Gaussian free turbulence

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A broad class of infinite-dimensional systems which conserve phase volume have integrals of motion with structure such that it is possible to give a complete analysis of the corresponding Gibbs distribution. In that case the ultraviolet catastrophe plays an important role. Free turbulence in such systems decays into a coherent state (a statistical attractor) and Gaussian noise. The Gaussian noise is determined by a universal singular correlator. We give examples of physically interesting systems. The statistical attractor of two-dimensional MHD contains current layers.

1. It is well known that free turbulence in some infinite-dimensional systems is solitonic.[1–4] The turbulent state decays into solitons and weakly nonlinear waves.[1] We obtain this result by a direct analysis of the Gibbs distribution. It turns out that these weakly nonlinear waves are Gaussian noise with a universal singular correlator. The coherent state which together with the Gaussian noise forms the free turbulence cannot always be called solitonic. We prefer the term Gaussian turbulence. The statistics of our nonlinear systems can be completely analyzed because of the ultraviolet catastrophe. We describe our procedure in detail using the example of one-dimensional weakly dispersive waves moving in one direction. After that we give results for one-dimensional weakly dispersive waves moving in both directions, for the degenerate Hasegawa–Mima equation (two-dimensional electron MHD) and for two-dimensional MHD. The latter two examples are interesting because the statistical attractor may be singular—it contains tangential discontinuities and current layers. Finally we formulate the conditions for the applicability of our approach in a general form and give examples of systems to which it cannot be applied.

Section 2 can be taken as yet another confirmation of the concept of solitonic turbulence. Sections 3, 4, and 5 (two-dimensional models and general statement) must be taken as purely conjectural. The fact is that the statistical theory of such infinite-dimensional systems contains many open problems, even in the case of one-dimensional evolutionary equations, not to mention multidimensional generalizations.[5]

2. To stress the direct connection of our approach with papers on solitonic turbulence[1–4] we start with an example close to Ref. 1. One-dimensional weakly dispersive waves moving in one direction are described by the Hamilton equation

$$\partial_t u = \partial_x(\delta H/\delta u), \tag{1}$$

$$H[u] = \int dx\left\{\frac{1}{2}u_x^2 + f(u)\right\}. \tag{2}$$

If the waves are weakly nonlinear one can restrict the expansion of f in powers of u to three terms. We arrive at the KdV equation. The KdV equation is completely integrable and has an infinite set of integrals of motion. We have for the function f in its general form two important integrals of motion—the energy H and the momentum

$$P[u] = \frac{1}{2}\int dx\, u^2. \tag{3}$$

We are interested in free turbulence in the system (1) in a periodic cell. More precisely, let us take initially some state $u_0(x)$ and let the system be left to itself. We must find the probability distribution $u(x)$ at large times. According to the statistical theory we can solve this problem as follows. Take the statistical sum in the form

$$Z(\alpha,\beta) = \int Du(x)\exp\{-\alpha P[u] - \beta H[u]\} \tag{4}$$

and choose the reciprocal temperatures α and β such that the average energy and momentum are equal to the energy and the momentum at the initial time:

$$\langle P[u]\rangle = P[u_0] \equiv P_0, \quad \langle H[u]\rangle = H[u_0] \equiv H_0. \tag{5}$$

We write conditions (5) in the form

$$Z(\alpha,\beta) = Y(P_0, H_0). \tag{6}$$

We first find the most probable state. This state minimizes the functional in the exponent in (4) and, hence, satisfies the equation

$$\alpha\delta P/\delta u + \beta\delta H/\delta u = 0. \tag{7}$$

Putting $\gamma = \alpha/\beta$ we rewrite (7) in the form

$$u_{xx} - \gamma u - f'(u) = 0. \tag{8}$$

The solution of (8) are solitons which we denote by $s(x)$. The phase of a soliton is arbitrary—together with $s(x)$, the solutions of (8) are $s(x+\theta)$, $\theta = \text{const}$. We choose γ such that $P[s] = P_0$.

