\xi,x)$ can thus formally be obtained by means of the Hamiltonian (3). This means physically that the propagation of wave packets and beams in a DFB system can be reduced to well known self-action processes (cf. Ref. 21).

The solutions (15), taking into account (16) to (18), are given in Appendix 1 (under the condition $L_{\rm nl} \sim L'_{\rm exc}$) for two cases—for a quasistationary change in the transverse beam profile and for the variations of the temporal envelope of a pulse. This makes is possible to calculate the fluctuations of radiation under conditions when self-focusing (self-defocusing) and self-compression (self-decompression) effects develop, respectively.[5] In this case we assume that at the entrance into the medium the mode A_h is a vacuum mode and that A_0 has the eigenvalue α_0.

The results of the numerical calculations for the mean square fluctuations $2\langle:\Delta Q^2:\rangle$ are shown in Figs. 2 and 3. The values of the necessary dimensionless parameters which are defined in Appendix 1 are given in the figure captions; they

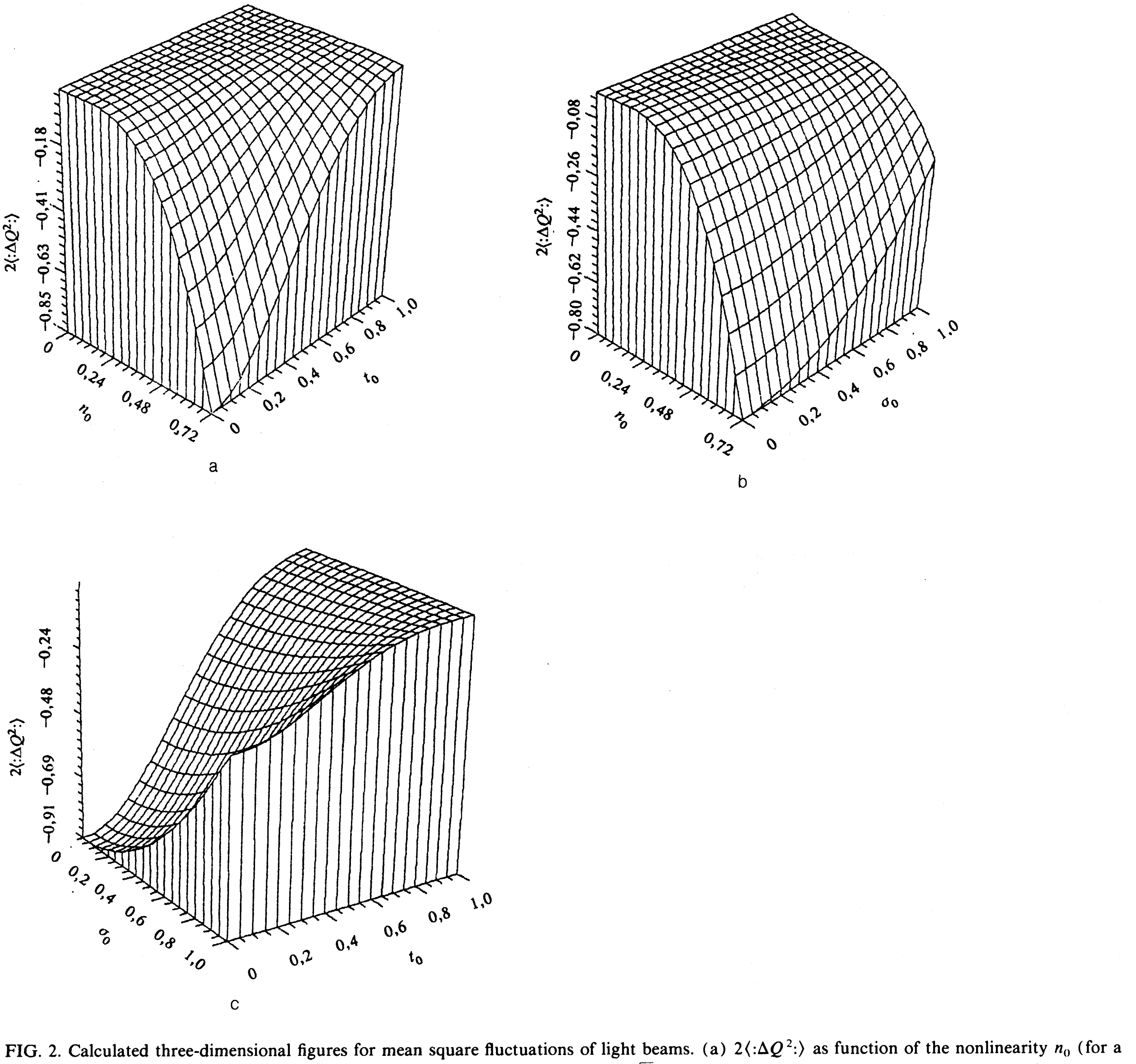

FIG. 2. Calculated three-dimensional figures for mean square fluctuations of light beams. (a) $2\langle:\Delta Q^2:\rangle$ as function of the nonlinearity n_0 (for a normalized value of the number of photons $|\alpha_0|^2$) and of the normalized time $t_0 \equiv t/\sqrt{2}\tau_u$ (τ_u is the length of the light pulse). Light in a coherent state corresponds to $\langle:\Delta Q^2:\rangle = 0$. The numerical values of the parameters are: $m_4 = 1$; $m_3 = 0.1$; $p_1 = 4.4\beta$; $p_2 = 3.6\beta$; $\delta_1 = \delta_1'|\alpha_1|^2 \equiv L'_{\rm exc} p_1 |\alpha_0|^2/2 = 2.2 L'_{\rm exc}\beta|\alpha_0|^2 = 2.2 L'_{\rm exc}/L_{\rm nl} \equiv 1.1 n_0$; $\delta_2 = \delta_2'|\alpha_2|^2 = 0.9 n_0$; $L'_{\rm exc} = 10^{-2}$ cm; $L_{\rm nl} = 2\times 10^{-2}/n_0$ cm; $d = 500\,\mu$m; $\eta_0 \equiv d/L'_{\rm exc} = 5$; $\tilde{\delta} = 86°$; $r_0 = 0.3\times 10^{-2}$ cm; $x = 0.02$ cm; $\theta_0 = 3\pi/4$; $L_{\rm exc} = 10^{-3}$ cm. (b) $2\langle:\Delta Q^2:\rangle$ as function of n_0 and σ_0. The values of the parameters are: $L'_{\rm exc} = 10^{-2}$ cm; $L_{\rm nl} = 2\times 10^{-2}/n_0$ cm; $\delta_1 = 1.1 n_0$; $\delta_2 = 0.9 n_0$; $d = 500\,\mu$m; $\eta_0 = 5$; $t_0 = 0.2$; $\sigma_0 = x/(2L_{\rm exc} L'_{\rm exc} \tan^2\tilde{\delta})^{1/2}$; $\tilde{\delta} = 86°$; $\theta_0 = 3\pi/4$; $L_{\rm exc} = 10^{-3}$ cm. (c) $2\langle:\Delta Q^2:\rangle$ as function of σ_0 and t_0. We assume that for $\sigma_0 = t_0 = 0$ the light is already in the squeezed state ($-1 < \langle:\Delta Q^2:\rangle < 0$). The numerical data are $L'_{\rm exc} = 10^{-2}$ cm; $L_{\rm nl} = 2\times 10^{-2}/n_0$ cm; $L_{\rm exc} = 10^{-3}$ cm; $d = 500\,\mu$m; $\delta_1 = 0.792$; $\delta_2 = 0.648$ ($n_0 = 0.72$); $\eta_0 = 5$; $\tilde{\delta} = 86°$; $\theta_0 = 3\pi/4$.

