On the nature of the spatial distribution of metric inhomogeneities in the general solution of the Einstein equations near a cosmological singularity

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The question of the effect of a stochastic oscillatory mode on the large-scale inhomogeneities of the metric of characteristic dimensions much larger than the size of the horizon is investigated. The description of the inhomogeneities is given in a canonical formalism which explicitly singles out four physically arbitrary functions. It is shown that in the process of evolution one of them, having the significance of the Arnowitt–Deser–Misner Hamiltonian density, remains constant, while the other three acquire the character of random functions of the coordinates. A statistical description of the latter is obtained.

1. INTRODUCTION

As is well known, one of the problems of relativistic cosmology is the formulation of initial conditions for the so-called standard model of the universe. In spite of the fact that the standard (hot) model gives an adequate description of the current state of our universe, there exist several observable facts (such as the flatness, the absence of a horizon, the occurrence of density fluctuations necessary for the formation of structure in the form of galaxies and their clusters), whose origin it is unable to explain. These facts enter the standard model in the form of initial conditions. The possibility that these facts could be given a rational explanation arose in the framework of models containing a phase with exponential growth of the universe, the so-called inflationary models.[1–3] The presence in these models of the de Sitter phase provides the mechanism by means of which the necessary initial conditions can be formulated naturally. Nevertheless it is still of interest to look for other possible mechanisms for the solution of this problem.

In the present article we consider the possibility of utilizing for such a mechanism the chaotic oscillatory mode discovered in Ref. 4. In a somewhat different context this idea was first proposed by Misner.[5] Misner considered a homogeneous type IX model and attempted to explain the isotropy of the relict radiation by supposing that the horizon could open in some directions in the vicinity of a singularity in the oscillatory mode of evolution of the universe. Although that attempt was later discredited, a similar idea could be given a different direction by taking into account the inhomogeneity of the gravitational field. The question can be posed as follows: can the large-scale structure of an inhomogeneous field become homogeneous and isotropic in the course of evolution? The oscillatory mode has been studied in detail in homogeneous models.[6–8] In this article we investigate the influence of this mode on the spatial distribution of large-scale inhomogeneities in the metric, and attempt to answer thereby the question posed above.

As is well known, the dynamics of a large-scale quasi-homogeneous gravitational field in the vicinity of singularities consists in an alternating series of "Kasner epochs"[1)] replacing each other.[4] In an individual Kasner epoch the metric (in leading order in $1/t$) has the form

$$ds^2 = dt^2 - (t^{2s_l} l_\alpha l_\beta + t^{2s_m} m_\alpha m_\beta + t^{2s_n} n_\alpha n_\beta) dx^\alpha dx^\beta, \quad (1.1)$$

where the Kasner vectors l, m, and n and the exponents s_l, s_m, and s_n are functions of the coordinates. The law of replacement of Kasner regimes determines the transformation of Kasner epochs (TKE) T: $(l,m,n,s_l,s_m,s_n) \to (l',m',n',s'_l,s'_m,s'_n)$ (for its explicit form see Ref. 4). We note the following characteristic properties of TKE.

1. In the "deep oscillations" approximation the transformation formulas for the exponents and the vector amplitudes turn out to be local (i.e., independent of spatial derivatives).[4,6]

2. TKE has the stochastic property.[6–8]

It follows from the second property, in particular, that for a sufficiently prolonged action of TKE the information about initial conditions is "forgotten" (the imprecision in the determination of the initial conditions grows exponentially with the number of elapsed Kasner eras), and the evolution of the field admits a statistical description independent of these conditions. It is natural to expect that the stochastic nature of TKE, together with the already indicated locality property, should lead to a monotonic decrease of the coordinate scale of the metric inhomogeneities and, in final analysis, to the formation of spatial chaos in the metric functions (we note that the spatial scale of the inhomogeneities can, nevertheless, turn out to be increasing when the cosmological space expansion is taken into account). Indeed, it was shown in Ref. 9, using as the example the general solution constructed in Ref. 10 with a scalar field, that as $t \to 0$ the action of TKE leads to the fractioning of the coordinate scale λ of the inhomogeneities of the Kasner exponents s_l ($\lambda \sim \lambda_0 2^{-N}$, where N is the number of elapsed Kasner epochs and λ_0 is the initial scale of the inhomogeneities). Moreover, during the last (monotonic) Kasner epoch, during which the cosmological collapse ends in the presence of a scalar field, the chaotic cellular structure of the exponents is formed with universal (for sufficiently large N) statistical properties.