We write an arbitrary state $u(x)$ in the form $u = s + v$ and we expand $H + \gamma P$ in powers of v up to the second order:

$$H[u] + \gamma P[u] = H_s + \gamma P_0 + \frac{1}{2}\int dx\{v_x^2 + (f''(s)+\gamma)v^2\}, \tag{9}$$

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where we have written $H_s \equiv H[s]$. We discuss the accuracy of the expansion (9) below. We now expand v in the eigenmodes of the equation

$$v_{xx} - (\gamma + f''(s))v = -\lambda v. \tag{10}$$

We know one of the eigenmodes: $\lambda_0 = 0, v_0 = s_x$. We assume that the soliton realizes a local minimum of the functional $H + \gamma P$. The remaining eigenvalues λ_n are then positive. The zeroth mode corresponds to a phase shift θ of the soliton and will be considered separately. We thus have

$$v(x) = \sum_{n=1}^{\infty} a_n v_n(x) \tag{11}$$

and we can rewrite (4) in the form

$$Z = \int d\theta \prod_n \int da_n \exp\{-\beta \sum_n \lambda_n a_n^2/2\}. \tag{12}$$

We then have

$$\langle H \rangle = H_s + (1/\beta) \sum_n (1 - \gamma/\lambda_n), \tag{13}$$

$$\langle P \rangle = P_0 + (\gamma/\beta) \sum_n (1/\lambda_n). \tag{14}$$

The sum (13) diverges (ultraviolet catastrophe). Temporarily we restrict the number of modes: $n \leqslant N$. We choose β such that $\langle H \rangle$ equals H_0, i.e., we put

$$\beta = (N - \gamma \sum_1^N \lambda_n^{-1})/(H_0 - H_s). \tag{15}$$

We then have

$$Z(\alpha, \beta) = Y(P_0 + \delta P, H_0), \tag{16}$$

where we have put

$$\delta P = (\gamma/\beta) \sum_1^N \lambda_n^{-1}. \tag{17}$$

As $N \to \infty$ we find that $\beta \to \infty$ and $\delta P \to 0$. Then, for $N \to \infty$ and β defined by Eq. (15) we get the required probability distribution. We can estimate the error in the expansion (9) as $N^{-1/2}$; for large N the expansion is exact. The temperature is equal to zero ($\beta \to \infty$), so that the noise level is equal to zero in any bounded spectral range. We discard the noise with $n \simeq 1$ and for the others we rewrite (10) in the form

$$v_{xx} = -\lambda v. \tag{18}$$

The Gaussian noise is here given by the correlator

$$\langle v_x(x_1) v_x(x_2) \rangle = 2(H_0 - H_s)\delta^*(x_1 - x_2), \tag{19}$$

where we have put

$$\delta^*(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases} \tag{20}$$

We then have

$$\langle v(x_1) v(x_2) \rangle = 0. \tag{21}$$

With probability unity the state u is the same as $s(x+\theta)$ for some θ. The soliton is found to be a statistical attractor.[1] The phase volume is dumped into Gaussian noise together with the excess energy. On the other hand, u_x is only distributed around s_x and u_{xx} and the leading derivatives are infinite in each point x.

There can be several local minima [solutions of Eq. (8)]. Each of them corresponds to a stable rather than a metastable state since the temperature is equal to zero. The system falls into one of these states and remains in it. There can be no local minima at all if f changes too fast with increasing u. In the case of the nonlinear Schrödinger equation a collapse then appears and the problem of free turbulence does not arise.[1] We do not know what happens in such a situation in (1).

We can obtain free turbulence of weakly dispersive waves moving in both directions using a similar procedure. Such waves are described by the equations

$$\partial_t a = \partial_x(\delta H/\delta b), \quad \partial_t b = \partial_x(\delta H/\delta a), \tag{22}$$

$$H[a, b] = \int dx \left\{\frac{1}{2} a_x^2 + \frac{1}{2} b_x^2 + f(a, b)\right\}. \tag{23}$$

Apart from H the momentum,

$$P = \int dx\, ab \tag{24}$$

is conserved. A coherent state is described by the equations

$$a_{xx} = -W_a, \quad b_{xx} = -W_b, \tag{25}$$

where we have put

$$W(a, b) = -f(a, b) - \gamma ab. \tag{26}$$

Choosing γ so that the coherent state contains the whole momentum P_0 we get Gaussian noise containing the rest of the energy:

$$\begin{gathered} \langle a_x(x_1) a_x(x_2) \rangle = (H_0 - H_s)\delta^*(x_1 - x_2), \\ \langle b_x(x_1) b_x(x_2) \rangle = (H_0 - H_s)\delta^*(x_1 - x_2), \\ \langle a_x(x_1) b_x(x_2) \rangle = 0. \end{gathered} \tag{27}$$

This is true if the coherent state realizes a local minimum of the functional $H + \gamma P$.