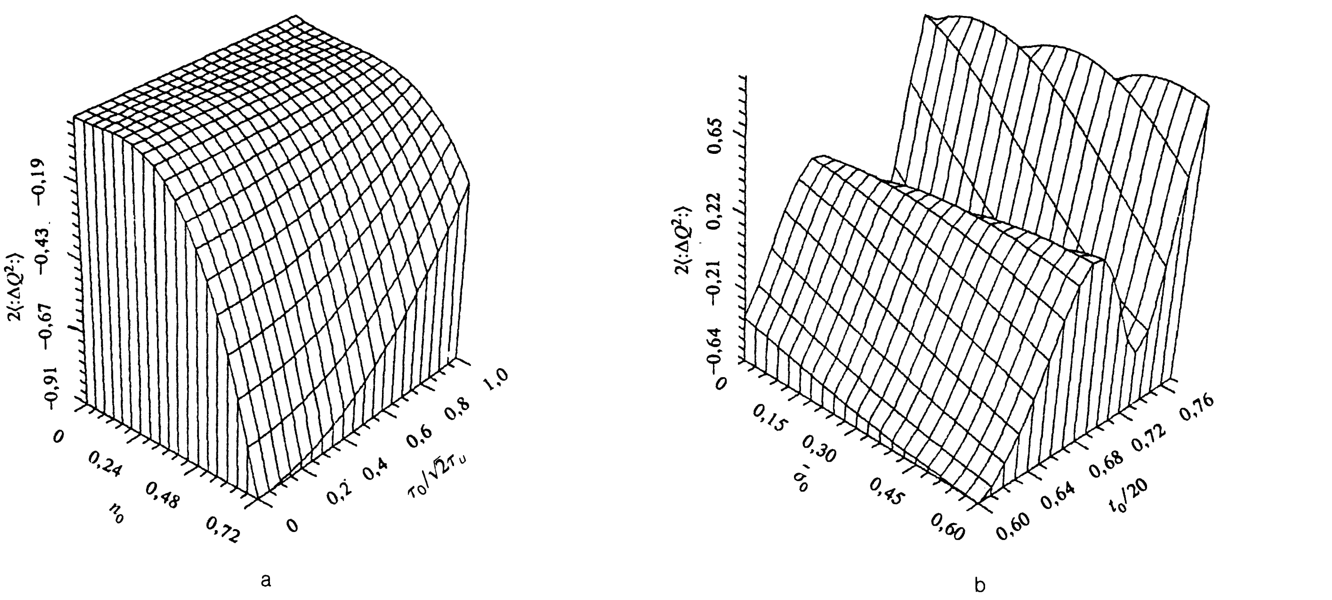

FIG. 3. Fluctuations for light pulses. (a) $2\langle:\Delta Q^2:\rangle$ as function of n_0 and $\tau_0/2\tau_u$. The values of the parameters are: $L_{nl} = 2\times 10^{-2}/n_0$ cm; $L_{exc} = 10^{-3}$ cm; $\sqrt{2}v\tau_u = 0.06$ cm; $\gamma_0 = 10^{-2}$ cm; $\delta_1 = 1.1n_0$; $\delta_2 = 0.9n_0$; $d = 500\,\mu$m; $\eta_0 = 5$; $x = 0.02$ cm; $\tilde{\delta} = 86°$; $\theta_0 = 3\pi/4$; $\tau_0 = t - z/v\cos\tilde{\delta} - x\sin\tilde{\delta}/v$, and τ_u is the initial pulse length. (b) $2\langle:\Delta Q^2:\rangle$ as function of $\tilde{\sigma}_0$ and $t_0/20$. The values of the parameters are: $\sqrt{2}v\tau_u = 0.06$ cm; $L_{nl} = 2\times 10^{-2}/n_0$ cm; $\gamma_0 = 10^{-2}$ cm; $n_0 = 0.65$; $\delta_1 = 0.715$; $\delta_2 = 0.585$; $L_{exc} = 10^{-3}$ cm; $\eta_0 = 5$; $\tilde{\delta} = 86°$; $\theta_0 = 3\pi/4$; $\tilde{\sigma}_0 = x\sin\tilde{\delta}/\sqrt{2}v\tau_u$; and $\eta_0 \equiv d/\gamma_0$.

correspond to actual optical media—CLC—and do not go beyond the framework of the assumptions made in the theory.

It is clear from Fig. 2 (quasistationary transformation of a spatial beam profile) that there is a similarity in the behavior of the fluctuations as function of the time t_0 ($t_0 \equiv t/\sqrt{2}\tau_u$) (a) and of the transverse coordinate σ_0 ($\sigma_0 = x/(2L_{exc}L'_{exc}\tan^2\tilde{\delta})^{1/2}$) (b) when the control parameter of the problem—the nonlinearity parameter n_0 (the normalized intensity: $n_0 \sim |\alpha_0|^2 \sim 1/L_{nl}$) changes. The space-time transformation of the fluctuations of the wave packet show up most clearly in the simultaneous dependence of $2\langle:\Delta Q^2:\rangle$ on t_0 and σ_0 (Fig. 2c).[6)] For $t_0 = 0$ ($\tau_u = \infty$) we have a stationary problem—a deeper squeezing is reached when $|\alpha_0|^2$ is increased. A characteristic property of quantum wave packets is the distribution of noise over their space-time profile. The minimum level of fluctuations occurs at the center of the beam (pulse), i.e., for $t_0 = 0$ ($t = 0$) and $\sigma_0 = 0$ ($x = 0$). In the other limit—$t_0 \to \infty$, $\sigma_0 \to \infty$ (which in our case corresponds to $t, x \to \infty$)[7)]—we see a degradation of the squeezing for the peripheral regions of the wave packet—$\langle:\Delta Q^2:\rangle \to 0$ (coherent level).

We note that the effects listed above depend significantly on the phase relations between the components (the local modes) of the wave packet, which are determined by the t_0- and σ_0-dependence of the nonlinear phase. In that case there appear oscillations in the temporal and spatial behavior of the fluctuations (in t_0 and σ_0). Moreover, such a behavior of the dispersion of the quadratures is similar to the results of the problem of pulsed squeezing of light when one neglects the dependence of the parameters on the transverse coordinate–plane-wave approximation ($L'_{exc} \to \infty$; cf. Ref. 24).

The calculated curves for the temporal profile of a pulse (in the compression problem we are dealing with purely temporal behavior—the dependence on the effective time τ_0) are given in Fig. 3. They are on the whole the same as the functions of Fig. 2, i.e., one can speak of a similar behavior of the fluctuations for light beams and pulses (cf. Figs. 3a and 2b for which the quantities $\tau_0/\sqrt{2}\tau_u$ and σ_0 turn out to be the analogous parameters) although for the compression problem one observes a deeper squeezing; its maximum level ($\tau_0 = 0$) is shifted relative to the center of the pulse and corresponds to $t = x\sin\tilde{\delta}/v$. One can see the above mentioned oscillations particularly clearly in Fig. 3b where the "true" time t_0 figures as variable.

The principal result appearing in the relations given here is the complex law for the redistribution of the fluctuations both along the length of a pulse and along the transverse profile of a beam which in the quantum case cannot at all be reduced to the monotonic "self-cleaning," from noise modulation, of the pulse (beam) during its nonlinear propagation (dumping of noise in an airfoil) as can normally occur in the classical case (e.g., for solitons; cf. Ref. 4); this is connected with the involvement of vacuum modes in the interaction process. We mentioned already that the choice of phase parameters in the problem is important here; they enable us to localize the noise level in any part of a propagating space-time light packet and minimize it at its lowest value (see, e.g., Fig. 3b). These problems were analyzed in detail in Refs. 23 and 24 where it was shown that an effective control of the noise level at the exit from the medium is possible in particular by an appropriate choice of the initial (preliminary) chirp in the pulse incident on it.

The effect of the finite space-time dimensions of the radiation leads to a degradation of the squeezed states in comparison to their level for plane-wave continuous radiation (cf. Refs. 23, 24); this is connected not only with the obvious factor of the decrease in intensity at the peripheral parts of the packet (and thereby with the retaining for them of the initial coherent state) but also with the spatial and temporal drift of the packets (dephasing of the radiation components) for the 0- and the h-modes, the energy exchange between which is the cause of the effects discussed here. In particular, under the conditions when $L_{nl} > L'_{exc} \gtrsim L_{exc}$ the nonlinearity of the medium is not able at all to affect significantly the level of the fluctuations.[8)] By itself a large nonlinearity ($L_{nl}/\eta \to 0$) can also destroy a squeezed radiation state through a violation of the optimal phase relations between its different components—the local modes.[25] However, this case cannot be analyzed in the framework of the approximations made in the model considered by us (see the next section).

4. SELF-SIMILAR SOLUTIONS IN THE FRAMEWORK OF THE SCHRÖDINGER FORMALISM

It is natural to consider quantum fluctuations for space-time wave packets in the Schrödinger representation. Such an analysis has, however, up to the present time only been done for optical solitons.[14,15,26] Our considerations are free of this restriction and take into account the change both in the shape of the light packet and also in its temporal profile. We start from the approach developed in Ref. 26.

We write the total state vector $|\varphi\rangle$ in the form (cf. Eq. (A13) in Appendix 1):

$$|\varphi\rangle = |\varphi_1\rangle|\varphi_2\rangle, \tag{19}$$

where the $|\varphi_{1,2}\rangle$ are the state vectors for the operators $a_{1,2}(y)$ which satisfy the Schrödinger equation

$$i\hbar \frac{d}{d\tau}|\varphi_{1,2}\rangle = H_{\mathrm{Sch}_{1,2}}|\varphi_{1,2}\rangle, \tag{20}$$

$H_{\mathrm{Sch}_{1,2}}$ is the Hamiltonian (3) of the system of bosons in the present representation; the following commutation relations hold:

$$[a_i(y); a_i^+(y')] = \delta_{ij}\delta(y-y'),$$
$$[a_i(y); a_j(y')] = 0, \quad i,j = 1,2. \tag{21}$$

We have for the $|\varphi_{1,2}\rangle$ states (in Fock space):

$$|\varphi_{1,2}\rangle = \sum_{n_{1,2}} h_{n_{1,2}} \int \frac{1}{\sqrt{n_{1,2}!}} \times g_{n_{1,2}}(y_1 \ldots y_n, \tau) \ldots a_{1,2}^+(y_1) \ldots . a_{1,2}^+(y_n) dy_1 \ldots . dy_n |0\rangle, \tag{22}$$

where we can assume the $g_{n_{1,2}}$ to be symmetric weight functions;[14] the normalization conditions are:

$$\sum_{n_{1,2}} |h_{n_{1,2}}|^2 = 1, \qquad \int |g_{n_{1,2}}(y_1 \ldots y_n, \tau)|^2 dy_1 \ldots dy_n = 1. \tag{23}$$

Using the above relations and writing the solution in the form (Hartree–Fock approximation)

$$g_{n_{1,2}}(y_1 \ldots y_n, \tau) = \prod_{i=1}^{n_{1,2}} \Psi_{n_{1,2}}^{(1,2)}(y_i, \tau), \tag{24}$$

where $\Psi_{n_{1,2}}^{(1,2)}(y_i,\tau)$ describes the $n_{1,2}$th state and can be found using the procedure of minimizing a functional (it is defined in Ref. 15 and reaches its minimum value when $\Psi_{n_{1,2}}^{(1,2)}(y_i,\tau)$ satisfies a classical nonlinear Schrödinger equation), we can transform (20) into

$$i\frac{\partial}{\partial\tau}\Psi_{n_{1,2}}^{(1,2)}(y_i,\tau) = -\frac{\partial^2}{\partial y^2}\Psi_{n_{1,2}}^{(1,2)}(y_i,\tau) + 2(n_{1,2}-1)\varkappa|\Psi_{n_{1,2}}^{(1,2)}(y_i,\tau)|^2\Psi_{n_{1,2}}^{(1,2)}(y_i,\tau). \tag{25}$$