In the absence of the scalar field, as well as in the problem of cosmological expansion, the study of the properties of the spatial distribution of the metric inhomogeneities with the help of the TKE turns out to be inconvenient. This is due in the first place to the fact that the change in the Kasner

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epochs at different points in space proceeds not simultaneously but on a certain hypersurface, which in the general case is not spacelike. Moreover, TKE has bad statistical properties (as manifested by the absence of an invariant measure; in this sense a representation factorized over the Kasner eras is more adequate, see Ref. 6). It turns out that to this end the canonical approach, proposed previously by Misner,[5] is more convenient. In this article a generalization of the Misner approach to the case of an inhomogeneous field is given (for simplicity the case with matter absent will be considered).

In the case of the homogeneous model the gravitational field can be described by a complex quantity z, which is characteristic of the degree of anisotropy of space, and by its canonical conjugate variable p characterizing the speed of variation of the anisotropy.[11,12] In the case of an inhomogeneous field the quantities z and p become functions of the space coordinates. At the singularity the dynamical system describing the inhomogeneous gravitational field degenerates into a direct product of a continuum of systems of homogeneous type (a result, obvious *a priori* from the already mentioned locality of the transformation formulas for the exponents and vector amplitudes and indicating the absence of an effect on the dynamics by the space derivatives).

In terms of the variables z and p the system of the homogeneous type represents the geodesic flow on parts of a Lobachevskiĭ plane having a finite phase-space volume. As is well known, the behavior of geodesics on a manifold with negative curvature is characterized by exponential instability[13,14] (during the motion along a geodesic the normal deviations grow no slower than the exponential of the traversed path, whose exponent equals the square root of the modulus of the curvature). This instability gives rise to the stochastic nature of the corresponding geodesic flow. The system possesses the mixing property and an invariant measure induced by the Liouville measure. The absence of influence of space derivatives on the dynamics of the inhomogeneous field simplifies the description of the spatial structure of the dynamic functions. In particular, the function $h=\frac{1}{2}(1-|z|^2)|p|$, which is the ADM (Arnowitt–Deser–Misner[11]) Hamiltonian density and characterizes the rate of variation of the spatial volume, remains constant in the process of evolution and, consequently, conserves its initial inhomogeneity. The remaining dynamical functions acquire the character of random functions of the coordinates. Moreover, the statistical properties of the temporal as well as spatial behavior of these functions turn out to be the same and are characterized by the invariant distribution (4.1).

2. THE HAMILTONIAN FORM OF THE THEORY OF GRAVITY

The basic variables in the canonical formulation of gravity are the Riemann metric $g_{\alpha\beta}$ specified on the 3-manifold S, and its conjugate momentum matrix $\Pi^{\alpha\beta}=\sqrt{g}(K^{\alpha\beta}-g^{\alpha\beta}K)$, where $K_{\alpha\beta}$ is the extrinsic curvature of S. The Lagrange function for the gravitational field has in Planck units the form

$$\Lambda=\int_S\left(\Pi^{\alpha\beta}\frac{\partial}{\partial t}g_{\alpha\beta}-NC+N^\alpha C_\alpha\right)d^3x, \tag{2.1}$$

where

$$C=\frac{1}{\sqrt{g}}\left[\Pi^\alpha_\beta\Pi^\beta_\alpha-\frac{1}{2}(\Pi^\alpha_\alpha)^2+g(-{}^3R)\right],$$
$$C_\alpha=-2\Pi^\beta_{\alpha;\beta}. \tag{2.2}$$

Here N and N^α are, respectively, an arbitrary function and a vector field on S, which are interpreted as the lapse function and displacement vector in space-time, constructed from the temporal evolution of S, and ${}^3R$ is the scalar curvature of S. In the canonical formalism N and N^α appear as Lagrange multipliers. Variation of the action with respect to them gives the equations of constraint (the so-called Hamiltonian and momentum constraints) $C=0$ and $C_\alpha=0$ which, together with the equations of motion for $(\Pi^{\alpha\beta}, g_{\alpha\beta})$, constitute the Einstein equations.