3. Large-scale motion in a rotating atmosphere or in a magnetized plasma and also fast large-scale oscillations of a two-dimensional plasma are described by the degenerate Hasegawa–Mima equation (equation for two-dimensional electron MHD)

$$(\partial_t + [\nabla \Delta \Psi, \nabla])\Psi = 0. \tag{28}$$

Here [,] denotes the z component of the vector product and we have $\Delta = \partial_x^2 + \partial_y^2$. Equation (28) conserves the phase volume:

$$\int d^2 r \delta \Psi(r)/\delta \Psi(r) = 0. \tag{29}$$

The integrals of motion are the energy,

$$E = \frac{1}{2} \int d^2r(\nabla\Psi)^2 \tag{30}$$

and an infinite set of incompressibility integrals

$$I_n = \int d^2r\Psi^n. \tag{31}$$

The Gibbs distribution is

$$\rho = \exp(-\alpha E - \sum \beta_n I_n). \tag{32}$$

Introducing the temperature function

$$\beta(z) = \alpha^{-1} \sum \beta_n z^n, \tag{33}$$

we can rewrite (32) in the form

$$\rho = \exp\{-\alpha [E + \int d^2r\beta(\Psi)]\}. \tag{34}$$

The coherent state Ψ_s is given by the equation

$$\Delta\Psi_s = \beta'(\Psi_s) \tag{35}$$

and the conditions

$$I_n[\Psi_s] = I_{n0}. \tag{36}$$

If Ψ_s realizes a local minimum of $E + \int d^2r\beta(\Psi)$ the free turbulent state Ψ decays into Ψ_s and Gaussian noise ϕ. The Gaussian noise does not contain the I_n and contains the rest of the energy. It is determined by the pair correlator

$$\langle \nabla\phi\nabla\phi \rangle = 2(E_0 - E_s)\delta^*. \tag{37}$$

This example is interesting in that the statistical attractor (35), (36) can contain tangential discontinuities (lines on which $\Delta\Psi = \infty$; see Appendix).

4. Two-dimensional MHD is given by the equations

$$(\partial_t + [\nabla\Psi, \nabla])\Delta\Psi = [\nabla\chi, \nabla]\Delta\chi, \tag{38}$$

$$(\partial_t + [\nabla\Psi, \nabla])\chi = 0. \tag{39}$$

Here Ψ is the stream function and χ is the magnetic current function so that we have for the velocity $\mathbf{v} = (-\Psi_y, \Psi_x)$ and for the field $\mathbf{B} = (-\chi_y, \chi_x)$. The integrals of motion are the energy

$$E = \frac{1}{2} \int d^2r\{(\nabla\Psi)^2 + (\nabla\chi)^2\}, \tag{40}$$

the freezing-in integrals

$$I_n = \int d^2r\, \chi^n \tag{41}$$

and the cross helicity

$$H = \int d^2r \nabla\Psi\nabla\chi. \tag{42}$$

The statistical attractor is given by the equations

$$\Delta\chi_s + \gamma\Delta\Psi_s = \beta'(\chi_s), \tag{43}$$

$$\gamma\Delta\chi_s + \Delta\Psi_s = 0 \tag{44}$$

and the conditions

$$I_n[\chi_s] = I_{n0}. \tag{45}$$

The Gaussian noise is given by the correlators

$$\langle \nabla\Psi\nabla\Psi \rangle = a\delta^*, \quad \langle \nabla\Psi\nabla\chi \rangle = b\delta^*, \quad \langle \nabla\chi\nabla\chi \rangle = a\delta^*. \tag{46}$$

The constants a, b and γ are determined by the initial energy, helicity, and magnetic energy of the attractor (which is determined by the initial values of the I integrals). We have

$$a = E_0 - (1 + \gamma^2)W_s, \quad b = H_0 + 2\gamma W_s, \tag{47}$$

W_s denotes the magnetic energy of the attractor,

$$W_s = \frac{1}{2} \int d^2r(\nabla\chi_s)^2. \tag{48}$$

It is independent of γ and given by the I_{n0} integrals. The ratio γ of the temperatures is found from the equation

$$W_s\gamma^3 - (E_0 + W_s)\gamma - H_0 = 0. \tag{48'}$$

As in Sec. 3, the nontrivial topology of the initial configuration causes singularities in the statistical attractor. The attractor contains current layers which are simultaneously tangential discontinuities.