Equation (25) is formally the same as the classical Eq. (2) under the substitution $\varkappa \to \varkappa(n_{1,2}-1)$; we can therefore use the results of Sec. 3 [see Eq. (15), $l=0$] and write down self-similar solutions for the wave functions for the two above-mentioned cases—of spatial and temporal profiles.[9)]

For brevity we write down the solutions only for the second case and directly for the modes $A_{0,h}(\eta,x)$ [under the condition (A2)]:

$$\langle\varphi|A_{0,h}^+(\eta,x)A_{0,h}(\eta,x)|\varphi\rangle = 0{,}5|\alpha_0|^2\exp(-0{,}5|\alpha_0|^2)\{0{,}5\sum_n \frac{|\alpha_0/\sqrt{2}|^{2n}}{n!} \times[|\Psi_{n+1}^{(1)}|^2 + |\Psi_{n+1}^{(2)}|^2] \pm \exp(-0{,}5|\alpha_0|^2) \times \mathrm{Re}\{[\sum_n \frac{|\alpha_0/\sqrt{2}|^{2n}}{n!n!}(\Psi_{n+1}^{(1)})^*\Psi_{n+1}^{(2)} + \sum_{n,m}\frac{(\alpha_0^*/\sqrt{2})^n(\alpha_0/\sqrt{2})^m}{n!m!}(\Psi_{n+1}^{(1)})^*\Psi_{m+1}^{(2)}]\exp(2i\eta/L_{exc})\}\}. \tag{26}$$

It is clear from (26) that a quantum pulse is a superposition of quantum partial pulses [the first two terms in (26)] and their products [the last two (cross-) terms]. In the quasiclassical limit[15] we have $n = n_0 \equiv |\alpha_0|^2 \gg 1$ and $\delta_{n_{1,2}} = \delta_{n_{0_{1,2}}} \equiv L'_{exc}p_{1,2}(n_{0_{1,2}}-1) \ll 1$, i.e., when the number of photons is large and fixed ($n = n_0$) this superposition is, in fact, replaced by a single phase-modulated pulse:[10)]

$$\langle\varphi|A_{0,h}^+(\eta,x)A_{0,h}(\eta,x)|\varphi\rangle \approx \frac{|\alpha_0|^2}{4}\{|\Psi_{n+1}^{(1)}|^2 + |\Psi_{n+1}^{(2)}|^2 \pm 2\mathrm{Re}[(\Psi_{n+1}^{(1)})^*\Psi_{n+1}^{(2)}\exp(2i\eta/L_{exc})]\}. \tag{27}$$

For $\delta_{n_{1,2}} \gtrsim 1$, $\eta/L'_{exc} > 1$ we may find (for $\psi_{n_1+1}^{(1)} > \psi_{n_2+1}^{(2)}$) that:

$$\psi_{n_1+1}^{(1)} \approx \frac{1}{2}\{1 + (1 + 8\delta_{n_1+1}\eta^2/L'^2_{exc})^{1/2}\}, \tag{28a}$$

$$\psi_{n_2+1}^{(2)} \approx \frac{1}{2}\{1 + \cos(2\sqrt{2}\delta_{n_2+1}\eta/L'_{exc})\}. \tag{28b}$$

We show in Fig. 4 the qualitative behavior of $\psi_{n_{1,2}+1}^{(1,2)}$ as function of η/L'_{exc} (cf. Ref. 22). $\psi_{n_1+1}^{(1)}$ thus describes the decompressed (self-defocused) wave packet $a_1(\eta,x)$ and $\psi_{n_2+1}^{(2)}$ describes the periodic compression [(self-focusing) of $a_2(\eta,x)$].

An analysis of the problem shows (see Ref. 22) that we

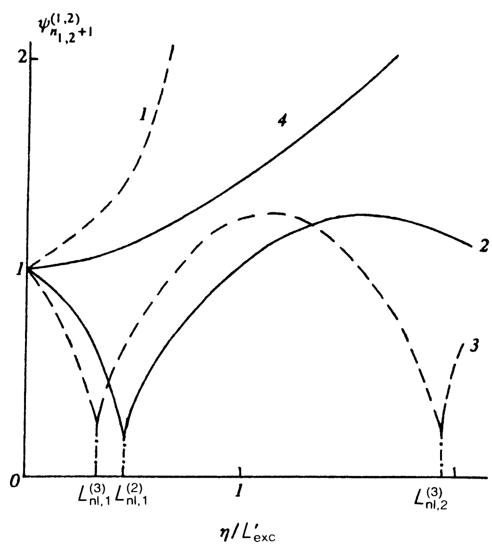

FIG. 4. Qualitative behavior of the wave functions $\psi^{(1)}_{n_1+1}$ (curve *1*: dispersive spreading) and $\psi^{(2)}_{n_2+1}$ (curves *2* and *3*: periodic compression, in the points $L_{nl,i}$) as functions of the spatial parameter η/L'_{exc}. For $p_1 = p_2$ ($m_{3,4} = 0$), $n_{1,2} = 0$—linear optics—we have: $\delta_{n_1+1} = \delta_{n_2+1} \equiv \delta = 0$ ($\psi^{(1)}_{n_1+1} = \psi^{(2)}_{n_2+1}$)—curve *4*: dispersive spreading. The curves *2* and *3* correspond, respectively, to the values $\delta = 3$ and $\delta = 5$ ($m_{3,4} = 0$) of the nonlinear parameter.

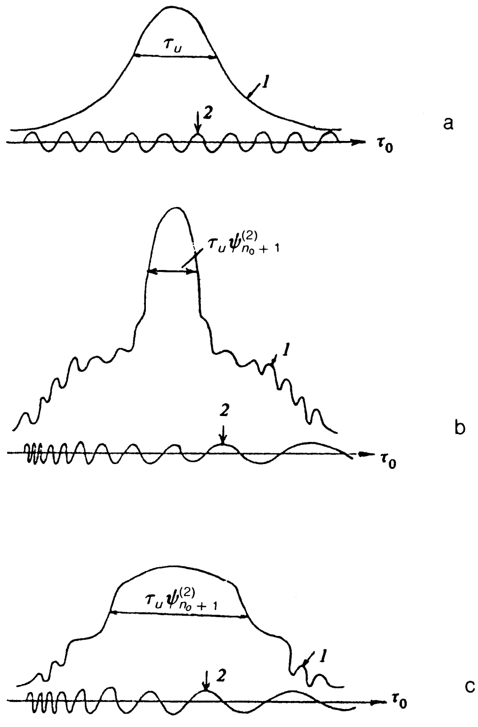

FIG. 5. Schematic picture of compression (b; for values of the parameter η/L'_{exc} corresponding to the minima in the curves *2* and *3* of Fig. 4) and decompression (c; for η/L'_{exc} values determining the maxima of the curves *2* and *3* in Fig. 4) for the incident pulse (a); note that *1* shows the envelope of the $A_{0,h}(\eta,x)$ packet and *2* its filling—local modes, characterizing the phase modulation and determined by (26) and (A1); the product $\tau_u\psi^{(2)}_{n_0+1}$ determines the exit pulse length (τ_u is the initial length), and $\tau_0 = t - z\cos\bar{\delta}/v - z\sin\bar{\delta}/v$ is the running coordinate.

can neglect in Eq. (27) the contribution from the last (interference, cross) term in comparison to the contribution from one of the first two terms, owing to the dispersive spreading of the packet $a_1(\eta,x)$. This interference term produces only small oscillations in the overall envelope of the pulse. The component $a_1(\eta,x)$ with $\psi^{(1)}_{n_1+1}$ then determines a pedestal (which does not play a significant role for the shape of the packet for large η/L'_{exc}) while the component $a_2(\eta,x)$ with $\psi^{(2)}_{n_2+1}$ is responsible for the compression of the packet (it is just this one which determines its shape). This is shown schematically in Fig. 5. At the points (in the parameter η/L'_{exc}) where $\psi^{(2)}_{n_1+1} \equiv \psi^{(2)}_{n_0+1}$ is a minimum we have a decrease in the length of the transmitted 0- and the scattered h-pulses, i.e., compression appears (Fig. 5b); these lengths are approximately equal to $\tau_u\psi^{(2)}_{n_0+1}$ (see Ref. 22) while at the points (in the parameter η/L'_{exc}) where $\psi^{(2)}_{n_2+1}$ is a maximum (~ 1) the pulses undergo dispersive spreading (see Fig. 5c); their length is also approximately determined by the quantity $\tau_u\psi^{(2)}_{n_0+1}$ [see Eq. (A20) in Appendix 2].

In the quantum case the behavior of the envelope of the wave packet may differ considerably from the quasiclassical limit; this is connected with the need to take into account the infinite set of functions $\psi^{(1,2)}_{n_{1,2}+1}$.[11] We discuss such a quantum theory of the compression of a pulse in Appendix 2, where we also give the main results. Here, however, we dwell briefly only on an analysis of the behavior of the fluctuations of the quadratures of the partial modes.