The existence of constraints reduces the number of independent variables in the Lagrangian (2.1) to four functions. To study the question on the behavior of the metric inhomogeneities it is necessary to extract in some fashion the independent variables explicitly. This can be achieved by solving the equations of constraint (in the following it will be convenient to keep the Hamiltonian constraint unsolved, thus keeping a freedom in the choice of time). The form of the metric (1.1) indicates how such variables can be extracted. To this end we parametrize the metric and the momenta in the form

$$g_{\alpha\beta}=l_\alpha l_\beta+m_\alpha m_\beta+n_\alpha n_\beta,$$
$$\Pi^{\alpha\beta}=p_l L^\alpha L^\beta+p_m M^\alpha M^\beta+p_n N^\alpha N^\beta, \tag{2.3}$$

where $L^\alpha=g^{\alpha\beta}l_\beta$, while p_l, p_m, and p_n are the eigenvalues of the mixed momentum matrix (which are scalar densities of unit weight under coordinate transformations in S), and l, m, and n evidently coincide with the Kasner vectors. We note that in the general case the representation (2.3) is unambiguous. As independent variables we can take p_l, p_m, p_n and a set of quantities canonically conjugate to them, which are evidently the logarithms of the scales of the vectors (2.3). The following procedure will be employed to extract them explicitly. Making use of the freedom in the choice of the coordinate system in S, we require that the conditions $(l,m)=(m,n)=(n,l)=0$ be satisfied (here and below all vector operations are performed in the same way as in a Euclidean space). In this system of coordinates we define the scale functions by the relation $\exp(q^l)=l^2=(l,l)$. Going over now to arbitrary coordinates, we obtain a parametrization of the vectors by nine functions in the form

$$l_\alpha=\exp(q^l/2)\tilde{l}_\alpha,\quad \tilde{l}_\alpha=U_{lK}\partial_\alpha\varphi^K\quad (K=l,m,n), \tag{2.4}$$

where $U_{LK}\in SO(3)$ is a matrix depending on three angles. Substituting (2.3) and (2.4) into (2.1) we obtain for the Lagrange function the expression

$$\Lambda=\int_S\left(\sum_A p_A\frac{\partial}{\partial t}q^A+\sum_A\pi_A\frac{\partial}{\partial t}\varphi^A-NC+N^\alpha C_\alpha\right)d^3x, \tag{2.5}$$

where the quantities π_A are expressed in terms of U_{LK} and p_l, p_m, and p_n by the relation

$$\pi_K=-2\partial_\alpha\left(\sum_{A,B}p_A U_{AK}U_{AB}\partial x^\alpha/\partial\varphi^B\right). \tag{2.6}$$

This relation can be solved for the quantities U_{LK} and in this way all the functions in the vectors (2.4) can be expressed in terms of the canonical variables only. In terms of the canonical variables the constraint equations take the form

$$C = \frac{1}{\sqrt{g}}\left\{\sum p_A^2 - \frac{1}{2}\left(\sum p_A\right)^2 + g(-{}^3R)\right\},$$

$$C_\alpha = \sum_A (p_A\partial_\alpha q^A + \pi_A\partial_\alpha \varphi^A), \tag{2.7}$$

where the scalar curvature ${}^3R$ is given by the expression

$$g(-{}^3R) = \sum (l \operatorname{rot} l)^2 - \frac{1}{2}\left(\sum l \operatorname{rot} l\right)^2 - 2\sum (l \operatorname{rot} m)(m \operatorname{rot} l) + 2\sum \sqrt{g}\nabla\left(\frac{l \operatorname{rot} m - m \operatorname{rot} l}{\sqrt{g}}[lm]\right) \tag{2.8}$$

(here the sum is taken over cyclic permutations), and in view of (2.4) and (2.6) is a function of π_A, p_A, q^A, and φ^A. Solving the momentum constraint equations for π_A and choosing φ^A as the new coordinates we can completely exclude these variables from the Lagrange function (2.5). In terms of the remaining independent variables the action for the inhomogeneous gravitational field takes the form

$$I_g = \int_S \left(\sum p_A \frac{\partial}{\partial t} q^A - NC(p, q)\right) d^3x dt. \tag{2.9}$$

3. THE ASYMPTOTIC FORM OF THE THEORY IN THE VICINITY OF A COSMOLOGICAL SINGULARITY

Asymptotically close to a singularity the system (2.9) has a rather simple model representation. To pass to this representation we parametrize the scale functions as follows[11,12]

$$q^A = -e^{-\tau}\eta_A(z), \quad \eta_A = \frac{1 + |z|^2 - 4\mathrm{Re}[z \exp(i\theta_A)]}{1 - |z|^2},$$

$$A = l, m, n, \tag{3.1}$$

where $z = z(x)$ is a complex function, $|z|^2 \leqslant 1$, and $\theta_A = (0, \pm 2\pi/3)$. Now the action (2.9) becomes