5. The conditions for the applicability of our scheme can in general form be formulated as follows. Let the system conserve the phase volume. Let its integrals have the form

$$I_n[u] = \int dx f_n(u, \partial u, ..., \partial^n u), \quad n = 0, ..., N-1, \tag{49}$$

$$E = \int dx\{g(u, ..., \partial^{N-1}u)(\partial^N u)^2 + h(u, ..., \partial^{N-1}u)\}. \tag{50}$$

The local minimum of E for fixed I_n will be a statistical attractor. The excess energy $E_0 - E_s$ is contained in the Gaussian noise; this noise does not contribute to the I_n. The Gaussian noise is given by the correlator

$$\langle \partial^N u \partial^N u \rangle = \mathrm{const}(E_0 - E_s)[g(u_s)]^{-1}\delta^*. \tag{51}$$

In conclusion we give two physically interesting examples where our approach does not work. The first example is trivial. A two-dimensional incompressible fluid is described by the Euler equation

$$(\partial_t + [\nabla\Psi, \nabla\Psi])\Delta\Psi = 0 \tag{52}$$

with the integrals

$$E = \frac{1}{2} \int d^2r(\nabla\Psi)^2, \quad I_n = \int d^2r(\Delta\Psi)^n. \tag{53}$$

One sees that the integrals (53) do not have the form (49), (50).

The second example is the Euler equation in Clebsch variables

$$\partial_t\lambda = -\delta H/\delta\mu, \quad \partial_t\mu = \delta H/\delta\lambda. \tag{54}$$

The Hamiltonian is equal to

$$H = \frac{1}{2} \int d^3r\, \mathbf{v}^2, \tag{55}$$

where the velocity $\mathbf{v}$ is determined by the conditions

$$\mathbf{v} = \lambda\nabla\mu - \nabla\phi, \quad \mathrm{div}\,\mathbf{v} = 0. \tag{56}$$

The integrals of (54) to (56) are H and

$$I_{n,m}[\lambda, \mu] = \int d^3r \lambda^n \mu^m. \tag{57}$$

The leading derivative occurring in the set of integrals is $\nabla\lambda$ and $\nabla\mu$ and the energy is quadratic in the leading derivative. Our procedure is inapplicable for another reason. It is not known whether there exist flows λ_s,μ_s which are a local minimum (maximum) of E for fixed $I_{n,m}$.

It is a pleasure for me to thank V. V. Yan'kov for discussions of this work and D. Ivanov for discussing the Appendix.

APPENDIX

We show that the solutions Ψ_s of (35) and (36) can contain tangential discontinuities. This problem has been discussed in detail in the theory of magnetic reconnections and current layers (Ref. 6 and the literature cited there). We give some nonrigorous considerations explaining the cause for the occurrence of a discontinuity.

According to (36) Ψ_s is obtained from the initial state Ψ_0 by a incompressible transformation [strictly speaking, this does not follow from (36) but is implied in (36)]. The energy (30) is then minimized. We are interested in the properties of such a mapping $O:\Psi_0\rightarrow\Psi_s$.

First let the topology of the lines of Ψ levels be trivial—a single maximum. The mapping then transforms all level lines into circles as shown in Fig. 1a. We now take an initial field Ψ_0 with two maxima (and a hyperbolic point; Fig. 1b). We assume that the energy in the exterior part of the separatrix is negligibly small. The mapping is then known from Fig. 1a; it is shown in Fig. 1c. We have not drawn the level lines in the external part of the separatrix bearing in mind that in that region we have $\nabla\Psi\rightarrow 0$. We can similarly consider another limiting case (Fig. 1d). The case of a general situation is somewhat intermediate between Fig. 1c and Fig. 1d (Fig. 1e). The common part of the separatrix is a tangential discontinuity—$\nabla\Psi$ changes its direction into the opposite one when it passes through it.

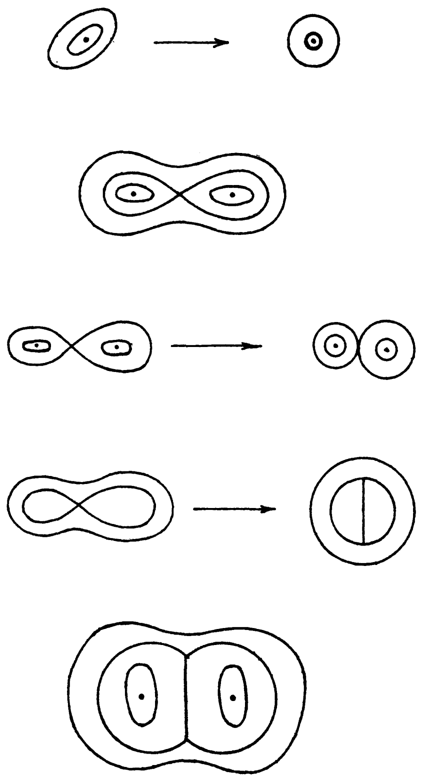

FIG. 1. The incompressible mapping M minimizes the energy.

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