A calculation of $\langle:\Delta Q^2_{1,2}:\rangle$ and $\langle:\Delta P^2_{1,2}:\rangle$ leads to the relations [cf. (A12)]:

$$\left.\begin{array}{l}\langle\varphi_{1,2}|:\Delta Q^2_{1,2}:|\varphi_{1,2}\rangle\\ \langle\varphi_{1,2}|:\Delta P^2_{1,2}:|\varphi_{1,2}\rangle\end{array}\right\} \approx 2\{\pm \mathrm{Re}[\sum_n \frac{|\alpha_{1,2}|^{2n}}{n!} \times \Psi^{(1,2)}_{n+1}\Psi^{(1,2)}_{n+2}\exp(-|\alpha_{1,2}|^2) - \sum_{n,m}\frac{|\alpha_{1,2}|^{2(n+m)}}{n!m!}\Psi^{(1,2)}_{n+1}\Psi^{(1,2)}_{m+1}\exp(-2|\alpha_{1,2}|^2)]\alpha^2_{1,2} + |\alpha_{1,2}|^2[\exp(-|\alpha_{1,2}|^2)\sum_n\frac{|\alpha_{1,2}|^{2n}}{n!}|\Psi^{(1,2)}_{n+1}|^2 - \exp(-2|\alpha_{1,2}|^2)\sum_{n,m}\frac{|\alpha_{1,2}|^{2(n+m)}}{n!m!}\Psi^{(1,2)}_{n+1}(\Psi^{(1,2)}_{m+1})^*]\}. \quad (29)$$

The behavior of the mean square fluctuations of the quadratures is determined in the framework of perturbation theory solely by the initial stage of the formation of the pulses (beams) in the DFB medium. Although the nonlinearity leads to a phase modulation of the wave packets, it hardly affects the change of its shape at all. It is clear that just this makes it possible to achieve a significant squeezing for the fluctuations of the quadratures—see Fig. 2. [Here it is important that the analysis is correct also for large L'_{exc}, i.e., small L_{nl} (cf. Ref. 24).] The situation here is analogous to the problem of quantum solitons. Since the effective phase modulation time is smaller than the time for the formation of a soliton while the characteristic time determining the effect of dispersion is larger than the latter (see Ref. 15); it is possible to obtain deep squeezing in the initial stage of soliton formation and this is usually analyzed (see Refs. 4 and 16).

However, later on the interaction between nonlinearity and dispersion turns out to be essential and this leads to a change in the shape of the packet when it leaves the DFB system (we return to this problem in Appendix 2).

For the partial pulse (beam) $a_1(\eta,x)$ the efficiency of the nonlinear interaction decreases since it is spread out so that the degree of squeezing for the corresponding quadrature will be small. However, for the self-focusing component $a_2(\eta,x)$ there will be squeezing. The magnitude of the mean square fluctuations of the quadratures for the resulting packets (the 0- and h-modes) will thus mainly be determined by the dispersion of the quadrature for $a_2(\eta,x)$. We note that also here the effects considered are appreciable for $\eta/L'_{\rm exc} > 1$. Moreover, Eq. (29) is very sensitive to the presence of phase relations between its terms which determine the local radiation modes; in particular, when they are dephased the squeezed states may be destroyed (cf. Ref. 25).

We emphasize also that in the quasiclassical limit there occur practically the same results as in the Heisenberg representation—see (11). One can easily check this by using perturbation theory for finding $\psi_n^{(1,2)}$ and $\Phi_n^{(1,2)}$—see Appendix 1—and neglecting the contribution from the terms $K_2^2|\alpha_0|^4$ and $|\varkappa_2|^2|\alpha_0|^4$ [see (6)]. This means physically that quantum interference is taken into account in (26) only between two nearest neighbors (in the index $n_{1,2}$).

In conclusion it is useful to estimate the contribution of the different terms in the sums in (26). For small $n_{1,2} \gtrsim 1$ one can neglect the contribution from the terms corresponding to those indices; they therefore do not affect the development of fluctuations and $\langle:\Delta Q^2:\rangle \approx 0$. Such a result is obvious—this condition corresponds to the inequality $\delta_{n_{1,2}+1} \sim L'_{\rm exc}/L_{\rm nl} \ll 1$, i.e., the 0- and h-modes are able to separate before there is an effective energy exchange realized between them.

When the $n_{1,2}$ increase, but under the condition that $\delta_{n_{1,2}+1} < 1$, this result can still be retained, provided $\eta/L'_{\rm exc} < 1$ (in that case $\eta/L_{\rm nl} < 1$).

One sees easily that in the case $\delta_{n_{1,2}+1} \gtrsim 1$, $\eta/L'_{\rm exc} > 1$, $\eta/L_{\rm nl} > 1$ $(1 < \eta/L_{\rm nl} < \eta/L'_{\rm exc})$ there may appear squeezed states.

For terms of the sum in (26) with indices $n_{1,2}$ such that $\delta_{n_{1,2}+1} \gtrsim 1$ (i.e., $L'_{\rm exc} > L_{\rm nl}$) the squeezed light may be formed for $\eta/L'_{\rm exc} < 1$ (independent of the value of the parameter $\eta/L_{\rm nl}$) although its efficiency is small for $\eta/L'_{\rm exc} < \eta/L_{\rm nl} < 1$.

We note that one can usually restrict oneself for numerical calculations[15] to summing only terms with $n \sim n_0$ where n_0 corresponds to the number of photons for which $\psi_{n_0+1}^{(2)}$ is a minimum (Fig. 5b)—compare the assumption in the derivation of (27).

As a matter of principle one is also interested in the problem of the cross-correlation of the fluctuations of the 0- and h-wave packets. At first sight it seems that it is rigidly defined by the partial operators $a_{1,2}(\eta,x)$. For the case $(l=0)$ considered by us there is no intermode quantum interference of these operators [see (15)] which, in particular, leads to independent equations for the wave functions $\psi_n^{(1,2)}$—see (25). This, however, does not mean at all that the fluctuations of the quadratures for the $a_{1,2}(\eta,x)$ or $A_{0,h}(\eta,x)$ modes are independent in the general case also for $l \neq 0$ for which the energy exchange and the transfer of fluctuations for the $A_{0,h}(\eta,x)$ modes is a consequence of the quantum interference of the $a_{1,2}(\eta,x)$ operators. For $l=0$ we have the appearance only of cross-terms for the packets $\Psi_n^{(1,2)}$ in the compression problem [see (26)]. Nonetheless our analysis shows also for $l \neq 0$ [the equations for the $a_{1,2}(\eta,x)$ are not decoupled—see (15)] that the physical picture of the behavior of the fluctuations does not change in principle.[24]

One can give a clear picture of the description of the correlation properties for the fluctuations of the 0- and h-modes by introducing the radius R of the quantum correlation (e.g., for the numbers of photons $n_{0,h}$ of these modes) taking into account their anticommutator (cf. Refs. 18 and 19):

$$R = (\langle n_0 n_h\rangle + \langle n_h n_0\rangle - 2\langle n_0\rangle\langle n_h\rangle)/2(\langle \Delta n_0^2\rangle\langle\Delta n_h^2\rangle)^{1/2}$$
$$\equiv (\langle\Delta n_0 \Delta n_h\rangle + \langle\Delta n_h \Delta n_0\rangle)/2(\langle\Delta n_0^2\rangle\langle\Delta n_h^2\rangle)^{1/2}, \quad (30)$$

where $\Delta n_{0,h} = n_{0,h} - \langle n_{0,h}\rangle$.

The qualitative discussions given here of the interdependence of the fluctuations in the quadratures of the $A_{0,h}(\eta,x)$ modes give for them, indeed, a value of R which is different from zero.

We note that the presence of an explicit correlation between the 0- and h-modes (and their fluctuations) and therefore the existence of energy exchange in the system is, nonetheless, compatible with the condition that their commutator vanish. Indeed, in our case one checks easily that $[n_0, n_h] = 0$. This is connected with the possibility of a simultaneous experimental measurement (by macroscopic apparatus) of two quantities—the numbers of photons in the given two modes [cf. (19)].

The introduction of a parameter R in the form (30) such that it describes the quantum correlation of the two modes is most expedient when one considers light which is squeezed in amplitude, i.e., in the number of photons (but not in the quadratures), e.g., in the problem of the fluctuations of a two-mode field.[29] However, this analysis goes beyond the framework of the present paper and we shall not discuss it (see Ref. 30).

In the present section we have thus shown that for the problem considered here the results of the analysis in the Schrödinger representation describing the process of changing the shape of the pulse (beam) are the most general ones; in the quasiclassical limit they go over into the corresponding approximate solutions in the Heisenberg representation. The physics of the effect consists in that a quantum wave packet in a DFB system is a superposition of a denumerable set of classical pulses (beams)—with different phase advances and different shapes—entering into the sum with different weight factors. As a result the initial pulse can when leaving the system undergo compression or decompression. We note in Appendix 2 that in contrast to the classical case where a complete cancellation of the chirp formed in the nonlinear medium is possible due to the spatial dispersion of the medium, no such cancellation occurs here. Taking the change in the shape of the wave packet into account leads to the necessity of analyzing the quantum relation between the pulse length τ and the width $\Delta\omega$ of its spectrum: $\Delta\omega\tau = \tilde{S} \geqslant 0.5$ [see Eq. (A19) in Appendix 2]. For a spectrally bounded pulse necessarily in a coherent state[23,24] the value of $\tilde{S}$ is a minimum (for a Gaussian profile $\tilde{S} = 0.5$).

The decompression of the wave packet does not make it possible that there exists a deep suppression of the mean square fluctuations of the quadratures, the behavior of which depends in principle on the nonlinear phase advances of the different terms of the pulses in (26) which are determined by the local radiation modes.