$$I_g = \int_S \left\{\mathrm{Re}\left(\bar{p}\frac{\partial}{\partial t}z\right) - h\frac{\partial}{\partial t}\tau - \frac{N}{6\sqrt{g}}e^{2\tau}[\varepsilon^2(z, p) + U - h^2]\right\}$$

$$\times d^3x\, dt, \tag{3.2}$$

where $U = 6e^{-2\tau}g(-{}^3R)$, and $\varepsilon^2(z,p) = \frac{1}{4}(1 - |z|^2)^2|p|^2$. As already noted above, the evolution of the metric near a singularity corresponding to the asymptote $\tau \to -\infty$ consists of alternating series of Kasner regimes (1.1). The Kasner regime corresponds to ignoring in (3.2) the potential in comparison to the kinetic term ($\varepsilon^2 \gg U$). Making use of the expression (2.8) for the potential, the condition for the applicability of the Kasner regime can be written in the form (see Ref. 4)

$$q^l(z, \tau) + \ln\left|\frac{\sqrt{6}}{\varepsilon}e^{-\tau}\tilde{l} \operatorname{rot} \tilde{l}\right| = -e^{-\tau}\eta_l(z) + \ln\left|\frac{\sqrt{6}}{\varepsilon}e^{-\tau}k\right| \ll 0 \tag{3.3}$$

(and similar inequalities for the vectors m and n). Here the coordinate scale over which the metric changes substantially is of the order of $1/k$. Therefore the inequalities (3.3) impose on the degree of inhomogeneity of the gravitational field definite restrictions. A rougher, but physically clearer, estimate of these restrictions can be obtained by setting $z = 0$. Then the conditions (3.3) will take the form $l_i \gg l_h$ (where l_i, l_h are characteristic spatial dimensions of the inhomogeneity and the horizon, respectively). A change of the Kasner regimes is caused by the violation of one of these inequalities. This happens when one of the quantities η_l reaches the minimum value $\eta_l^{\min}(z,\tau) \propto -e^{\tau}\ln|(\sqrt{6}/\varepsilon)e^{-\tau}\tilde{l} \operatorname{curl} \tilde{l}|$ (which corresponds to the maximal value of q_l).[4] As $\tau \to -\infty$ we can use, to leading order, the "deep-oscillations" approximation, i.e., set $\eta_A^{\min} = 0$ and replace quantities of the type $\exp(q^A)$ in U by their asymptotic values $\exp(q^A)_{\tau \to -\infty} \Xi_A(z) = \{0$ for $\eta_A(z) > 0$ and ∞ for $\eta_A(z) < 0\}$. We then obtain for U the model representation

$$U \underset{\tau \to -\infty}{\to} V(z) = \sum_A \Xi_A(z), \tag{3.4}$$

which depends only on $z(x)$. Choosing as time the quantity τ (i.e., setting the lapse function N to be $3\sqrt{g}e^{-2\tau}/h$) and making use of the asymptotic expression (3.4) for the potential, we can reduce the action to the form

$$I_g = \int_S [\mathrm{Re}(\bar{p}\frac{\partial}{\partial \tau}z) - h(p, z)]d^3x\, d\tau, \tag{3.5}$$

where $h(p,z) = [\varepsilon^2(z,p) + V(z)]^{1/2}$. The expression (3.5) formally coincides with the action for the homogeneous model,[5,11,12] but differs from this model in that the basic variables are already functions of the spatial coordinates. The absence of spatial derivatives from (3.5) leads to the result that the corresponding dynamic system is a direct product of a continuum of systems of the "homogeneous" type.

4. HOMOGENEOUS GRAVITATIONAL FIELD

For a homogeneous gravitational field the system (3.5) represents a "billiard" on a part of the Lobachevskiĭ plane. To see this it is convenient to make use of the Poincaré model of the Lobachevskiĭ plane on the upper complex half-plane $H = \{W = U + iV, V \geqslant 0\}$, the quantity W being related to z by $z = 1 + i\sqrt{3}W/(1 - i\sqrt{3}W)$. The line $V = 0$ is called the absolute and its points lie at infinity. The geodesics in H are given by semi-circles with centers on the absolute, or by rays perpendicular to the absolute. The billiard constitutes the region $K \in H$, bounded by the geodesic triangle $\partial K = \{\eta_A(z) = 0, A = l,m,n\} = \{|W| = 1, U = \pm 1\}$. The area of the billiard is finite and equal to π. The Kasner regime corresponds to the motion of a point along a geodesic inside the region K, and a change in the Kasner regime occurs upon reflection from the potential walls. As is well known, geodesic flow on a manifold with negative curvature possesses the mixing property (i.e., stochasticity).[13,14] This means that an arbitrary initial measure relaxes to an invariant one which in this case has the form