CONCLUSION

In the present paper we have thus been the first to consider the principal problem of the formation of squeezed states of light for space-time light packets propagating in a nonlinear DFB system under Bragg resonance conditions.

It is shown in Ref. 31 that the effective length and degree of squeezing depend on a parameter proportional to the nonlinearity constant of the medium and to the pumping strength and inversely proportional to its spectral width. For LC the nonlinearity constant is very large which makes it possible to reach a high degree of squeezing for a low radiation strength.

Especially efficient are LC in the form of thin oriented $100\,\mu$m thick layers. Simple estimates show (see Ref. 1) that the use of such layers makes it possible to carry out experiments for observing nonclassical states of light in the field of relatively low-power lasers with high coherence properties.

Most promising here are hybrid schemes with two elements in series, the first a strongly nonlinear medium (a nematic LC) in which the light induces an effective lattice—the second term in Eq. (1) is operative—and for the second a DFB system (CLC). Such schemes are traditional when we have compression (nonlinear fibers and dispersive delay lines). For specially prepared nematic LC cells (see Ref. 9) it turns out to be possible to induce a refractive-index lattice and then to obtain quantum states of the field in a He–Ne laser field of milliwatt (and even less) power. Of course, the number of photons of the incident radiation is still assumed to be large (quasiclassical limit) so that there do not arise any problems of obtaining a strong squeezing for low pumping powers. A detailed discussion of this problem and also an actual experimental scheme for obtaining an effective squeezing of radiation in the presence of DFB is of interest by itself. We note, nonetheless, that especially interesting possibilities in the field of generating extreme states of light are opened up when one uses lasers with an extreme spatial coherence determined by the natural (spontaneous) laser noise[27] which thus can be suppressed in the case when squeezed light is formed.

Of course, there arises then the problem of the real characteristics (e.g., the temperature stability) of the DFB system itself (and of its natural fluctuations) and as a consequence the problem of the widths of the Bragg reflection resonance curves and also those of the coherence properties of the incident radiation, of the presence of loss in the medium, and so on. All these factors lead to a degradation of the squeezed states in the propagating pulse. Moreover, schemes for detecting the squeezed light (e.g., a balance homodyning scheme) require a separate analysis.[2] Of great interest in this case is the use of a polarization interferometer on the basis of which one may realize an original and highly effective heterodyning scheme for squeezed light.[28]

We express our gratitude to our teacher S. A. Akhmanov who died prematurely in 1991 and with whom we discussed the present work several times; the idea itself of studying the problems touched upon in this paper arose from discussions in the nonlinear optics seminar at Moscow State University over which he presided.

APPENDIX 1

Wave-packet operators (light beams)

We look for the solution of (15) with $l=0$ in the form [cf. (5)]

$$a_{1,2}(\eta,x)=\frac{1}{(f_{1,2}(\eta))^{1/2}}\exp\left\{-\frac{x^2}{2L_{\rm exc}L'_{\rm exc}\tan^2\tilde{\delta}f^2_{1,2}(\eta)}\pm i\frac{x^2f'_{1,2}(\eta)}{2L_{\rm exc}\tan^2\tilde{\delta}f_{1,2}(\eta)}+i\Phi_{1,2}(\eta)\right\}a_{1,2},\qquad(\mathrm{A1})$$

where the operators $a_{1,2}(\eta,x)\equiv a_{1,2}(z,x)$ (d is the thickness of the medium) correspond to the radiation leaving the medium, while the $a_{1,2}\equiv a_{1,2}(\eta,x)|_{\eta=0}$ correspond to the radiation entering the medium; $f'(\eta)\equiv df(\eta)/d\eta$. The boundary conditions for the $A_0(\eta,x)$ mode are:

$$\alpha_0(\eta,x)|_{\eta=0}=\exp(-x^2\cos^2\tilde{\delta}/2r_0^2)\alpha_0,\qquad(\mathrm{A2})$$

where r_0 is the beam radius on entering; the $A_h(\eta,x)$ mode is a vacuum mode ($\alpha_h(\eta,x)|_{\eta=0}=0$). The commutation conditions are

$$[a_1(\eta,x);a_1^+(\eta,x)]=A,\qquad[a_2(\eta,x);a_2^+(\eta,x)]=B,\qquad(\mathrm{A3})$$

where A and B are some operators with real positive values of their averages, $\langle A\rangle$ and $\langle B\rangle$ (cf. Ref. 13).

The relations between the characteristic spatial scales of the problem are as follows: $L_{\rm exc}\lesssim L_{\rm nl}\sim L'_{\rm exc}$ where $L'_{\rm exc}=r_0^2/L_{\rm exc}\sin^2\tilde{\delta}$; they determine the length over which the effect of the finite size of the beam exerts influence (spatial drift of the 0- and h-modes).[12]

The requirement that the operators $f_{1,2}(\eta)$ and $\Phi_{1,2}(\eta)$ be Hermitian leads to the equations [substitution of (A1) into (15)]:

$$\frac{L'^2_{\rm exc}}{f_1}\frac{d^2f_{1,2}}{d\eta^2}=\frac{1}{f^4_{1,2}}\pm\delta'_{1,2}a^+_{1,2}\frac{1}{f^3_{1,2}}a_{1,2},$$
$$-2L'_{\rm exc}\frac{d\Phi_{1,2}}{d\eta}=\pm\frac{1}{f^2_{1,2}}+\delta'_{1,2}a^+_{1,2}\frac{1}{f^3_{1,2}}a_{1,2},\qquad(\mathrm{A4})$$

where $\delta'_{1,2}=L'_{\rm exc}p_{1,2}$ and $p_{1,2}$ is a nonlinear parameter defined in (15).

The system (A4) has Hermitian solutions only in the framework of perturbation theory—when we expand in the small operator parameter $\delta'_{1,2}a^+_{1,2}a_{1,2}$, i.e., we assume that $\delta'_{1,2}|a_{1,2}|^2\ll1$, and neglect small (non-Hermitian) terms. We then obtain a relation like (6):

$$f_1(\eta)\approx\mu_1+\mu_2\delta'_1a^+_1a_1+\mu_3\delta'^2_1(a^+_1a_1)^2,$$
$$f_2(\eta)\approx\nu_1+\nu_2\delta'_2a^+_2a_2+\nu_3\delta'^2_2(a^+_2a_2)^2,\qquad(\mathrm{A5})$$

where the $\mu_{1,2,3}=\mu_{1,2,3}(\eta)$ and $\nu_{1,2,3}=\nu_{1,2,3}(\eta)$ are numerical functions which satisfy the following equations [substitution of (A5) into (A4), neglecting small terms and satisfying the boundary conditions $f_{1,2}(\eta)|_{\eta=0}=1$ and $df_{1,2}(\eta)/d\eta|_{\eta=0}=0$]:

$$L'^2_{\rm exc} d^2\mu_1/d\eta^2 \approx \frac{1}{\mu_1^3}, L'^2_{\rm exc} d^2\mu_2/d\eta^2 \approx \frac{-3\mu_2}{\mu_1^4} + \frac{1}{\mu_1^2},$$

$$L'^2_{\rm exc} d^2\mu_3/d\eta^2 \approx \frac{-2\mu_2}{\mu_1^3}. \quad (A6)$$

From (A6) we find

$$\mu_1 \approx (1+\eta_0^2)^{1/2}, \quad \mu_2 \approx \eta_0 \operatorname{arctg}\eta_0 - 0{,}5\ln(1+\eta_0^2),$$

$$\mu_3 \approx 0{,}5(1+\eta_0^2)^{1/2}\ln(1+\eta_0^2) \quad (A7a)$$

$$+ \int \frac{\operatorname{arctg}\eta_0}{(1+\eta_0^2)^{1/2}} d\eta_0 - \eta_0 \ln[\eta_0 + (1+\eta_0^2)^{1/2}],$$

where $\eta_0 \equiv \eta/L'_{\rm exc}$ and where we have used the approximation $3\mu_2/\mu_1^2 < 1$, $(6\mu_2^2 - 3\mu_1\mu_3)/\mu_1^2 < 2\mu_2$.

After similar transformations we get for $\nu_{1,2,3}$:

$$\nu_1 = \mu_1, \quad \nu_2 = -\mu_2, \quad \nu_3 = -\mu_3. \quad (A7b)$$

The physical meaning of the relations obtained here consists in the following: The parameters $\mu_1(\eta)$ and $\nu_1(\eta)$ characterize the linear drift (in the transverse direction) as the modes $a_{1,2}(\eta,x)$ propagate; $\mu_{2,3}(\eta)$ and $\nu_{2,3}(\eta)$ correspond to the nonlinear drift.

The procedure described here can also be applied for finding the nonlinear phase operator $\Phi_{1,2}(\eta)$ in (A1). We then find successively:

$$\Phi_1(\eta) \approx \sigma_1(\eta) + \sigma_2(\eta)\delta'_1 a_1^+ a_1 + \sigma_3(\eta)\delta'^2_1(a_1^+a_1)^2,$$

$$\Phi_2(\eta) \approx \lambda_1(\eta) + \lambda_2(\eta)\delta'_2 a_2^+ a_2 + \lambda_3(\eta)\delta'^2_2(a_2^+a_2)^2, \quad (A8)$$

$$-2L'_{\rm exc}\, d\sigma_1/d\eta \approx 1/\mu_1^2, \quad -2L'_{\rm exc} d\sigma_2/d\eta \approx 1/\mu_1 - 2\mu_2/\mu_1^3,$$

$$-2L'_{\rm exc}/d\eta \approx -\mu_2/\mu_1^2, \quad (A9)$$

$$\sigma_1 \approx -0{,}5\operatorname{arctg}\eta_0, \quad \sigma_2 \approx 0{,}5\ln[\eta_0(1+\eta_0)^{1/2}]$$

$$-\frac{\operatorname{arctg}\eta_0}{(1+\eta_0^2)^{1/2}} - 0{,}5\eta_0\frac{\ln(1+\eta_0^2)}{(1+\eta_0^2)^{1/2}}, \quad (A10a)$$

$$\sigma_3 \approx 0{,}5\{0{,}5\ln(1+\eta_0^2)\operatorname{arctg}\eta_0 - \int \frac{\ln(1+\eta_0^2)}{1+\eta_0^2} d\eta_0\},$$

$$\sigma_1 = -\lambda_1, \quad \sigma_2 = \lambda_2, \quad \sigma_3 = -\lambda_3. \quad (A10b)$$

We used here the boundary conditions $\Phi_{1,2}(\eta)\,|_{\eta=0} = 0$.

If we use (A7) and (A10), Eqs. (A5) and (A8) determine the partial modes [(the wave-packet operators) $a_{1,2}(\eta,x)$ in (A1)].

Fluctuations

We determine the Hermitian quadratures

$$Q_{0,h}(\eta,x) = A_{0,h}(\eta,x) + A^+_{0,h}(\eta,x),$$

$$P_{0,h}(\eta,x) = i\{A^+_{0,h}(\eta,x) - A_{0,h}(\eta,x)\}. \quad (A11)$$

We have for the mean square fluctuations $\langle \Delta Q^2_{0,h}\rangle$ and $\langle \Delta P^2_{0,h}\rangle$:

$$\langle \Delta Q^2_{0,h}\rangle = 0{,}5\{\langle \Delta Q_1^2\rangle + \langle \Delta Q_2^2\rangle\},$$

$$\langle \Delta P^2_{0,h}\rangle = 0{,}5\{\langle \Delta P_1^2\rangle + \langle \Delta P_2^2\rangle\}, \quad (A12)$$

where $\langle \Delta Q^2_{1,2}\rangle$ and $\langle \Delta P^2_{1,2}\rangle$ are the mean square fluctuations of the quadratures of the partial modes and the quantum averaging is carried out over the initially coherent states:

$$\langle\Phi|\ldots|\Phi\rangle = {}_h\langle 0|\langle\alpha_0(x)|\ldots|\alpha_0(x)\rangle|0\rangle_h$$

$$= \langle\alpha_1(x)|\alpha_2(x)|\ldots|\alpha_2(x)\rangle|\alpha_1(x)\rangle, \quad (A13)$$

where $|\alpha_0(x)\rangle$ is the wave function of the eigenstate of the operator $A_0(\eta,x)\,|_{\eta=0}$; $|0\rangle_h$ that of the vacuum state for $A_h(\eta,x)\,|_{\eta=0}$ and $|\alpha_{1,2}(x)\rangle$ those for the partial wave-packet operators $a_{1,2}(\eta,x)\,|_{\eta=0}$,[13)]

We assume that $m_4 = 1$ $(l=0)$—see (15); the $\langle \Delta Q^2_{0,h}\rangle$ and $\langle \Delta P^2_{0,h}\rangle$ fluctuations which do not contain quantum interference terms (cf. Ref. 22) are the same for the 0- and the h-modes; we denote them by $\langle \Delta Q_0^2\rangle = \langle \Delta Q_h^2\rangle \equiv \langle \Delta Q^2\rangle$ and $\langle \Delta P_0^2\rangle = \langle \Delta P_h^2\rangle \equiv \langle \Delta P^2\rangle$.

Temporal characteristics (light pulses)

In the nonstationary problem (quasistatic case) there is at the entrance into the medium radiation with a temporal (Gaussian) profile of its envelope, i.e., we must put $A_0(\eta,\xi,x)\,|_{\eta=0} \equiv A_0(x,t) \equiv A_0$ and $a_{1,2}(x,t) \equiv \alpha_{1,2}$, where the averages for the corresponding operators are determined in the form [we use (16)]:

$$\langle\alpha_0(x,t)|A_0(x,t)|\alpha_0(x,t)\rangle = \alpha_0(x)\exp(-t^2/2\tau_u^2),$$

$$\langle\alpha_{1,2}(x,t)|a_{1,2}(x,t)|\alpha_{1,2}(x,t)\rangle = \alpha_{1,2}(x)\exp(-t^2/2\tau_u^2), \quad (A14)$$

τ_u is the initial length of the pulse.

The boundary conditions are in this case the following [cf. (A2)]:

$$\alpha_0(\eta,\xi,x)\,|_{\eta=0} = \exp(-x^2\cos^2\tilde\delta/2r_0^2)\exp(-t^2/2\tau_u^2)\alpha_0 \quad (A15)$$

and they must lead to a time-dependence also for the operators $f_{1,2} \equiv f_{1,2}(\eta,t)$ and $\Phi_{1,2} \equiv \Phi_{1,2}(\eta,t)$ in (A1)—the case of quasistatic self-focusing (self-defocusing).[14)]

If we are only interested in the effects of transforming the temporal envelope of the light pulse (its compression and decompression)[15)] we have instead of (A1) for the propagating quantum wave packets the following solutions:

$$a_{1,2}(z,x,t) = \frac{1}{(f_{1,2}(z))^{1/2}}$$

$$\times\exp\left\{-\frac{\tau_0^2}{2\tau_u^2 f_{1,2}(z)} \pm i\frac{\tau_0^2\gamma_0 f'_{1,2}(z)}{2\tau_u^2 f_{1,2}(z)} + i\Phi_{1,2}(z)\right\}, \quad (A16)$$

where the effective time coordinate is $\tau_0 = t - z/v\cos\tilde\delta - x\sin\tilde\delta/v$, $\gamma_0^{-1} = L_{\rm exc}\sin^2\tilde\delta\tan^2\tilde\delta/v^2\tau_u^2$, τ_u determines the length of the pulse, and the operators $f_{1,2}(z)$ and $\Phi_{1,2}(z)$ are functions only of the propagation coordinate z. We obtain the solution of (A16) under the boundary conditions

$$\alpha_0(z,x,t)\,|_{z=0} = \exp[-(t - x\sin\tilde\delta/v)^2/2\tau_u^2]\alpha_0,$$

$$\alpha_h(z,x,t)\,|_{z=0} = 0, \quad (A17)$$

$$\alpha_{1,2}(z,x,t)\,|_{z=0} = \exp[-(t - x\sin\tilde\delta/v)^2/2\tau_u^2]\alpha_{1,2}.$$

It is clear that the transition to the spatial problem is performed by means of the substitution $\gamma_0 \to L'_{\rm exc}$, $\tau_0^2/2\tau_u^2 \to x^2/2L_{\rm exc}L'_{\rm exc}\tan^2\tilde{\delta}$.

Comparison of (A1) and (A16) thus enables us to speak about a complete space-time analogy for the two cases considered by us.

APPENDIX 2

The uncertainty relations for the quadrature components P and Q, [see (A11) and (A12)],

$$\langle{:}\Delta Q^2{:}\rangle\langle{:}\Delta P^2{:}\rangle \gtrsim 0, \quad (A18)$$

are, strictly speaking, justified for single-mode fields in the stationary case. Using (A18) for quantum wave packets propagating in a medium with a cubic nonlinearity therefore requires additional comment.

Indeed, the appearance of phase modulation for an ensemble of wave packets, which is characteristic for the quantum case [see (26)] and for which the propagation velocities of the components and also the spatial scales $L_{\rm nl} \sim 1/(n_{1,2}-1)$ are different, leads to a complex transformation of the envelope and makes the problem considerably more complicated. This set of packets, in fact, reduces only in the quasiclassical limit to a single pulse (beam) with some effective value of $L_{\rm nl}$ [see (27)].

It is therefore convenient, for instance, in the problem of solitons, which are stable objects, to start from the momentum—coordinate[2] or (when one takes phase modulation into account) from the particle number—phase[15] uncertainty relations. However, in our case it is apparently more natural to turn to the relation connecting the length τ of the envelope of the packet with the width $\Delta\omega$ of its spectrum, i.e., to the time-energy[19] uncertainty relation:[16)]

$$\Delta\omega\tau = \bar{S} \geq 0{,}5, \quad (A19)$$

where

$$\tau = (\bar{t}^2 - (\bar{t})^2)^{1/2}, \qquad \Delta\omega = (\bar{\omega}^2 - (\bar{\omega})^2)^{1/2},$$

$$\bar{t}^n = W_0^{-1}\int_{-\infty}^{+\infty} t^n\langle a^+(t)a(t)\rangle dt, \qquad \bar{\omega}^n = W_0^{-1}\int_{-\infty}^{+\infty}\omega^n S_0(\omega)d\omega,$$

$$W_0 = \int_{-\infty}^{+\infty}\langle a^+(t)a(t)\rangle dt, \qquad S_0(\omega) = 2\pi\langle a^+(\omega)a(\omega)\rangle,$$

$\langle a^+(t)a(t)\rangle$ is the envelope of the radiation wave packet; W_0 and $S_0(\omega)$ are its energy and spectral density, respectively; $a^+(\omega)$ and $a(\omega)$ are the Fourier transforms of $a^+(t)$ and $a(t)$ [we do not distinguish here between the notation for the partial and the normal (0- and h-) radiation modes].