$$d\mu_{inv}(z, \vartheta) = \frac{1}{\pi}\frac{d^2z}{(1 - |z|^2)^2}\frac{d\vartheta}{2\pi}, \tag{4.1}$$

where the angle ϑ determines the direction of the velocity and is connected with the momentum variables by $p = h\exp(i\vartheta)/(1-|z|^2)$, while h is an integral of the motion. The characteristic relaxation time is determined by the entropy of the system, which in this case is equal to unity (this means that the "coarse" element of phase space grows with time like $\overline{\Delta\Gamma} \sim \Delta\Gamma_0 \exp(\Delta\tau)$. It can be shown that the results obtained here within the framework of the homogeneous model give, upon passage to the Kasner variables, the same stationary distribution as in Refs. 6–8. However, the present approach is more convenient for the study of the spatial distribution in the inhomogeneous case.

5. INHOMOGENEOUS GRAVITATIONAL FIELD

In the presence of inhomogeneities in the gravitational field the quantities h, z, and ϑ are functions of the spatial coordinates. The absence in (3.5) of derivatives with respect to spatial coordinates results in the coordinates playing a passive role. This role reduces to an additional coordinate dependence of the initial conditions for the homogeneous field. This makes possible a translation of the results for the homogeneous field to the inhomogeneous case. In the process of evolution the function $h(x)$ remains constant. The characteristic scale of the inhomogeneities of the functions $z(x)$ and $\vartheta(x)$ (as a result of mixing) behaves on the average like $1/k = (dz/dx)^{-1} \sim (1/k_0)\exp(-\Delta\tau)$ and after a sufficiently long time $\Delta\tau \to \infty$ the quantities z, and ϑ become random functions of the coordinates. The spatial distribution of these functions can be described by their average values $\langle z(x)\rangle$, $\langle e^{i\vartheta(x)}\rangle$ and the corresponding correlation functions of the type $\langle z(x)\bar{z}(y)\rangle$. Here the angular brackets that denote averaging can be interpreted in two ways: either as averaging with the help of the distribution function that describes the lack of precision in the determination of the initial data, or as averaging over a certain coordinate volume ΔV of the space.[2)] In the latter case it is convenient to make use of the n-point distribution function defined by the relation

$$\rho(x_1, ..., x_n, z_1, ..., z_n, \vartheta_1, ..., \vartheta_n) = \frac{1}{\Delta V}\int_{\Delta V} \prod_{i=1}^{n} \delta(z_i - z(x_i+\xi))\delta(\vartheta_i - \vartheta(x_i+\xi))d^3\xi. \quad (5.1)$$

The presence of mixing leads to relaxation of the initial distribution function (5.1) to the limiting one (for $\Delta\tau \to \infty$), which is expressed in the form of a direct product of the measures (4.1)

$$d\mu = \prod_{i=1}^{n} d\mu(z_i, \vartheta_i). \quad (5.2)$$

From (4.1) and (5.2) we readily obtain the limiting expression for the averages and the correlating function

$$\langle z(x)\rangle = \langle \exp[i\vartheta(x)]\rangle = 0, \quad \langle z(x), \bar{z}(y)\rangle = \langle |z|^2\rangle\delta(x,y), \quad (5.3)$$

where the applicability of the Dirac delta-function is in actuality restricted on the small-scale side by the inequalities (3.3).

6. CONCLUDING REMARKS

In this manner the large-scale structure of the quasi-homogeneous gravitational field is determined by the single function of the spatial coordinates $h(x)$. In this sense the dynamics of the field acquires the character of the quasi-isotropic Lifshitz-Khalatnikov solution.[15] The function $h(x)$, which is (in the vicinity of a singularity) an integral of the motion, carries the information on the primordial degree of inhomogeneity of the space. We note that this does not yet allow one to make definitive conclusions about the inhomogeneity or homogeneity of the Universe at a later stage of evolution. Here the main contribution to $h(x)$ comes not from the gravitational field but from the field of ordinary matter; and the investigation of this question requires additional research.

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[1)]The correctness of the representation of the dynamics of the gravitational field by a sequence of Kasner epochs of the form (1.1) imposes the restrictions (3.3) on the degree of inhomogeneity of the field (see Ref. 4).

[2)]Note that the coordinate volume must not be too small. Otherwise there would not be enough time for the establishment of an invariant distribution and (5.2) will not be true.

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