Although the interpretation of Eq. (A19) in quantum theory is not that simple (see, e.g., Ref. 19), for the problem considered by us it is not significant. Therefore, using (26) (for simplicity we neglect the cross terms) we get for τ and $\Delta\omega$ in the case of the 0- and h-modes:

$$\Delta\tau_{0,h} \approx \left\{\exp\left(-\frac{|\alpha_0|^2}{2}\right)\sum_m \frac{|\alpha_0/\sqrt{2}|2m}{m!} \times[(\psi_{m+1}^{(1)}\tau_u)^2 + (\psi_{m+1}^{(2)}\tau_u)^2]\frac{1}{4}\right\}^{1/2}, \quad (A20)$$

$$\Delta\omega_{0,h} \approx \left\{\exp\left(-\frac{|\alpha_0|^2}{2}\right)\sum_m \frac{|\alpha_0/\sqrt{2}|^{2m}}{m!} \times\left[\left(\frac{1}{\beta_{m+1}^{(1)}\tau_u}\right)^2 + \left(\frac{1}{\beta_{m+1}^{(2)}\tau_u}\right)^2\right]\frac{1}{2}\right\}^{1/2},$$

where $1/\beta_{m+1}^{(1,2)}\tau_u = \{1 + (\gamma_0\psi_{m+1}^{(1,2)}\psi_{m+1}^{(1,2)\prime})^2\}^{1/2}/\psi_{m+1}^{(1,2)}\tau_u$ is the effective width of the spectrum of the $(m+1)$st component of the packet (its length is τ_u). It is clear from (A20) that the lengths of the transmitted and the scattered packets and also the widths of their spectra are determined by the lengths $\tau_{m+1}^{(1,2)} \sim \psi_{m+1}^{(1,2)}\tau_u$ and the spectra $1/\beta_{m+1}^{(1,2)}\tau_u$ of the partial pulses.

However, strictly speaking, it is impossible in the quantum case to neglect the cross-terms for the partial components since the envelope of the packet is formed by the superposition of the component pulses with different $n_{1,2}$—see (26). (In the quasiclassical case when one requires that $|\alpha_0|^2 \gg 1$ the interference term is unimportant.) The more exact expression for (A20) therefore contain additional terms of combinations of $\psi_{m+1}^{(1,2)}$ and $(\psi_{m+1}^{(1,2)})'$ which can be called cross-durations (spectra) of the partial components. Hence, in the quantum case there appears an additional mechanism for the change in the shape of the envelope. Moreover, for different $\psi_{m+1}^{(2)}$ in (26) there is an optimal length over which, for instance, there occurs compression of that component (see Fig. 4). The total effect therefore leads to a more complex dependence for compression than the simple periodicity in the propagation coordinate as is observed in the classical problem (see Ref. 22).[17)]

The transformation of the shape of the quantum wave packet (for the $A_0(\eta,x)$ mode), $\langle A_0^+(\eta,x)A_0(\eta,x)\rangle$, has thus its special features as compared to the classical case; its envelope is arranged in an "uncertainty shell" with a nonuniform width (along the length of the packet) which is determined by the superposition of component pulses and connected with a phase modulation of the radiation which appears (as the result of quantum interference) when it propagates in a nonlinear medium with a strong spatial dispersion—Fig. 6. Of course, these processes also determine the behavior of the fluctuations of both the amplitude (connected with the uncertainty in the amplitude of the oscillations[18)] filling with the optical frequency the "space" inside the envelope) and in phase (connected with the uncertainty in the position of the trains of these oscillations) (Ref. 17).[18)]

The fact itself of the appearance of phase modulation of the radiation due to the nonlinearity of the medium is, of course, characteristic also for the classical discussion. In that case it may be cancelled thanks to the dispersive properties of the medium and this leads, in particular, to compression in the temporal problem and to the appearance of a spectrally bounded pulse at the exit from the system [for which one can only write down Eq. (A19) which uniquely connects the values of τ and $\Delta\omega$]. On the other hand, cancellation is also possible due to an *a priori* chosen chirp of the initial pulse.[24]

However, in the quantum case such a total cancellation is not realized—this follows already from Eqs. (26) and (29)—i.e., no spectrally bounded pulse in the classical sense

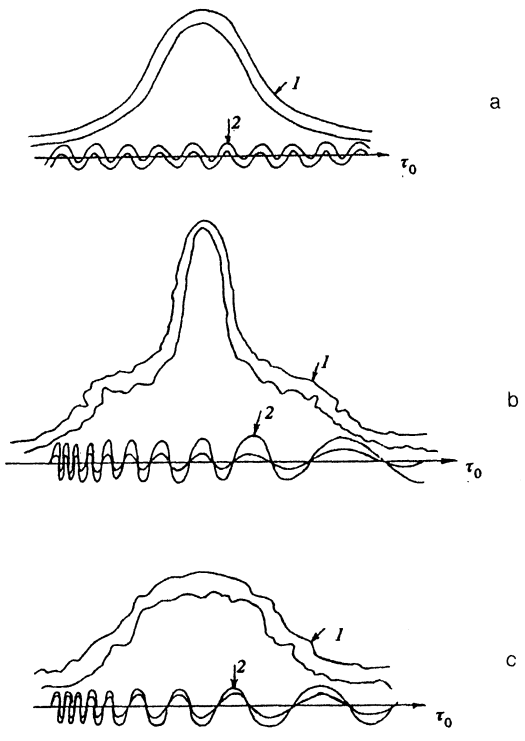

FIG. 6. Qualitative behavior of the wave functions of space-time packets for the problem in the Schrödinger representation: (a): a spectrally bounded pulse is incident when entering the medium; (b) and (c): transformation of the shape of the envelope on leaving: (b): compression, (c): spreading. Notation: 1 is the "uncertainty shell" for the envelope $\langle A_0^+(t,z,x)A_0(t,z,x)\rangle$ of the wave packet; 2 shows the absence (a) or presence (b,c) of quantum noise and phase modulation [the result of the summation in (26) over separate local modes for each $n_{1,2}$].

is formed. However, under well defined conditions a partial cancellation occurs (see Ref. 24) and here we can also speak of spectral boundedness of the wave packet on average; the fluctuations are then $\langle:\Delta Q^2:\rangle \sim \langle:\Delta P^2:\rangle \approx 0$ for any t and ω. It is important that for this case the "uncertainty shell" is uniform in t (or ω).[19)]

The state of the wave packets can thus be identified as being coherent when there is no phase modulation, but when there is phase modulation it leads to the appearance of additional quantum noise (nonuniform along the width of the packet). We are therefore dealing here with the presence of "nonclassical" chirp. The decreasing fluctuations (strength of the fluctuation spectrum) and then the transition to a squeezed state affects naturally also the shape of the envelope of the wave packet (and hence of the true spectrum) and in that sense one can speak, for instance, of its compression beyond the limits imposed by the spectral boundedness.

It is of interest to study these effects for video pulses when there is not mechanism for changing the length of the pulse by means of phase modulation; for squeeze light the coefficient $\tilde{S}$ in (A19) itself changes then its value (in this case it is necessary to write down the energy $\Delta E \equiv \hbar\Delta\omega$ instead of $\Delta\omega$).

In connection with this last statement one can again consider the problem of the stability of optical solitons to quantum fluctuations; a recent discussion (see Ref. 32) connects the effect obtained here, of the spreading of similar structures, exclusively with the quantum uncertainty in the position of the soliton peak. However, in such an analysis one should pay special attention to the role of phase modulation. Its influence, essential for the envelope of the wave packet $\Gamma \equiv \langle A_0^+(\eta,x)A_0(\eta',x')\rangle$ (the field correlation function) does, of course, not appear in the intensity correlation functions $G_0 \equiv \langle A_0^+(\eta,x)A_0(\eta,x)A_0^+(\eta',x')A_0 \times (\eta',x')\rangle$—a well known fact in classical statistical optics.[12] However, the latter quantity, G, which determines the correlation of the photons [cf. (30)], characterizes the stability of the packet.[32]

1) It is customary to speak in these cases of the geometries of Laue and of Bragg scattering.

2) We note also that the waves producing a lattice in the medium ($\mathbf{E}_2$ and $\mathbf{E}_3$ or $\mathbf{E}_1$ and $\mathbf{E}_3$) may in principle be of a nature which is not optical (e.g., acoustical—photon-phonon interactions); in that case the transition to a naturally periodic medium in (1) is most obvious.

3) For a free radiation field (without sources) in a cavity, when the different modes are quantum-mechanical oscillators, one is led to the representation of boson creation and annihilation operators; in that case $\hat{C} = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$.[17-19]

4) The effects of the diffractive spreading of beams has been neglected, i.e., $d \ll L_{\rm diffr} \equiv k_{0,h} r_0^2/2$, where r_0 is the initial beam radius; moreover, we assume that $L_{\rm exc} < d$.

5) Both these cases are mathematically similar under a transformation of the parameters: $L'_{\rm exec} \leftrightarrow v^2\tau_u/L_{\rm exc}\sin^2\tilde{\delta}\tan^2\tilde{\delta}$, $x^2/2L_{\rm exc}L'_{\rm exc}\times\tan^2\tilde{\delta} \leftrightarrow \tau_0^2/2\tau_u^2$, where $\tau_0 = t - z/v\cos\tilde{\delta} - x\sin\tilde{\delta}/v$ and τ_u is the initial length of the pulse.

6) For $\sigma_0 = t_0 = 0$ we assume that there is initially squeezed light.

7) The case $t_0 \to \infty$ goes at $\tau_u \to 0$ beyond the framework of the quasistationary approximation (the nonlinear response of the medium is assumed to be instantaneous) considered by us; however, the case $\sigma_0 \to \infty$ corresponds just to the value $x \to \infty$, i.e., to a weak beam field (the limiting values $\sigma_0 = 0$ and $\sigma \to \infty$ correspond to the values $x = 0$ and $x \to \infty$ since it is assumed in the given approximation that the quantity $L'_{\rm exc}$ is practically fixed ($L'_{\rm exc} \sim L_{\rm nl} > L_{\rm exc}$).

8) For small values of $L'_{\rm exc}$ one must take into account the diffractive terms of (15), which we have neglected.

9) The self-similar solutions $\Psi_{n_{1,2}}^{(1,2)}(y,\tau)$ presume that one can find functions $\psi_n^{(1,2)}(\tau)$ and $\Phi_n^{(1,2)}(\tau)$ determining the width and the phase of the propagating wave packet [analogous to the operators $f_{1,2}(\eta)$ and $\Phi_{1,2}(\eta)$ in (A1)] and satisfying relations such as (A4).

10) We note that one can use this approximation only after a quantum averaging over the states of the system (this was not taken into account in Ref. 15).

11) This contribution from the vacuum modes leads to the existence of a continuous set of spatial parameters $L_{{\rm nl},i}$ which determine the position of the minima of the function $\psi_{n_{1,2}}^{(2)}(\eta/L'_{\rm exc})$ (cf. Fig. 4).

12) The plane-wave limit (see Ref. 24) corresponds to the condition $L'_{\rm exc} \gg L_{\rm exc}$, $L_{\rm nl}$ ($r_0 \to \infty$).

13) We note that the last equality in (A13) is valid only when the wave packets $A_{0,h}(\eta,x)$ are uncorrelated on entering the medium.

14) The quantity τ_u is assumed to be longer than the characteristic time for the nonlinear response of the medium; the interdependence of the spatial and the temporal parameters of the problem is neglected [there are no mixed derivatives $\partial^2/\partial t\partial z$ in (15)].

15) The problem was solved in the classical case in Ref. 22.

16) In the classical case its analog is a property of the Fourier transformation; of course, (A19) can also be written for the spatial case.[12]

17) For different $n_{1,2}$ the spread of these dependences is confined to the band between the dependence for the linear case (when chirp is present *a priori*)[3] and that for the nonlinear[22] case.

18) These oscillations are often called local modes.[14]

19) One can write the operator of such a wave packet as a sum of a regular (c-number) and an operator (δ-correlated) part.[4,24]

[1] A. P. Alodjants, S. M. Arakelian, and Yu. S. Chilingarian, Proc. SPIE **1402**, 175 (1991); Kvantovaya Elektron. (Moscow) **18**, 967 (1991) [Sov. J. Quantum Electron. **21**, 876 (1991)].

[2] D. F. Walls and P. Zoller, Phys. Rev. Lett. **47**, 709 (1981); S. Kielich and M. K. Tanas, Opt. Acta **32**, 1023 (1985); D. F. Smirnov and A. S. Troshin, Usp. Fiz. Nauk **153**, 233 (1987) [Sov. Phys. Usp. **30**, 851 (1987)].

[3] R. B. Alaverdian, S. M. Arakelian, L. P. Gevorkian *et al.*, Phys. Lett. **A151**, 317 (1990).
[4] A. V. Belinskiĭ and A. S. Chirkin, Zh. Eksp. Teor. Fiz. **98**, 407 (1990) [Sov. Phys. JETP **71**, 228 (1990)]; A. V. Belinskiĭ, Zh. Eksp. Teor. Fiz. **99**, 748 (1991) [Sov. Phys. JETP **72**, 414 (1991)].
[5] V. L. Vinetskiĭ, N. V. Kukhtarev, S. G. Odulov *et al.*, Usp. Fiz. Nauk **129**, 113 (1979) [Sov. Phys. Usp. **22**, 742 (1979)].
[6] S. M. Arakelyan, Usp. Fiz. Nauk **153**, 579 (1987) [Sov. Phys. Usp. **30**, 1041 (1987)]; S. M. Arakelian, Yu. S. Chilingarian *et al.*, J. Phys. (Paris) **50**, 1393 (1989).
[7] Zs. Bor, IEEE J. Quantum Electron. **16**, 517 (1980); Zs. Bor and A. Muller, IEEE J. Quantum Electron. **22**, 1524 (1986); R. Grousson, M. L. Roblin, F. Gires *et al.*, Proc. SPIE **812**, 147 (1987).
[8] J.-C. Diels, W. Dietel, J. J. Fontaine *et al.*, J. Opt. Soc. Am. **B2**, 680 (1985).
[9] S. M. Arakelyan and Yu. S. Chilingaryan, *Nonlinear Optics of Liquid Crystals*, Nauka, Moscow (1984).
[10] R. B. Alaverdyan, S. M. Arakelyan, and Yu. S. Chilingaryan, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 366 (1985) [JETP Lett. **42**, 451 (1985)].
[11] P. D. Maker and R. W. Terhune, Phys. Rev. **A137**, 801 (1965).
[12] S. A. Akhmanov, Yu. E. D'yakov, and A. S. Chirkin, *Introduction to Statistical Radiophysics and Optics*, Nauka, Moscow (1981).
[13] L. V. Prokhorov, Usp. Fiz. Nauk **154**, 299 (1988) [Sov. Phys. Usp. **31**, 151 (1988)].
[14] B. Yurke and H. J. Potasek, J. Opt. Soc. Am. **B6**, 1227 (1989).
[15] Y. Lai and H. A. Haus, Phys. Rev. **A40**, 844, 854 (1989); H. A. Haus, R. Watanabe, and Y. Yamamoto, J. Opt. Soc. Am. **B6**, 1138 (1989).
[16] P. D. Drummond and S. J. Carter, J. Opt. Soc. Am. **B4**, 1515 (1987); S. J. Carter, P. D. Drummond, M. D. Reid *et al.*, Phys. Rev. Lett. **58**, 1841 (1987); P. D. Drummond, Quantum Optics, Vol. V (Springer Proceedings in Physics, **41**, 57 (1989)); B. Yurke, Phys. Rev. **A32**, 300 (1985).
[17] D. N. Klyshko, *Photons and Nonlinear Optics*, Nauka, Moscow (1977).
[18] G. J. Milburn, D. F. Walls, and M. D. Levenson, J. Opt. Soc. Am. **B1**, 390 (1984).
[19] V. V. Dodonov and V. K. Man'ko, Proc. Lebedev Physics Institute, Acad. Sci. USSR **183**, 5 (1987) [English translation published by Nova Science Publishers, New York].
[20] V. P. Bykov and G. V. Shepelev, *Radiation by Atoms near Material Bodies*, Nauka, Moscow (1986).
[21] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[22] G. A. Lyakhov and V. A. Makarov, Izv. Vysshh. Uchebn. Zaved. Radiofiz. **22**, 1453 (1979); **24**, 373 (1981) [Radiophys. Qu. Electron. **22**, 1014 (1980); **24**, 262 (1981)]; L. P. Geborkyan, G. A. Makarov, and E. B. Cherepetskaya, Kvantovaya Elektron. (Moscow) **16**, 2478 (1989) [Sov. J. Quantum Electron. **19**, 1591 (1989)].
[23] A. P. Alodzhants, S. M. Arakelyan, and Yu. S. Chilingaryan, Izv. Akad. Nauk SSSR Ser. Fiz. **55**, 357 (1991); Kvantovaya Elektron. (Moscow) **18**, 626 (1991) [Sov. J. Quantum Electron. **21**, 570 (1991)]; Quantum Opt. **4**, 62 (1992).
[24] A. P. Alodzhants, A. M. Arakelyan, and Yu. S. Chilingaryan *et al.*, Izv. Akad. Nauk SSSR Ser. Fiz. **56**, 25 (1992); A. P. Alodjants, S. M. Arakelyan, and Yu. S. Chilingaryan, Laser Phys. **2**, 341 (1992).
[25] S. A. Akhmanov, A. V. Belinskiĭ, and A. S. Chirkin, Kvantovaya Elektron. (Moscow) **15**, 873 (1988) [Sov. J. Quantum Electron. **18**, 560 (1988)]; Opt. Spektrosk. **66**, 738 (1989) [Opt. Spectrosc. (USSR) **66**, 432 (1989)].
[26] B. Yoon and J. W. Negele, Phys. Rev. **A16**, 1451 (1977).
[27] S. M. Arakelyan, S. A. Akhmanov, V. G. Tunkin, and A. S. Chirkin, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 571 (1974) [JETP Lett. **19**, 299 (1974)].
[28] S. M. Arakelyan, L. E. Arushanyan, and Yu. S. Chilingaryan, Zh. Eksp. Teor. Fiz. **80**, 1186 (1981) [Sov. Phys. JETP **53**, 607 (1981)].
[29] S. Reynaud, C. Fabre, and E. Jiacobino, J. Opt. Soc. Am. **B4**, 1520 (1987).
[30] A. P. Alodzhants, G. A. Dzheĭranyan, L. P. Gevorkyan, and S. M. Arakelyan, Kvantovaya Elektron. (Moscow) **20** (1993)—to be publ.
[31] A. N. Oraevskiĭ, Zh. Eksp. Teor. Fiz. **95**, 59 (1989) [Sov. Phys. JETP **68**, 34 (1989)].
[32] A. V. Belinskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **53**, 73 (1991); **54**, 569 (1991) [JETP Lett. **53**, 74 (1991)]; **54**, 571 (1991)]; D. Yu. Kuznetsov, Pis'ma Zh. Eksp. Teor. Fiz. **54**, 566 (1991) [JETP Lett. **54**, 568 (1991)].

Translated by D. ter Haar